LARGE SCALE SINGULARLY PERTURBED
BOUNDARY VALUE PROBLEMS

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ABSTRACT

In this paper an alternative approach to the method of asymptotic expansions for the study of a singularly perturbed linear system with multiparameters and multiple time scales is developed. The method consists of developing a linear non-singular transformation that transforms an arbitrary n -time scale system into a diagonal form. Furthermore, a dichotomy transformation is employed to decompose the faster subsystems into stable and unstable modes. Fast, slow, stable and unstable modes decomposition processes provide a modern technique to find an approximate solution of the original system in terms of the solution of an auxiliary system. This method yields a constructive and computationally attractive way to investigate the system.

Key Words: Diagonalization process, decoupled system, dichotomy transformation,
fast sub-system, fundamental matrix solution, multiparameter,
optimal control, stable and unstable systems.

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1 Introduction

Singular perturbations of two point boundary value problem is an active subject of research with long history. By employing the asymptotic expansion of such systems under strong conditions on the coefficient matrices, existence, uniqueness and approximations of solutions of such systems are studied in [12,13,14]. In [2,5,15,16], under less demanding conditions on the coefficient matrices, boundary value problems for two-time scale linear systems are analyzed. Furthermore, the fast and slow mode decomposition approach [2,9,16] provides a modern alternative technique to study the singularly perturbed systems. Moreover using a dichotomy transformation to decompose the faster subsystems into stable and unstable modes yields a constructive and computationally attractive procedure to investigate such problems [10,15,17,18].

In this paper, by following a hierarchical order reduction scheme of a joint multiparameters and multi-time scale singular perturbation of linear system [6,8,11], a linear non-singular transformation which totally decouples an arbitrary n-time scale and multiparameter linear singularly perturbed system, is developed. Furthermore, a dichotomy transformation is employed to decompose the faster subsystems into stable and unstable modes. Sufficient conditions are given for existence and uniqueness of solution of an auxiliary boundary value problem which guarantees the existence, uniqueness and approximation of solution of given boundary value problem. The obtained conditions are given in terms of known coefficient matrices.

This paper is organized as follows: In section 2, by following the argument [6,8,11], a joint multi-time scale multiparameter singularly perturbed two point boundary value problem is formulated. For the sake of convenience and simplicity a few notations are defined in section 3. These notations will be used throughout this paper. In section 4, by following the decoupling procedure in [6], a totally decoupling process is briefly discussed. Moreover, the asymptotic behaviour and the representation of the transformation in terms of the given coefficient matrices is given. In section 5, by developing a dichotomy transformation which decomposes the faster subsystems of totally decoupled system into stable and unstable subsystems, the existence, uniqueness and approximation of the original problems are investigated. An example illustrating the decomposing procedures and finding an approximate solution is presented in section 6. This example exhibits the generalization and extension of the work of Wilde and Kokotovic [17,18].

2 Problem Formulation

Consider a linear time-varying system of differential equations

$$e_i^j \dot{X}_{ij} = \sum_{p=1}^n \sum_{k=1}^{r_p} A_{ik}^{jp}(t) X_{pk} \quad (2.1)$$

with the boundary condition

$$\sum_{p=1}^n \sum_{k=1}^{r_p} Q_{ik}^{jp}(\epsilon) X_{pk}(0) + \sum_{p=1}^n \sum_{k=1}^{r_p} R_{ik}^{jp}(\epsilon) X_{pk}(1) = B_i^j(\epsilon) \quad (2.2)$$

where $\epsilon = (\epsilon_1^1 \epsilon_1^2 \dots \epsilon_1^{r_1} \dots \epsilon_n^1 \epsilon_n^2 \dots \epsilon_n^{r_n})$, $X_{ij} \in \mathbb{R}^{n_{ij}}$, $j \in J(1, r_i)$, $i \in J(1, n)$; r_i 's are positive integers

and $J(a, b) = \{a, a+1, \dots, b\}$; $a, b \in \mathbb{Z}^+ \cup \{0\}$, $n_i = \sum_{j=1}^{r_i} n_{ij}$; the dimension of the entire system (2.1) is

$N = \sum_{i=1}^n n_i$. In (2.1) all the coefficient matrices are continuous on $\mathbb{R}^+$ and have appropriate

dimensions. For $j \in J(1, r_i)$ and $i \in J(1, n)$, parameters ϵ_i^j 's are positive real numbers. For fixed

$i \in J(1, n)$, ϵ_i^j 's have the same order with respect to j . This means that for fixed $i \in J(1, n)$, the following inequality is valid:

$$\underline{\epsilon}_i \leq \frac{\epsilon_i^j}{\epsilon_i^k} \leq \bar{\epsilon}_i, \quad j, k \in J(1, r_i). \quad (2.3)$$

where $\underline{\epsilon}_i, \bar{\epsilon}_i$'s are some positive numbers. Furthermore, for any $j \in J(1, r_i)$, ϵ_i^j 's have different order with respect to i .

Assumption A 1.2.1:

$$\left\{ \begin{array}{l} \epsilon_1 = 1 \\ \lim_{\epsilon_i \rightarrow 0} \frac{\epsilon_{i+1}}{\epsilon_i} = 0 \quad \text{for } i \in J(2, n-1) \end{array} \right. \quad (2.4)$$

where $\epsilon_i = (\epsilon_i^1 \epsilon_i^2 \dots \epsilon_i^{r_i})^{\frac{1}{r_i}}$.

Using (2.4), we can rewrite the system (2.1)–(2.2) as

$$\epsilon_i \dot{X}_i = \sum_{j=1}^n D_i A_{ij}(t) X_j, \quad (2.5)$$

with the boundary condition

$$\sum_{j=1}^n Q_{ij}(\epsilon) X_j(0) + \sum_{j=1}^n R_{ij}(\epsilon) X_j(1) = B_i(\epsilon), \quad i \in J(1, n) \quad (2.6)$$

where $X_i = (X_{i1}^T \ X_{i2}^T \ \dots X_{ir_i}^T)^T$; $i \in J(1,n)$; $A_{ip} = (A_{ik}^{jp}(t))_{n_i \times n_p}$; $Q_{ip}(\epsilon) = (Q_{ik}^{jp}(\epsilon))_{n_i \times n_p}$,

$$R_{ip}(\epsilon) = (R_{ik}^{jp}(\epsilon))_{n_i \times n_p}, \quad B_i(\epsilon) = (B_i^j(\epsilon))_{n_i \times 1}; \quad i, p \in J(1,n).$$

For simplicity we omit the argument of the coefficient matrix functions. The matrices in (2.5) are block matrices that are formed in an obvious way from the coefficient matrices in (2.1) with D_i 's being defined by:

$$D_i = \text{diag} \left\{ \begin{bmatrix} \epsilon_i \\ 1 \\ \epsilon_i \end{bmatrix} I_{i1} \begin{bmatrix} \epsilon_i \\ 2 \\ \epsilon_i \end{bmatrix} I_{i2} \dots \begin{bmatrix} \epsilon_i \\ r_i \\ \epsilon_i \end{bmatrix} I_{ir_i} \right\}, \quad i \in J(1,n). \quad (2.7)$$

where I_{ir_i} are the identity matrices of appropriate dimensions. In view of (2.3) and (2.4), the elements of the D_i matrices are bounded, that is,

$$\xi_i \leq \frac{\epsilon_i}{\epsilon_i} \leq \tilde{\epsilon}_i, \quad j \in J(1,r_i), \quad i \in J(1,n). \quad (2.8)$$

where the bounds in (2.8) depend on the corresponding bounds in (2.3). Thus system (2.5)–(2.6) is equivalent to multiparameter multiple time scale system (2.1)–(2.2).

3 Notations and Definitions.

For the sake of simplicity and convenience to our future presentation, let us introduce the following notations.

For $m \in J(0,n-1)$, $r \in J(0,m)$, $l \in J(r,n)$

$$X_{lr}^m = (X_l^T(m), X_2^T(m), \dots, X_{l-r}^T(m), X_{l-r+1}^{T(m-1)}, X_{l-r+2}^{T(m-2)}, \dots, X_1^T(0)),$$

$$X_{lr}^{m0} = X_{lr}^m(0), \quad X_{lr}^{m1} = X_{lr}^m(1),$$

where $(X_1^T(0) \ X_2^T(0) \ \dots \ X_n^T(0))^T = (X_1^T \ X_2^T \ \dots \ X_n^T)^T$,

$$Q(0,\epsilon) = (Q_{ij}(\epsilon))_{n \times n}, \quad R(0,\epsilon) = (R_{ij}(\epsilon))_{n \times n}, \quad B(\epsilon) = (B_i(\epsilon))_{n \times 1}.$$

$$I_r = I_{n_r \times n_r}, \quad I^r = I_r \sum_{i=1}^r n_i \times \sum_{i=1}^r n_i, \quad I(r) = I \sum_{i=r}^n n_i \times \sum_{i=r}^n n_i,$$

and for $m \in J(0, n-2)$

$$L_m = (L_{n-m-1} \ L_{n-m-2} \ \dots \ L_{n-m-n-m-1}) ,$$

$$M_m = (M_{1 \ n-m}^T \ M_{2 \ n-m}^T \ \dots M_{n-m-1 \ n-m}^T)^T .$$

Define

$$\tilde{A}_{ij}^m = \begin{cases} D_i A_{ij} & , \text{ for } m = 0 \text{ and } ij \in J(1, n) \\ \tilde{A}_{ij}^{m-1} + \tilde{A}_{i \ n-m+1}^{m-1} L_{n-m+1 \ j} & , \text{ for } m \in J(1, n-1) \text{ and } ij \in J(1, n-m); \end{cases}$$

and $\hat{A}^m$ is defined by

$$\hat{A}_{ij}^m = \begin{cases} D_i A_{ij} & , \text{ for } m = 0 \text{ and } ij \in J(1, n) \\ \hat{A}_{ij}^{m-1} + \hat{A}_{i \ n-m+1}^{m-1} [\hat{A}_{n-m+1 \ n-m+1}^{m-1}]^{-1} \hat{A}_{n-m+1 \ j}^{m-1} & , \text{ for } m \in J(1, n-1) \\ & \text{and } ij \in J(1, n-m). \end{cases}$$

