STABILITY OF VOLTERRA SYSTEM WITH IMPULSIVE EFFECT¹

M. RAMAMOZHANA RAO

*Department of Mathematics
I.I.T. Kanpur- 208016, INDIA*

S. SIVASUNDARAM

*Department of Mathematics and Physical Sciences
College of Engineering and Aviation Sciences
Embry-Riddle Aeronautical University
Daytona Beach, FL 32014, USA*

ABSTRACT

Sufficient conditions for uniform stability and uniform asymptotic stability of impulsive integrodifferential equations are investigated by constructing a suitable piecewise continuous Lyapunov-like functionals without the decrease property. A result which establishes no pulse phenomena in the given system is also discussed.

Key words: Uniform stability, asymptotic stability, Lyapunov functional, fundamental matrix, integral curves, surfaces.

AMS (MOS) subject classification: 34A10, 45D05

¹Received: February, 1990. Revised: January, 1991.

1. INTRODUCTION: The stability analysis of ordinary differential equations with impulsive effect has been the subject of many investigations [1, 2,4] in recent years and various interesting results are reported. However, much has not been developed in this direction of integro-differential equations with impulsive effect except for a few [3, 5] in which the impulsive integral inequalities are used. The purpose of this paper is to investigate sufficient conditions for uniform stability and uniform asymptotic stability of Linear integro-differential equations by employing the piecewise continuous Liapunov functional without the decrescent property. It is also proved that every solution of the integro-differential system meets any given surface exactly once and thus there exists no pulse phenomena in the system.

Let the hyper surfaces σ_k be defined by the equations

$$\sigma_k \equiv t = \tau_k(x), \quad 0 < \tau_1(x) < \dots < \tau_k(x) < \dots$$

where $\tau_k(x) \rightarrow \infty$ as $k \rightarrow \infty$.

Pc^+ denote the class of piecewise continuous functions from

$R_+^2 \rightarrow R^{n^2}$ with discontinuities of the first kind at $t \neq \tau_k(x)$, $k = 1, 2, \dots$ and left continuous at $t = \tau_k$.

Let $\tau_0(x) \equiv 0$ for $x \in R_+$ and

$$G_k = \{(t, x) \in I \times R^n: \tau_{k-1}(x) < t < \tau_k(x)\}, \quad k = 1, 2, \dots$$

The function $V: I \times R^+ \rightarrow R$ belongs to class V_0 if:

- (i) The function V is continuous on each of the sets G_k and $V(t, 0) = 0$

(ii) For each $k = 1, 2, \dots$ and $(t_0, x_0) \in G_k$ there exists finite limits

$$V(t_0 - 0, x_0) = \lim_{\substack{(t, x) \rightarrow (t_0, x_0) \\ (t, x) \in G_k}} V(t, x); \quad V(t_0, x_0) = \lim_{\substack{(t, x) \rightarrow (t_0, x_0) \\ (t, x) \in G_k}} V(t, x)$$

and $V(t_0 - 0, x_0) = V(t_0, x_0)$ is satisfied.

Also if $(t_0, x_0) \in G_k$ then $V(t_0 + 0, x_0) = V(t_0, x_0)$

Let $V \in V_0$ For $(t, x) \in \bigcup_{k=1}^{\infty} G_k$, D^+V is defined as

$$D^+V(t, x) = \lim_{h^- \rightarrow 0^+} \sup_{h^-} \frac{1}{h^-} [V(t + h, x(t + h)) - V(t, x(t))]$$

2. Consider the impulsive integro-differential system

$$x' = A(t)x + \int_{t_0}^t k(t, s)x(s)ds, \quad t \neq \tau_k(x), \quad k=1, 2, \dots$$

$$\Delta x|_{t=\tau_k(x)} = I_k(x), \quad x(t_0) = x_0 \quad (2.1)$$

where $A \in PC^+[R_+, R^{n \times n}]$, $K \in PC^+[R_+^2, R^{n \times n}]$, and $I_k(0) = 0$, $t \geq t_0$, $k=1, 2, \dots$

$$\text{Let us consider: } \begin{aligned} x' &= A(t)x & t \neq \tau_k(x) \\ \Delta x|_{t=\tau_k(x)} &= B_k(x) \end{aligned} \quad (2.2)$$

where $\det(I + B_k) \neq 0$.

Not let $\phi_k(t, s)$ be the fundamental matrix of the linear system

$$x' = A(t)x, \quad (\tau_{k-1} < t < \tau_k) \quad (2.3)$$

Then the solution of the linear system (2.2) can be written in the form

$$x(t, t_0, x_0) = \psi(t, t_0 + 0) x_0, \text{ where}$$

$$\psi(t, s) = \begin{cases} \phi_k(t, s) & \text{for } \tau_{k-1} < s < t < \tau_k \\ \phi_{k+1}(t, t_k) (I + B_k) \phi_k(t_k, s) & \text{for } \tau_{k-1} < s < \tau_k < t < \tau_{k+1} \\ \phi_k(t, t_k) (I + B_k)^{-1} \phi_{k+1}(t_k, s) & \text{for } \tau_{k-1} < s < \tau_k < t < \tau_{k+1} \end{cases}$$

The following Lemma gives sufficient conditions for the absence of beating.

Lemma 2.1: Let the following conditions be satisfied for $|x| < \rho$

- (i) $|\phi_k(t, s)| \leq \alpha e^{\lambda(t-s)}$ for $0