$\hat{A}^m$ can be represented by

$$\hat{A}^m = \begin{bmatrix} \hat{A}_{11}(m) & \hat{A}_{12}(m) \\ \hat{A}_{21}(m) & \hat{A}_{22}(m) \end{bmatrix} ,$$

where $\hat{A}_{11}(m) = (\hat{A}_{ij}^m)_{(n-m-1) \times (n-m-1)}$, $\hat{A}_{22}(m) = \hat{A}_{n-m \ n-m}^m$ and $m \in J(0, n-2)$.

For $m \in J(0, n-2)$, we denote

$$A^m = \begin{bmatrix} A_{11}(m) & A_{12}(m) \\ \frac{1}{\epsilon_{n-m}} A_{12}(m) & \frac{1}{\epsilon_{n-m}} A_{22}(m) \end{bmatrix} ,$$

where $A_{11}(m) = (\frac{1}{\epsilon_i} \tilde{A}_{ij}^m)_{(n-m-1) \times (n-m-1)}$,

$$A_{12}(m) = (\frac{1}{\epsilon_1} \tilde{A}_{1 \ n-m}^T \ \frac{1}{\epsilon_2} \tilde{A}_{2 \ n-m}^T \ \dots \ \frac{1}{\epsilon_{n-m-1}} \tilde{A}_{n-m-1 \ n-m}^T)^T ,$$

$$A_{21}(m) = (\tilde{A}_{n-m-1}^m \ \tilde{A}_{n-m-2}^m \ \dots \ \tilde{A}_{n-m-n-m-1}^m) ,$$

$$A_{22}(m) = \tilde{A}_{n-m \ n-m}^m .$$

For $m \in J(0, n-2)$, we denote

$$\bar{D}_m = \text{diag}(\epsilon_1 I_1 \quad \epsilon_2 I_2 \quad \dots \quad \epsilon_{n-m-1} I_{n-m-1}) ;$$

$$\hat{L}_m = -\hat{A}_{22}(m)^{-1} \hat{A}_{21}(m) ;$$

$$\hat{M}_m = \hat{A}_{12}(m) \hat{A}_{22}(m)^{-1}.$$

4 Diagonalisation Process.

In this section our aim is to develop a procedure to totally decouple the original system (2.5). This can be done by carrying over the procedure exactly in analogous manner as in [6]. The only difference is that the boundary conditions are changed in the process due to the application of transformation. By following a procedure analogous to section 4 in [6], at m -th step, we will have,

$$\left\{ \begin{array}{l} S_{n-m \text{ c}} : \quad \dot{X}_{n-m}^m = A^m X_{n-m}^m, \\ S_{n-m \text{ d}} \quad \epsilon_{n-i} \dot{X}_{n-i}(i+1) = (A_{22}(i) - \epsilon_{n-i} L_i A_{12}(i)) X_{n-i}(i+1), \text{ for } i \in J(0, m-1). \end{array} \right. \quad (4.1)$$

$$Q(m, \epsilon) X_{nm-1}^{m \ 0} + R(m, \epsilon) X_{nm-1}^{m \ 1} = B(\epsilon), \quad m \in J(0, n-2) \quad (4.2)$$

Coupled system (4.1 $S_{n-m \text{ c}}$) can be rewritten as

$$\left\{ \begin{array}{l} \dot{X}_{n-m-1}^m = A_{11}(m) X_{n-m-1}^m + A_{12}(m) X_{n-m}(m), \\ \epsilon_{n-m} \dot{X}_{n-m}(m) = A_{21}(m) X_{n-m-1}^m + A_{22}(m) X_{n-m}(m), \end{array} \right. \quad (4.3)$$

$$\epsilon_{n-i} \dot{X}_{n-i}(i+1) = (A_{22}(i) - \epsilon_{n-i} L_i A_{12}(i)) X_{n-i}(i+1), \text{ for } i \in J(0, m-1).$$

$$Q(m, \epsilon) X_{nm-1}^{m \ 0} + R(m, \epsilon) X_{nm-1}^{m \ 1} = B(\epsilon) \quad (4.4)$$

Assumption A 4.1 : $A_{22}(m)$ is non-singular.

Now we apply Chang's transformation [1] defined by

$$X_{n-m}^{m+1} = \tilde{T}_{n-m} X_{n-m}^m \quad (4.5)$$

to system (4.3) with (4.4),

where $\tilde{T}_n = T_n$,

$$\tilde{T}_{n-m} = \begin{bmatrix} T_{n-m} & 0 \\ 0 & I(n-m) \end{bmatrix}, \text{ for } m \in J(1, n-2)$$

and

$$T_{n-m} = \begin{bmatrix} I^{n-m-1} + \epsilon_{n-m} M_m L_m - \epsilon_{n-m} M_m \\ -L_m & I_{n-m} \end{bmatrix}, \text{ for } m \in J(0, n-2). \quad (4.6)$$

and L_m, M_m satisfy the following differential equations:

$$\begin{cases} \epsilon_{n-m} \dot{L}_m = -\epsilon_{n-m} L_m (A_{11}(m) + A_{12}(m) L_m) + A_{21}(m) + A_{22}(m) L_m, \\ \epsilon_{n-m} \dot{M}_m = -M_m (A_{22}(m) - \epsilon_{n-m} L_m A_{12}(m)) + \epsilon_{n-m} (A_{11}(m) + A_{12}(m) L_m) M_m + A_{12}(m). \end{cases} \quad (4.7)$$

Then we will have

$$\begin{cases} \dot{X}_{n-m-1}^{m+1} = (A_{11}(m) + A_{12}(m) L_m) X_{n-m-1}^{m+1}, \\ \epsilon_{n-i} \dot{X}_{n-i}^{(i+1)} = (A_{22}(i) - \epsilon_{n-i} L_i A_{12}(i)) X_{n-i}^{(i+1)}, \text{ for } i \in J(0, m) \end{cases} \quad (4.8)$$

$$Q(m+1, \epsilon) X_{nm}^{m+1}{}^0 + R(m+1, \epsilon) X_{nm}^{m+1}{}^1 = B(\epsilon) \quad (4.9)$$

where $Q(m+1, \epsilon) = Q(m, \epsilon) \tilde{T}_{n-m}^{-1}$, $R(m+1, \epsilon) = R(m, \epsilon) \tilde{T}_{n-m}^{-1}$.

Furthermore, this fast and slow mode decoupling procedure terminates after $(n-1)$ steps. In the light of this, m in (4.1) and (4.2) belongs to $J(1, n-1)$. At the end of $(n-1)$ steps the original system (2.4) with (2.5) will be completely decoupled.

This totally decoupled system is rewritten as

$$\begin{cases} \dot{U}_1 = A_{11}(n-1) U_1, \\ \epsilon_{n-m} \dot{U}_{n-m} = [A_{22}(m) - \epsilon_{n-m} L_m A_{12}(m)] U_{n-m}, \text{ for } m \in J(0, n-2). \end{cases} \quad (4.10)$$

$$Q(n-1, \epsilon) U^0 + R(n-1, \epsilon) U^1 = B(\epsilon), \quad (4.11)$$

Remark 4.1: From (4.7), for $m \in J(0, n-2)$, T_{n-m} is determined by the following differential equations.

$$\begin{cases} \epsilon_{n-m} \dot{L}_m = -\epsilon_{n-m} L_m (A_{11}(m) + A_{12}(m) L_m) + A_{21}(m) + A_{22}(m) L_m, \\ \epsilon_{n-m} \dot{\bar{M}}_m = -\bar{M}_m (A_{22}(m) - \epsilon_{n-m} L_m A_{12}(m)) + \epsilon_{n-m} \bar{D}_m (A_{11}(m) + A_{12}(m) L_m) \bar{D}_m^{-1} \bar{M}_m \\ \quad + \bar{D}_m A_{12}(m). \end{cases} \quad (4.12)$$

Where $\bar{M}_m = \bar{D}_m M_m$.

Remark 4.2: We remark that instead of applying transformations $\tilde{T}_n, \tilde{T}_{n-1}, \dots, \tilde{T}_{n-m}$ successively, to the original system $(m+1)$ times, we can apply a composition of transformation

$$T^{n-m} = \tilde{T}_{n-m} \circ \tilde{T}_{n-m+1} \circ \dots \circ \tilde{T}_n$$

to the original system (2.5). An application of this composite transformation T^{n-m} to (2.5) gives rise to (4.8). The structure of T^{n-m} is as follows.

$$T^{n-m} = \begin{bmatrix} T_{n-m} & 0 \\ 0 & I(n-m) \end{bmatrix} \circ \begin{bmatrix} P_{n-m+1}^1(1) & C_{n-m+1}(1) \\ R_{n-m+1}(1) & P_{n-m+1}^2(1) \end{bmatrix} \circ \begin{bmatrix} P_{n-m+2}^1(2) & C_{n-m+2}(2) \\ R_{n-m+2}(2) & P_{n-m+2}^2(2) \end{bmatrix} \circ \dots \circ \begin{bmatrix} P_n^1(m) & C_n(m) \\ R_n(m) & P_n^2(m) \end{bmatrix}$$

where, for all $m \in J(0, n-2)$ and $q \in J(0, m-1)$, $P_{n-q}^1(k)$, $P_{n-q}^2(k)$, $C_{n-q}(k)$, $R_{n-q}(k)$ are given by

for $1 \leq k \leq n-q-1$,

$$\left[P_{n-q}^1(k) \right]_{i,j} = \begin{cases} I_i + \epsilon_{n-q} M_{i \ n-q} L_{n-q \ j} & , \text{for } i=j, 1 \leq i,j \leq n-q-k, \\ \epsilon_{n-q} M_{i \ n-q} L_{n-q \ j} & , \text{for } i \neq j, 1 \leq i,j \leq n-q-k. \end{cases}$$

$$\left[R_n(1) \right]_{i,j} = -L_{n \ j}, \text{ for } i=1, 1 \leq j \leq n-q-1,$$

$$\left[R_{n-q}(1) \right]_{i,j} = \begin{cases} -L_{n-q \ j} & , \text{for } i=1, 1 \leq j \leq n-q-1, \\ 0 & , \text{for } 2 \leq i \leq q+1, 1 \leq j \leq n-q-1. \end{cases}$$

for $2 \leq k \leq n-q-1$

$$[R_{n-q}(k)]_{ij} = \begin{cases} \epsilon_{n-q} M_{n-q-k+i} L_{n-q} j & , \text{for } i=j, 1 \leq i,j \leq n-q-k. \\ [R_{n-q}(k-1)]_{ij} & , \text{for } 2 \leq i \leq k+q, 1 \leq j \leq n-q. \end{cases}$$

$$[C_n(1)]_{i,1} = -\epsilon_n M_{i,n} , \text{ for } 1 \leq i \leq n-q-1,$$

$$[C_{n-q}(1)]_{ij} = \begin{cases} -\epsilon_{n-q} M_{i,n-q} & , \text{for } 1 \leq i \leq n-q-1, j=1, \\ 0 & , \text{for } 1 \leq i \leq n-q+1, 2 \leq j \leq q+1. \end{cases}$$

for $2 \leq k \leq n-q-1$

$$[C_{n-q}(k)]_{ij} = \begin{cases} \epsilon_{n-q} M_{i,n-q} L_{n-q} n-q-k+j & , \text{for } 1 \leq i,j \leq n-q-k, j=1, \\ [C_{n-q}(k-1)]_{ij} & , \text{for } 2 \leq i \leq n-q-k, 2 \leq j \leq k+q. \end{cases}$$

$$P_n^2(1) = I_n,$$

$$P_{n-q}^2(1) = I(n-q),$$

$$[P_n^2(2)]_{ij} = \begin{cases} I_{n-1} + \epsilon_n M_{n-1,n} L_{n,n-1} & , \text{for } i=1, j=1, \\ -L_{n,n-1} & , \text{for } i=2, j=1, \\ -\epsilon_n M_{n-1,n} & , \text{for } i=1, j=2, \\ I_n & , \text{for } i=2, j=2, \end{cases}$$

$$[P_{n-q}^2(2)]_{ij} = \begin{cases} I_{n-q} + \epsilon_{n-q} M_{n-q-1,n-q} L_{n-q} n-q-1 & , \text{for } i=1, j=1, \\ -L_{n-q} n-q-1 & , \text{for } i=2, j=1, \\ -\epsilon_{n-q} M_{n-q-1,n-q} & , \text{for } i=1, j=2, \\ [P_{n-q}^2(1)]_{ij} & , \text{for } i=2, j=2, \\ 0 & , \text{for } 3 \leq i \leq q+2, j=1 \text{ and } i=1, 3 \leq j \leq q+2, \end{cases}$$

$$\begin{aligned}
\left[P_n^2(k) \right]_{ij} &= \begin{cases} I_{n-k+1} + \epsilon_n M_{n-k+1} L_{n-k+1} & , \text{for } i=1, j=1, \\ \epsilon_n M_{n-k+i} L_{n-k+j} & , \text{for } 2 \leq i \leq k-1, j=1, \text{ and } i=1, 2 \leq j \leq k-1, \\ -L_{n-k+1} & , \text{for } i=k, j=1, \\ -\epsilon_n M_{n-k+1} & , \text{for } i=1, j=k, \\ \left[P_n^2(k-1) \right]_{ij} & , \text{for } 2 \leq i, j \leq k, \end{cases} \\
\left[P_{n-q}^2(k) \right]_{ij} &= \begin{cases} I_{n-q-k+1} + \epsilon_{n-q} M_{n-q-k+1} L_{n-q-k+1} & , \text{for } i=1, j=1, \\ \epsilon_{n-q} M_{n-q-k+i} L_{n-q-k+j} & , \text{for } 2 \leq i \leq k-1, j=1, \text{ and } i=1, 2 \leq j \leq k-1, \\ -L_{n-q-k+1} & , \text{for } i=k, j=1, \\ -\epsilon_{n-q} M_{n-q-k+1} & , \text{for } i=1, j=k, \\ \left[P_{n-q}^2(k-1) \right]_{ij} & , \text{for } 2 \leq i, j \leq k, \\ 0 & , \text{for } k+1 \leq i \leq k+q, j=1 \text{ and } i=1, k+1 \leq j \leq k+q, \end{cases}
\end{aligned}$$

We note that T^2 , transforms the original system (2.5)–(2.6) into totally decoupled system (4.10)–(4.11). Furthermore, we observe that

$$S_{n-m} = \begin{bmatrix} I^{n-m-1} & \epsilon_{n-m} M_m \\ L_m & I_{n-m} + \epsilon_{n-m} L_m M_m \end{bmatrix} \quad (4.13)$$

is the inverse of T_{n-m} , for $m \in J(0, n-2)$, and hence $\tilde{T}_{n-m}$ is invertible. Denoting $\tilde{S}_{n-m}$ as the inverse of $\tilde{T}_{n-m}$, $S^2 = \tilde{S}_n \circ \tilde{S}_{n-1} \circ \dots \circ \tilde{S}_2$ is an inverse of T^2 . Therefore the solution of the original system can be given in terms of the solution of the totally decoupled multi time scale system (4.10)–(4.11) by

$$X(t) = S^2(t) U(t)$$

where $U(t)$ is the solution of (4.10)–(4.11).

The validity of the transformation is established, by establishing the existence, uniqueness, boundedness and other fundamental properties of (4.7) for $m \in J(0, n-2)$ in [7]. For the sake of completeness, we present the desired assumptions.

Assumption A4.1: For every $m \in J(0, n-2)$, the absolute value of the real part of all eigen values of $\hat{A}_{22}(m)$ are greater than or equal to $2\alpha_m$ where $\alpha_m > 0$.

Assumption A4.2: Coefficient matrix functions A_{ij} , $ij \in J(1, n)$ in system (2.4) and $\hat{A}_{22}^{-1}(m)$, $m \in J(0, n-2)$ are bounded.

Assumption A4.3: $\hat{A}_{22}^{-1}(m)$, $\hat{A}_{21}(m)$, $\hat{A}_{12}(m)$, $\hat{A}_{22}^{-1}(m)$, $m \in J(0, n-2)$, are bounded and satisfies the following lipschitz condition.

$$\left\| \hat{A}_{22}^{-1}(m, t) \hat{A}_{21}(m, t) - \hat{A}_{22}^{-1}(m, s) \hat{A}_{21}(m, s) \right\| \leq l_m |t-s| \text{ for } 0 \leq t, s \leq 1,$$

$$\left\| \hat{A}_{12}(m, t) \hat{A}_{22}^{-1}(m, t) - \hat{A}_{12}(m, s) \hat{A}_{22}^{-1}(m, s) \right\| \leq m_m |t-s| \text{ for } 0 \leq t, s \leq 1,$$

for some positive real numbers l_m and m_m .

Remark 4.3: From the boundedness of coefficient matrices A_{ij} , $ij \in J(1, n)$, and the definition of $A_{11}(0)$, $A_{12}(0)$, $A_{21}(0)$ and $A_{22}(0)$, it is clear that $\epsilon_n A_{11}(0)$, $\epsilon_n A_{12}(0)$, $A_{21}(0)$ and $A_{22}(0)$ are bounded. If L_0 is bounded then it is clear that $\epsilon_{n-1} A_{11}(1)$, $\epsilon_{n-1} A_{12}(1)$, $A_{21}(1)$ and $A_{22}(1)$ are bounded. Continuing this process, one can conclude that if L_{m-1} is bounded, then it follows that $\epsilon_{n-m} A_{11}(m)$, $\epsilon_{n-m} A_{12}(m)$, $A_{21}(m)$ and $A_{22}(m)$ are bounded.

Moreover, if $L_0 = \hat{L}_0 + O(\frac{\epsilon_n}{\epsilon_{n-1}})$ then

$$A_{22}(1) = \hat{A}_{22}(1) + O(\frac{\epsilon_n}{\epsilon_{n-1}}), A_{21}(1) = \hat{A}_{21}(1) + O(\frac{\epsilon_n}{\epsilon_{n-1}}),$$

$$A_{12}(1) = \hat{A}_{12}(1) + D_1 O(\frac{\epsilon_n}{\epsilon_{n-1}}), A_{11}(1) = \hat{A}_{11}(1) + D_1 O(\frac{\epsilon_n}{\epsilon_{n-1}}).$$
 Continuing in

this manner, one can conclude that if $L_m = \hat{L}_m + \sum_{k=n-m}^n O(\frac{\epsilon_k}{\epsilon_{k-1}})$, then

$$A_{22}(m+1) = \hat{A}_{22}(m+1) + \sum_{k=n-m}^n O(\frac{\epsilon_k}{\epsilon_{k-1}}), A_{21}(m+1) = \hat{A}_{21}(m+1) + \sum_{k=n-m}^n O(\frac{\epsilon_k}{\epsilon_{k-1}}),$$

$$A_{12}(m+1) = \hat{A}_{12}(m+1) + D_{m+1} \sum_{k=n-m}^n O(\frac{\epsilon_k}{\epsilon_{k-1}}), A_{11}(m+1) = \hat{A}_{11}(m+1) + D_{m+1} \sum_{k=n-m}^n O(\frac{\epsilon_k}{\epsilon_{k-1}}).$$

Remark 4.4: From the approximations of L_m and M_m , it can be observed that

$$T^2 = \hat{T}^2 + \sum_{k=2}^n O(\frac{\epsilon_k}{\epsilon_{k-1}})$$

where

$$\hat{T}^2 = \hat{T}_2 \circ \dots \circ \hat{T}_n,$$

and

$$\hat{T}_{n-m} = \begin{bmatrix} \begin{bmatrix} I^{n-m-1} & 0 \\ -\hat{L}_m & I_{n-m} \end{bmatrix} & 0 \\ 0 & 0 & I(n-m) \end{bmatrix}.$$

Similarly

$$S^2 = \hat{S}^2 + \sum_{k=2}^n O(\frac{\epsilon_k}{\epsilon_{k-1}})$$

where

$$\hat{S}^2 = \hat{S}_2 \circ \dots \circ \hat{S}_n,$$

and

$$\hat{S}_{n-m} = \begin{bmatrix} \begin{bmatrix} I^{n-m-1} & 0 \\ \hat{L}_m & I_{n-m} \end{bmatrix} & 0 \\ 0 & 0 & I(n-m) \end{bmatrix}.$$

5 Existence and approximation.

In this section we give sufficient conditions to establish the existence of solution and to find an approximate solution of the original problem (2.5)-(2.6). The obtained conditions are easy to verify and computationally feasible. This is accomplished by developing a dichotomy transformation which decompose the faster sub-systems in the totally decoupled system (4.10) into stable and unstable sub-systems.

For this purpose let us introduce a few notations.

For $m \in J(0, n-2)$,

$$A_{22}(m) - \epsilon_{n-m} L_m A_{12}(m) = \begin{bmatrix} F_{11}(m) & F_{12}(m) \\ F_{21}(m) & F_{22}(m) \end{bmatrix}.$$

where $F_{11}(m), F_{12}(m), F_{21}(m)$ and $F_{22}(m)$ are of dimensions $k_m \times k_m$, $k_m \times (n_m - k_m)$,

$(n_{n-m}-k_m) \times k_m$ and $(n_{n-m}-k_m) \times (n_{n-m}-k_m)$ respectively. k_m 's are number of eigenvalues of $\hat{A}_{22}(m)$ with negative real part.

$$\hat{A}_{22}(m) = \begin{bmatrix} \hat{F}_{11}(m) & \hat{F}_{12}(m) \\ \hat{F}_{21}(m) & \hat{F}_{22}(m) \end{bmatrix},$$

where $\hat{F}_{ij}(m)$'s have the same dimensions as of $F_{ij}(m)$'s for $i,j \in J(1,2)$.

$$U_{n-m} = \begin{bmatrix} U_{n-m \ 1} \\ U_{n-m \ 2} \end{bmatrix}, W_{n-m} = \begin{bmatrix} W_{n-m \ 1} \\ W_{n-m \ 2} \end{bmatrix}, \hat{W}_{n-m} = \begin{bmatrix} \hat{W}_{n-m \ 1} \\ \hat{W}_{n-m \ 2} \end{bmatrix}.$$

where $U_{n-m \ 1}$, $W_{n-m \ 1}$, $\hat{W}_{n-m \ 1}$ are of dimensions $k_m \times 1$ and $U_{n-m \ 2}$, $W_{n-m \ 2}$, $\hat{W}_{n-m \ 2}$ are of dimensions $(n_{n-m}-k_m) \times 1$.

Now we consider the totally decoupled system (4.10). Faster sub-systems in (4.10) can be rewritten as

$$\epsilon_{n-m} \begin{bmatrix} \dot{U}_{n-m \ 1} \\ \dot{U}_{n-m \ 2} \end{bmatrix} = \begin{bmatrix} F_{11}(m) & F_{12}(m) \\ F_{21}(m) & F_{22}(m) \end{bmatrix} \begin{bmatrix} U_{n-m \ 1} \\ U_{n-m \ 2} \end{bmatrix}, m \in J(0, n-2). \quad (5.1)$$

We apply a transformation defined by

$$\begin{bmatrix} W_{n-m \ 1} \\ W_{n-m \ 2} \end{bmatrix} = E_{n-m} \begin{bmatrix} U_{n-m \ 1} \\ U_{n-m \ 2} \end{bmatrix}$$

to system (5.1),

where

$$E_{n-m} = \begin{bmatrix} I_{k_m} + P_m N_m & -P_m \\ -N_m & I_{n_{n-m}-k_m} \end{bmatrix}$$

and N_m , P_m satisfy the following differential equations

$$\begin{cases} \epsilon_{n-m} \dot{N}_m = -N_m(F_{11}(m) + F_{12}(m)N_m) + F_{21}(m) + F_{22}(m)N_m, \\ N_m(0) = N_m^0, \end{cases} \quad (5.2)$$

$$\begin{cases} \epsilon_{n-m} \dot{P}_m = -P_m(F_{22}(m) - N_m F_{12}(m)) + (F_{11}(m) + F_{12}(m)N_m)P_m + F_{12}(m) \\ P_m(1) = P_m^1. \end{cases} \quad (5.3)$$

Then we will have

$$\epsilon_{n-m} \begin{bmatrix} \dot{W}_{n-m} 1 \\ \dot{W}_{n-m} 2 \end{bmatrix} = \begin{bmatrix} F_{11}(m) + F_{12}(m)N_m & 0 \\ 0 & F_{22}(m) - N_m F_{12}(m) \end{bmatrix} \begin{bmatrix} W_{n-m} 1 \\ W_{n-m} 2 \end{bmatrix}. \quad (5.4)$$

For the validity of the transformation E_{n-m} , we will consider the algebraic equation corresponding to (5.2),

$$0 = -\bar{N}_m \hat{F}_{11}(m) + \hat{F}_{22}(m) \bar{N}_m - \bar{N}_m \hat{F}_{12}(m) \bar{N}_m + \bar{F}_{21}(m) \quad (5.5)$$

and assume the following assumption for a solution $\bar{N}_m(t)$ of (5.5) for $0 \leq t \leq 1$.

Assumption A5.1: For $m \in J(0, n-2)$ equation (5.5) has a bounded solution $\bar{N}_m(t)$, $t \in [0, 1]$ and it has a bounded derivative for every $t \in [0, 1]$. Furthermore, assume that $\hat{F}_{11}(m) + \hat{F}_{12}(m) \bar{N}_m$ and $\hat{F}_{22}(m) - \bar{N}_m \hat{F}_{12}(m)$ are unstable and stable matrices, respectively.

Theorem 5.1: Let the assumptions A2.1, A4.1, A4.2, A4.3 and A5.1 be satisfied. Then equation (5.2) has a solution N_m and

$$N_m = \bar{N}_m + \sum_{k=n-m}^n O\left(\frac{\epsilon_k}{\epsilon_{k-1}}\right), \quad (5.6)$$

where $\bar{N}_m$ is as in A5.1.

Moreover, $F_{11}(m) + F_{12}(m)N_m$ and $F_{22}(m) - N_m F_{12}(m)$ are unstable and stable matrices, respectively.

Proof: Let us consider the following differential equation

$$\epsilon_{n-m} \Delta \dot{P}_m = -\Delta N_m (F_{11}(m) + F_{12}(m) \bar{N}_m) + (F_{22}(m) - \bar{N}_m F_{12}(m)) \Delta N_m - \Delta N_m F_{12}(m) \Delta N_m + l_m \quad (5.7)$$

$$\Delta N_m(0) = N_m(0) - \bar{N}_m(0),$$

where

$$l_m = -\bar{N}_m(O_m + O_m \bar{N}_m) + O_m \bar{N}_m + O_m - \epsilon_{n-m} \dot{\bar{N}}_m,$$

and

$$O_m = \sum_{k=n-m}^n O\left(\frac{\epsilon_k}{\epsilon_{k-1}}\right).$$

and we will show that

$$\|\Delta N_m\| \leq \sum_{k=n-m}^n O\left(\frac{\epsilon_k}{\epsilon_{k-1}}\right),$$

and

$$N_m = \bar{N}_m + \Delta N_m.$$

We note that (5.7) has a unique solution.

Let $\phi_1(t,s)$ and $\phi_2(t,s)$ are fundamental matrix solutions of

$$\epsilon_{n-m} \dot{Z} = (F_{11}(m) + F_{12}(m)\bar{N}_m) Z,$$

$$\epsilon_{n-m} \dot{Y} = (F_{22}(m) - \bar{N}_m F_{12}(m)) Y, \text{ , respectively.}$$

From assumption A5.1, the approximation of $F_{ij}(m)$ and Lemma 1 in [4], we have

$$\left\{ \begin{array}{l} \|\phi_1(t,s)\| \leq K_1 \exp \left[-\frac{\alpha_1}{\epsilon_{n-m}}(s-t) \right], \quad 0 \leq t \leq s \leq 1, \\ \|\phi_2(t,s)\| \leq K_2 \exp \left[-\frac{\alpha_2}{\epsilon_{n-m}}(t-s) \right], \quad 0 \leq s \leq t \leq 1. \end{array} \right. \quad (5.8)$$

where K_1, K_2, α_1 and α_2 are some positive numbers independent of ϵ_{n-m} .

A solution of (5.7) can be expressed as follows.

$$\Delta N_m = \phi_2(t,0) \Delta N_m(0) \phi_1(0,t) + \int_0^t \phi_2(t,s) \frac{1}{\epsilon_{n-m}} [\Delta N_m(s) F_{12}(m)(s) \Delta N_m(s) + l_m(s)] \phi_1(s,t) ds. \quad (5.9)$$

Now we will show that if

$$\|\Delta N_m(0)\| \leq \rho_1,$$

then

$$\|\Delta N_m(t)\| \leq \rho, \text{ for all } 0 \leq t \leq 1,$$

where ρ, ρ_1 will be defined later in this proof.

From (5.8) and (5.9), we have

$$\|\Delta N_m(t)\| \leq K \rho_1 \exp \left[-\frac{\alpha}{\epsilon_{n-m}} t \right] + \int_0^t K \exp \left[-\frac{\alpha}{\epsilon_{n-m}} (t-s) \right] \frac{1}{\epsilon_{n-m}} [\|\Delta N_m\| \rho k_{12} + k O_m] ds. \quad (5.10)$$

where $K = k_1 + k_2, \alpha = \alpha_1 + \alpha_2, \|F_{12}\| \leq k_{12}$ and $k = 2 \|\bar{N}_m\| + \|\bar{N}_m\|^2 + \|\dot{\bar{P}}_m\|$.

set

$$m(t) = \exp \left[\frac{\alpha}{\epsilon_{n-m}} t \right] \|\Delta N_m(t)\| \quad (5.11)$$

From (5.10) and (5.11) we will have

$$m(t) \leq K \rho_1 + \int_0^t \frac{1}{\epsilon_{n-m}} [K k_{12} \rho m(s) + K k O_m \exp(\frac{\alpha}{\epsilon_{n-m}} s)] ds.$$

From the above integral inequality, we obtain the following estimate.

$$m(t) \leq K \rho_1 \exp[\frac{K k_{12} \rho}{\epsilon_{n-m}} t] + \frac{K k O_m}{\alpha - \epsilon_{n-m} K k_{12}} [\exp[\frac{\alpha}{\epsilon_{n-m}} t] - \exp[\frac{K k_{12} \rho}{\epsilon_{n-m}} t]]$$

From (5.11) and the above inequality, we will have,

$$\|\Delta N_m\| \leq K \rho_1 \exp[-\frac{\alpha - K k_{12} \rho}{\epsilon_{n-m}} t] + K k \frac{O_m}{\alpha - K k_{12} \rho} \quad (5.12)$$

Choose

$$\rho = \min(\frac{\alpha}{2 K k_{12}}, \frac{4 K k}{\alpha} O_m),$$

and

$$\rho_1 = \min(\frac{\rho}{4 K}, \frac{\rho}{4}).$$

From the choice of ρ_1 , (5.12), we get,

$$\|\Delta N_m(t)\| \leq \rho = \sum_{k=n-m}^n O(\frac{\epsilon_k}{\epsilon_{k-1}}).$$

This completes the boundedness of solution of (5.7). It is obvious that N_m defined by

$$N_m(t) = \bar{N}_m(t) + \Delta N_m(t)$$

is a solution of (5.2). This together with the boundedness of $\bar{N}_m$ and ΔN_m implies the existence of bounded solution N_m of (5.2). Unstability of $F_{11}(m) + F_{12}(m)N_m$ and stability of $F_{22}(m) - N_m F_{12}(m)$ follows directly from assumption A5.1 and the approximation of $F_{ij}(m)$ and N_m with $\hat{F}_{ij}(m)$ and $\bar{N}_m$.

Assumption A5.2: Algebraic equation corresponding to (5.3)

$$0 = (\hat{F}_{11}(m) + \hat{F}_{12}(m)\bar{N}_m) \bar{P}_m - \bar{P}_m (\hat{F}_{22}(m) - \bar{N}_m \hat{F}_{12}(m)) + \hat{F}_{12}(m) \quad (5.13)$$

has a bounded solution $\bar{P}_m$, and $\bar{P}_m$ has a bounded derivative on $0 \leq t \leq 1$.

Theorem 5.2: Under the assumptions of Theorem 5.1 and A5.2, equation (5.3) has a solution P_m and it can be approximated by solution $\bar{P}_m$ of (5.13).

Proof: Proof is analogous to the proof of Theorem 5.1. Therefore, we omit the details.

Now we will apply a transformation to the totally decoupled system (4.10) so that it decomposes every faster sub-systems into stable and unstable subsystems and leaves the slowest subsystem as it is.

Let

$$E = \text{diag} \{ I_{n_1}, E_2, \dots, E_n \}.$$

We apply the transformation E defined by

$$(W_1^T \ W_{21}^T \ W_{22}^T \ \dots W_{n1}^T \ W_{n2}^T)^T = E (U_1^T \ U_{21}^T \ U_{22}^T \ \dots U_{n1}^T \ U_{n2}^T)^T$$

to totally decoupled system (4.10) with boundary condition (4.11). Then, from (5.1)-(5.4), we will have

$$\begin{aligned} \dot{W}_1 &= A_{11}(n-1) W_1, \\ \epsilon_{n-m} \begin{bmatrix} \dot{W}_{n-m \ 1} \\ \dot{W}_{n-m \ 2} \end{bmatrix} &= \begin{bmatrix} F_{11}(m) + F_{12}(m)N_m & 0 \\ 0 & F_{22}(m) - N_m F_{12}(m) \end{bmatrix} \begin{bmatrix} W_{n-m \ 1} \\ W_{n-m \ 2} \end{bmatrix}, \quad m \in J(0, n-2) \end{aligned} \quad (5.14)$$

$$Q(n-1, \epsilon) E^{-1}(0) W(0) + R(n-1, \epsilon) E^{-1}(1) W(1) = B \quad (5.15)$$

Now we establish the existence and uniqueness of a bounded solution of the totally decoupled system (4.10) with boundary condition (4.11) and the original boundary value problem (2.5) - (2.6). For this purpose we prove the existence and uniqueness of a bounded solution for the decomposed system (5.14) - (5.15).

Theorem 5.3: Let the assumptions A2.1, A4.1, A4.2, A4.3, A5.1 and A5.2 be satisfied. let $\nabla(\epsilon)$ defined by

$$\begin{aligned} \nabla(\epsilon) &= Q(n-1, \epsilon) E^{-1}(0) \text{diag}\{\phi_1(0,0), \phi_{21}(0,0), \phi_{22}(0,1), \dots, \phi_{n1}(0,0), \phi_{n2}(0,1)\} \\ &\quad + R(n-1, \epsilon) E^{-1}(1) \text{diag}\{\phi_1(1,0), \phi_{21}(1,0), \phi_{22}(1,0), \dots, \phi_{n1}(1,0), \phi_{n2}(1,0)\} \end{aligned} \quad (5.16)$$

has an inverse, and its inverse is bounded.

where, $\phi_1(t,s)$, $\phi_{n-m \ 1}(t,s)$, $\phi_{n-m \ 2}(t,s)$, $m \in J(0, n-2)$ are fundamental matrix solutions of

$$\dot{W}_1 = A_{11}(n-1) W_1,$$

$$\epsilon_{n-m} \dot{W}_{n-m \ 1} = [F_{11}(m) + F_{12}(m)N_m] W_{n-m \ 1}, \quad (5.17)$$

$$\epsilon_{n-m} \dot{W}_{n-m \ 2} = [F_{22}(m) - N_m F_{12}(m)] W_{n-m \ 2}, \quad m \in J(0, n-2),$$

respectively. Then

(i) fundamental matrix solutions $\phi_1(t,s)$, $\phi_{n-m-1}(t,s)$, $\phi_{n-m-2}(t,s)$, $m \in J(0, n-2)$ of (5.17) satisfy

$$\left\{ \begin{array}{l} \|\phi_{n-m-1}(t,s)\| \leq K_{n-m-1} \exp \left[-\frac{\alpha_{n-m-1}}{\epsilon_{n-m}}(s-t) \right], \text{ for } 0 \leq t \leq s \leq 1, \\ \|\phi_{n-m-2}(t,s)\| \leq K_{n-m-2} \exp \left[-\frac{\alpha_{n-m-2}}{\epsilon_{n-m}}(t-s) \right], \text{ for } 0 \leq s \leq t \leq 1, m \in J(0, n-2). \end{array} \right. \quad (5.18)$$

and $\|\phi_1(t,s)\| \leq K_1 \exp [\alpha_1 |t-s|]$ for $0 \leq t, s \leq 1$,

where, for $m \in J(0, n-2)$, K_{n-m-1} , K_{n-m-2} , α_{n-m-1} , α_{n-m-2} , K_1 and α_1 are some positive integers, independent of ϵ_{n-m} . Moreover

$$\left\{ \begin{array}{l} W_1(t) = \phi_1(t,0) \eta_1, \\ W_{i-1}(t) = \phi_{i-1}(t,0) \eta_{i-1}, \\ W_{i-2}(t) = \phi_{i-2}(t,1) \eta_{i-2}, i \in J(2, n), \end{array} \right. \quad (5.19)$$

is a bounded solution of (5.14) for any $(\eta_1^T \eta_{2-1}^T \eta_{2-2}^T \dots \eta_{n-1}^T \eta_{n-2}^T)^T \in \mathbb{R}^N$;

(ii) the decomposed boundary value problem (5.14)-(5.15) has a unique bounded solution which can be expressed as (5.19) with

$$(\eta_1^T \eta_{2-1}^T \eta_{2-2}^T \dots \eta_{n-1}^T \eta_{n-2}^T)^T = \nabla(\epsilon)^{-1} B(\epsilon).$$

where $\nabla(\epsilon)$ is as defined in (5.16).

(iii) the original boundary value problem (2.5)-(2.6) has a unique bounded solution and it is given by

$$X(t) = S^2(t) E^{-1}(t) W(t).$$

where S^2 as in Remark 4.2 and $W(t) = (W_1(t) W_{21}(t) W_{22}(t) \dots W_{n1}(t) W_{n2}(t))^T$.

Proof: From the boundedness of $A_{11}(n-1)$ and from the stable, unstable modes as in Theorem 5.1, proof of (i) follows immediately. Let us prove (ii). From the choice of initial values, it can be shown that (5.19) satisfies the boundary condition (5.15). From (5.18) and from the boundedness of $\nabla(\epsilon)^{-1}$, we will have boundedness of the solution (5.19) of boundary value problem (5.14)-(5.15). Moreover, non-singularity of $\nabla(\epsilon)$ guarantees the unique choice of initial values. That is, it guarantees the uniqueness of solution of (5.14)-(5.15). This completes the proof of (ii). Now we complete the proof of the Theorem by proving (iii). Since $W(t)$ is the solution of the transformed decomposed system (5.14)-(5.15), $U(t) = E^{-1}(t) W(t)$ is the solution of the totally decoupled system (4.10) with (4.11). Since $S^2 = (T^2)^{-1}$ transforms (4.10) with (4.11) to the original boundary value problem (2.5)-(2.6), $X(t) = S^2(t) E^{-1}(t) W(t)$ is the solution of (2.5)-(2.6). Boundedness of solution $X(t)$ follows from

the boundedness of $W(t), E^{-1}(t)$ and $S^2(t)$ for $0 \leq t \leq 1$. Uniqueness of the solution $X(t)$ follows from uniqueness of $U(t)$ and the non-singularity of E and T^2 .

Our objective in the rest of this section is to find an approximate solution of the totally decomposed system (5.14)-(5.15) by a solution of a boundary value problem with the given coefficient matrices and hence to find an approximate solution of the original boundary value problem (2.5)-(2.6). For this purpose let us consider an auxiliary system corresponding to the totally decomposed system (5.14) as follows;

$$\dot{W}_1 = \hat{A}_{11}(n-1) \hat{W}_1,$$

$$\epsilon_{n-m} \begin{bmatrix} \dot{W}_{n-m} 1 \\ \dot{W}_{n-m} 2 \end{bmatrix} = \begin{bmatrix} \hat{F}_{11}(m) + \hat{F}_{12}(m) \bar{N}_m & 0 \\ 0 & \hat{F}_{22}(m) - \bar{N}_m \hat{F}_{12}(m) \end{bmatrix} \begin{bmatrix} \hat{W}_{n-m} 1 \\ \hat{W}_{n-m} 2 \end{bmatrix}, \quad m \in J(0, n-2) \quad (5.20)$$

$$\hat{Q}(n-1) \bar{E}^{-1}(0) \hat{W}(0) + \hat{R}(n-1) \bar{E}^{-1}(1) \hat{W}(1) = B \quad (5.21)$$

where $\bar{E} = \text{diag}\{I_{n_1}, \bar{E}_2, \dots, \bar{E}_n\}$

and

$$\bar{E}_{n-m} = \begin{bmatrix} I_{k_m} + \bar{N}_m \bar{N}_m & -\bar{N}_m \\ -\bar{N}_m & I_{n_n - m - k_m} \end{bmatrix}.$$

We observe that the fundamental matrix solution of (5.20) approximates the fundamental matrix solution of (5.14). Now we assume the following assumptions.

Assumption A5.3: For $m \in J(0, n-2)$, the coefficient matrices $(\hat{F}_{11}(m) + \hat{F}_{12}(m) \bar{N}_m)$ and $(\hat{F}_{22}(m) - \bar{N}_m \hat{F}_{12}(m))$ of (5.20) satisfies lipchitz condition for $0 \leq t \leq 1$.

Assumption A5.4: $\bar{\nabla}$ defined by,

$$\bar{\nabla} = \hat{Q}(n-1) \bar{E}^{-1}(0) \text{diag}\{I_{n_1}, I_{k_{n-2}}, 0, \dots, I_{k_0}, 0\} + \hat{R}(n-1) \bar{E}^{-1}(1) \text{diag}\{\hat{\phi}_1(1,0), 0, \dots, 0, I_{n_n - k_0}\}$$

has bounded inverse, where $\hat{\phi}_1(t,s)$ is the fundamental matrix solution of

$$\dot{W}_1 = A_{11}(n-1) \hat{W}_1.$$

Theorem 5.4: Let the assumptions A2.1, A4.1, A4.2, A4.3, A5.1, A5.2, A5.3 and A5.4 be satisfied.

Then,

(i) the solution in (5.19) of (5.14)-(5.15) can be approximated by

$$\begin{aligned}\hat{W}_1(t) &= \hat{\phi}_1(t,0) \hat{W}_1(0), \\ \hat{W}_{n-m-1}(t) &= \exp[(\hat{F}_{11}(m) + \hat{F}_{12}(m)\bar{N}_m)(0)\frac{t}{\epsilon_{n-m}}] \hat{W}_{n-m-1}(0), \\ \hat{W}_{n-m-2}(t) &= \exp[(\hat{F}_{22}(m) - \bar{N}_m\hat{F}_{12}(m))(1)\frac{(1-t)}{\epsilon_{n-m}}] \hat{W}_{n-m-1}(1), \quad m \in J(0, n-2),\end{aligned}\tag{5.22}$$

where $\hat{\phi}_1(t,0)$ is as in assumption A5.4,

$$(\hat{W}_1^{0T} \hat{W}_{21}^{0T} \hat{W}_{22}^{1T} \dots \hat{W}_{n1}^{0T} \hat{W}_{n2}^{1T})^T = \bar{V}^{-1}B.$$

and

$$(\hat{F}_{11}(m) + \hat{F}_{12}(m)N_m)(i) = \hat{F}_{11}(m,i) + \hat{F}_{12}(m,i)N_m(i), \text{ for } i=0,1.$$

(ii) the solution $X(t)$ of the original boundary value problem (2.5)-(2.6) can be approximated by $\hat{S}^2(t)\bar{E}^{-1}(t)\hat{W}(t)$.

Proof: From Theorems 5.1 and 5.2, we observe that $E(t)$ can be approximated by $\bar{E}(t)$. Furthermore, we notice that the fundamental matrix solutions of (5.14) can be approximated by the fundamental matrix solutions given in (5.22). To complete the proof of (i) it is enough to show η_{ij} 's $i \in J(2,n)$, $j \in J(1,2)$ in (5.19) can be approximated by $\hat{W}_i^0$, $\hat{W}_i^1$, $i \in J(2,n)$, respectively. This statement follows from the fact that

$$\phi_{i2}(0,1) \cong \sum_{k=n-i}^n O(\frac{\epsilon_k}{\epsilon_{k-1}}), \text{ for } i \in J(0, n-2).$$

Let us prove (ii). From Theorem 5.3, the solution of the original boundary value problem (2.5)-(2.6) is $X(t) = S^2(t)E(t)W(t)$. From Remark 4.4, $\hat{S}^2(t)$ approximates $S^2(t)$, from (i) $\bar{E}(t)$, $\bar{W}(t)$ approximates $E(t)$, $W(t)$ respectively. Hence $\hat{S}^2(t)\bar{E}^{-1}(t)\hat{W}(t)$ approximates $X(t)$. This completes the proof of the theorem.

Remark 5.1: We note that the dichotomy transformation is determined by the system of differential equation in (5.2) and (5.3). The results of this section generalize the results of Wilde and Kokotovic [17,18] in a systematic way. This fact can be justified in the succeeding sections.

6 Application

In this section we consider an optimal control problem. The preceeding results are applied to find an approximate solution of the boundary value problem associated with the necessary optimality condition.

Let us consider an optimal control problem.

$$\dot{X}_1 = A_{11}(t) X_1 + A_{12}(t) X_2 + B_1(t) U, \quad (6.1)$$

$$\epsilon_2 \dot{X}_2 = A_{21}(t) X_1 + A_{22}(t) X_2 + B_2(t) U.$$

with the performance index

$$J = \frac{1}{2} [X_1^T \ X_2^T] \begin{bmatrix} I_{n_1} & 0 \\ 0 & \epsilon_2 I_{n_2} \end{bmatrix} \Pi^{\epsilon_2} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \Big|_{t=1} + \frac{1}{2} \int_0^1 \{ [X_1^T \ X_2^T] Q(t) \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + U^T R(t) U \} dt. \quad (6.2)$$

and $[X_1^T(0) \ X_2^T(0)] = [X_1^0{}^T \ X_2^0{}^T]$.

where state variables X_1 , X_2 and control variables U are of n_1 , n_2 and m dimensional vectors, respectively; Π^{ϵ_2} is positive semidefinite matrix for all $0 \leq \epsilon_2 \leq \epsilon_2^*$, and

$$\Pi^{\epsilon_2} = \begin{bmatrix} \Pi_{11} & \epsilon_2 \Pi_{12} \\ \Pi_{12}^T & \Pi_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{12}(t) & Q_{22}(t) \end{bmatrix}, \quad R = \begin{bmatrix} R_{11}(t) & R_{12}(t) \\ R_{12}(t) & R_{22}(t) \end{bmatrix}.$$

Assumption A 6.1: For all $0 \leq t \leq 1$, $0 \leq \epsilon_2 \leq \epsilon_2^*$, the coefficient matrices A_{ij} , Q_{ij} , R_{ij} for $i, j \in J(1, 2)$; B_i for $i \in J(1, 2)$ are twice continuously differentiable with respect to t . Furthermore, we assume that R is symmetric positive definite and Q is symmetric positive semi definite.

With the assumption A 6.1, for the necessary optimality condition, the variables $[X_1^T \ X_2^T]^T$ and the adjoint variables $[\lambda_1^T \ \lambda_2^T]^T$ must satisfy the follwing boundary value problem [1, 17]:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{\lambda}_1 \end{bmatrix} = \begin{bmatrix} A_{11} & -S_{11} \\ -Q_{11} & -A_{11}^T \end{bmatrix} \begin{bmatrix} X_1 \\ \lambda_1 \end{bmatrix} + \begin{bmatrix} A_{12} & -S_{12} \\ -Q_{12} & -A_{21}^T \end{bmatrix} \begin{bmatrix} X_2 \\ \lambda_2 \end{bmatrix} \quad (6.3)$$

$$\epsilon_2 \begin{bmatrix} \dot{X}_2 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} A_{21} & -S_{21} \\ -Q_{21} & -A_{21}^T \end{bmatrix} \begin{bmatrix} X_1 \\ \lambda_1 \end{bmatrix} + \begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix} \begin{bmatrix} X_2 \\ \lambda_2 \end{bmatrix}$$

with boundary condition

$$H_1 X(0) + H_2 X(1) = H_3 \quad (6.4)$$

where

$$H_1 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\Pi_{11} & I_{n_1} & -\epsilon_2 \Pi_{12} & 0 \\ 0 & 0 & 0 & 0 \\ -\Pi_{12}^T & 0 & -\Pi_{22} & I_{n_2} \end{bmatrix}, \quad H_3 = \begin{bmatrix} X_1^0 \\ 0 \\ X_2^0 \\ 0 \end{bmatrix}$$

$$X(0) = \begin{bmatrix} X_1^0 \\ \lambda_1^0 \\ X_2^0 \\ \lambda_2^0 \end{bmatrix}, \quad X(1) = \begin{bmatrix} X_1^1 \\ \lambda_1^1 \\ X_2^1 \\ \lambda_2^1 \end{bmatrix} \text{ and } \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} R^{-1} \begin{bmatrix} B_1^T & B_2^T \end{bmatrix}$$

Assumption A 6.2: For all $t \in [0,1]$,

$$\text{rank}[B_2, A_{22} B_2, \dots, A_{22}^{n-2} B_2] = n_2,$$

$$\text{rank}[C_2^T, A_{22}^T C_2, \dots, A_{22}^{n-2} C_2] = n_2,$$

where C_2 satisfies $C_2^T C_2 = Q_{22}$.

Remark 6.1: From the assumptions A 6.1 and A 6.2, it is known [18,19] that the algebraic system

$$0 = -KA_{22} - A_{22}^T K + KS_{22}K - Q_{22}, \quad (6.5)$$

has a symmetric negative definite solution $\bar{N}$, and $-(A_{22} - S_{22}\bar{N})$ is stable. From this, it can be observed that

$$\begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix}^{-1} = \begin{bmatrix} D^{-1} + D^{-1}S_{22}D^{-1T}\bar{N} & -D^{-1}S_{22}D^{-1T} \\ \bar{N}D^{-1} + \bar{N}D^{-1}S_{22}D^{-1T}\bar{N} + D^{-1T}\bar{N} & -D^{-1T} - \bar{N}D^{-1}S_{22}D^{-1T} \end{bmatrix}$$

where $D = A_{22} - S_{22}\bar{N}$.

From assumptions A6.1, A6.2 and Remark 6.1, it is obvious that assumptions A4.1, A4.2 and

A4.3 are valid relative to system (6.3)–(6.4). Therefore, the decoupled system corresponding to this system (6.3)–(6.4) is given by

$$\begin{aligned} \begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} &= \left\{ \begin{bmatrix} A_{11} & -S_{11} \\ -Q_{11} & -A_{11}^T \end{bmatrix} + \begin{bmatrix} A_{12} & -S_{12} \\ -Q_{12} & -A_{21}^T \end{bmatrix} \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \right\} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \\ \epsilon_2 \begin{bmatrix} \dot{U}_3 \\ \dot{U}_4 \end{bmatrix} &= \left\{ \begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix} - \epsilon_2 \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} A_{12} & -S_{12} \\ -Q_{12} & -A_{21}^T \end{bmatrix} \right\} \begin{bmatrix} U_3 \\ U_4 \end{bmatrix} \end{aligned} \quad (6.6)$$

with the boundary condition

$$H_1 T^{-1}(0) U(0) + H_2 T^{-1}(1) U(1) = H_3 \quad (6.7)$$

where

$$T = \begin{bmatrix} I_{2n_1} + \epsilon_2 M L & -\epsilon_2 M \\ -L & I_{2n_2} \end{bmatrix}, \quad L = \begin{bmatrix} \hat{L}_{11} & \hat{L}_{12} \\ \hat{L}_{21} & \hat{L}_{22} \end{bmatrix} + O(\epsilon_2)$$

and

$$\begin{bmatrix} \hat{L}_{11} & \hat{L}_{12} \\ \hat{L}_{21} & \hat{L}_{22} \end{bmatrix} = \begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix}^{-1} \begin{bmatrix} A_{21} & -S_{21} \\ -Q_{21} & -A_{21}^T \end{bmatrix}$$

Furthermore, we remark that the assumptions A6.1 and A6.2, in the context of Remark 6.1, imply the validity of assumptions A5.1 and A5.2. This together with the applications of Theorems 5.1 and 5.2, a totally decomposed system similar to (5.20) can be written as

$$\begin{aligned} \begin{bmatrix} \dot{W}_1 \\ \dot{W}_2 \end{bmatrix} &= \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \\ \epsilon_2 \begin{bmatrix} \dot{W}_3 \\ \dot{W}_4 \end{bmatrix} &= \begin{bmatrix} A_{22} - S_{22} N & 0 \\ 0 & -(A_{22}^T - N S_{22}) \end{bmatrix} \begin{bmatrix} W_3 \\ W_4 \end{bmatrix} \end{aligned} \quad (6.8)$$

with boundary condition

$$H_1 \hat{T}^{-1}(0) \bar{E}(0) \dot{W}(0) + H_2 \hat{T}^{-1}(1) \bar{E}(1) \dot{W}(1) = H_3 \quad (6.9)$$

where

$$H_1 \hat{T}^{-1}(0) \bar{E}(0) = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hat{L}_{11}(0) & \hat{L}_{12}(0) & I_{n_2} & P(0) \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H_2 \hat{T}^{-1}(1) \bar{E}(1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\Pi_{11} & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Pi_{12}^T - \Pi_{22} \hat{L}_{11}(1) + \hat{L}_{21}(1) & -\Pi_{22} \hat{L}_{12}(1) + \hat{L}_{22}(1) & -\Pi_{22} + N(1) & -\Pi_{22} P(1) + I_{n_2} + P(1)N(1) \end{bmatrix}$$

and

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & -S_{11} \\ -Q_{11} & -A_{11}^T \end{bmatrix} - \begin{bmatrix} A_{12} & -S_{12} \\ -Q_{12} & -A_{21}^T \end{bmatrix} \quad (6.10)$$

In order to apply Theorem 5.4, we need to verify the assumptions A5.3 and A5.4. We note that assumption A6.1 implies A5.3. Therefore, we need to verify the validity of assumption A5.4. For this purpose, it is enough to verify the invertibility of the following matrix.

$$\begin{bmatrix} I_{n_1} & I_{n_1} & 0 & 0 \\ -\Pi_{11} \phi_{11}(1) + \phi_{12}(1) & -\Pi_{11} \phi_{11}(1) + \phi_{12}(1) & 0 & 0 \\ \hat{L}_{11}(0) & \hat{L}_{11}(0) & 0 & P(0) \\ [-\Pi_{12}^T - \Pi_{22} \hat{L}_{11}(1) + \hat{L}_{21}(1)] \phi_{11}(1) + [-\Pi_{22} \hat{L}_{12}(1) + \hat{L}_{22}(1)] \phi_{21}(1) & [-\Pi_{12}^T - \Pi_{22} \hat{L}_{11}(1) + \hat{L}_{21}(1)] \phi_{11}(1) + [-\Pi_{22} \hat{L}_{12}(1) + \hat{L}_{22}(1)] \phi_{21}(1) & -\Pi_{22} + N(1) & 0 \end{bmatrix}$$

where $\begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix}$ is the fundamental matrix solution of the slow system in (6.8).

Since $P(0)$ and $-\Pi_{22} + N(1)$ are invertible, it is enough to verify the invertibility of

$$\begin{bmatrix} I_{n_1} & 0 \\ -\Pi_{11}\phi_{11}(1) + \phi_{21}(1) & -\Pi_{11}\phi_{12}(1) + \phi_{22}(1) \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\Pi_{11} & I_{n_1} \end{bmatrix} \begin{bmatrix} \phi_{11}(1) & \phi_{12}(1) \\ \phi_{21}(1) & \phi_{22}(1) \end{bmatrix}. \quad (6.11)$$

To verify the above statement, we present the following lemma.

Lemma 6.1: Let the assumptions A6.1 and A6.2 hold. Then the recatti differential equation

$$\dot{K}_1 = -K_1 G_{11} - G_{11}^T K_1 + K_1 G_{12} K_1 - G_{21}, \quad K_1 = \Pi_{11} \quad (6.12)$$

has a solution. Moreover, matrix given in equation (6.11) is invertible.

Proof: From the structure of G_{ij} , $i, j=1,2$ in (6.10), after several algebraic computations, we can conclude that

$$G_{22} = -G_{11}^T \quad (6.13)$$

$$\begin{cases} G_{12} = -[B_{11} + S_{12}KD^{-1}B_{12} - A_{12}D^{-1}B_{12}]R^{-1} [B_{11} + S_{12}KD^{-1}B_{12} - A_{12}D^{-1}B_{12}]^T \\ G_{21} = -[Q_{11} + (\xi_1 + \xi_2)Q_{12}^T + Q_{12}(\xi_1^T + \xi_2^T) + (\xi_1 + \xi_2)Q_{22}(\xi_1^T + \xi_2^T)] \end{cases} \quad (6.14)$$

where

$$\xi_1 = Q_{12}D^{-1}K^{-1}, \quad \xi_2 = A_{12}^T KD^{-1}K^{-1}.$$

From the above identities (6.13) and (6.14), it is known that [18,19], (6.12) has a solution. To complete the proof of the lemma, let us consider the following differential equation

$$\begin{bmatrix} \dot{\hat{W}}_1 \\ \dot{\hat{W}}_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \hat{W}_1 \\ \hat{W}_2 \end{bmatrix} \quad (6.15)$$

Now we apply a transformation defined by

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = T_1 \begin{bmatrix} \hat{W}_1 \\ \hat{W}_2 \end{bmatrix} \quad (6.16)$$

to totally decouple the system (6.15),

where

$$T_1 = \begin{bmatrix} I_{n_1} + K_1 K_2 & -K_2 \\ -K_1 & I_{n_1} \end{bmatrix}.$$

This decoupling procedure requires that K_1 satisfy equation (6.12) and K_2 satisfy the following differential equation

$$\dot{K}_2 = (G_{11} + G_{12} K_1) K_2 + K_2 (G_{11}^T + K_1 G_{12}) + G_{12}, \quad K_2(0) = K_2^0.$$

Then we will have

$$\begin{bmatrix} \dot{Z}_1 \\ \dot{Z}_2 \end{bmatrix} = \begin{bmatrix} G_{11} - G_{12} K_1 & 0 \\ 0 & -(G_{11}^T - K_1 G_{12}) \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

A fundamental matrix solution of the above linear system is given as

$$\begin{bmatrix} \Psi_{11}(t,0) & 0 \\ 0 & \Psi_{22}(t,0) \end{bmatrix}$$

From this and (6.16), a fundamental matrix of (6.15) is of the form

$$\begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} + K_1(t) K_2(t) & -K_2(t) \\ -K_1(t) & I_{n_1} \end{bmatrix} \begin{bmatrix} \Psi_{11}(t,0) & 0 \\ 0 & \Psi_{22}(t,0) \end{bmatrix} \begin{bmatrix} I_{n_1} & K_2(0) \\ K_1(0) & I_{n_1} + K_1(0) K_2(0) \end{bmatrix}$$

This together with $K_1(1) = \Pi_{11}$, yields

$$\begin{bmatrix} I_{n_1} & 0 \\ -\Pi_{11} \phi_{11}(1) + \phi_{21}(1) & -\Pi_{11} \phi_{12}(1) + \phi_{22}(1) \end{bmatrix} = \begin{bmatrix} I_{n_1} & K_2(1) \Psi_{22}(1) \\ 0 & \Psi_2(1) \end{bmatrix} \begin{bmatrix} I_{n_1} & K_2(0) \\ K_1(0) & I_{n_1} + K_1(0) K_2(0) \end{bmatrix}$$

It is obvious that the above matrix is invertible. This completes the proof of the lemma.

Now by the application of Theorem 5.4, the expression in (5.23) and (5.24) with respect to (6.2) and (6.3) are given by

$$\hat{W}_1(0) = X_1^0,$$

$$\hat{W}_2(0) = [\Pi_{11}\phi_{12}(1) - \phi_{22}(1)]^{-1} [-\Pi_{11}\phi_{11}(1) + \phi_{21}(1)] X_1^0,$$

$$\hat{W}_3(0) = P^{-1}(0) [X_2^0 - \bar{X}_2^0],$$

$$\hat{W}_4(0) = [\Pi_{22} - N(1)]^{-1} \{\bar{\lambda}_2(1) - \Pi_{12}^T \bar{X}_1(1) - \Pi_{22} \bar{X}_2(1)\}$$

and for $0 \leq t \leq 1$,

$$X_1(t) = \bar{X}_1(t) + O(\epsilon_2)$$

$$\lambda_1(t) = \bar{\lambda}_1(t) + O(\epsilon_2)$$

$$X_2(t) = \bar{X}_2(t) + \hat{W}_3(t) + P(t)\hat{W}_4(t) + O(\epsilon_2)$$

$$\lambda_2(t) = \bar{\lambda}_2(t) + N(t)\hat{W}_3(t) + [I_{n_2} + N(t)P(t)] \hat{W}_4(t) + O(\epsilon_2)$$

where

$$\bar{X}_1(t) = \hat{W}_1(t),$$

$$\bar{\lambda}_1(t) = \hat{W}_2(t),$$

$$\begin{bmatrix} \bar{X}_2(t) \\ \bar{\lambda}_2(t) \end{bmatrix} = \begin{bmatrix} \hat{L}_{11}(t) & \hat{L}_{12}(t) \\ \hat{L}_{21}(t) & \hat{L}_{22}(t) \end{bmatrix} \begin{bmatrix} \bar{X}_1(t) \\ \bar{\lambda}_1(t) \end{bmatrix}.$$

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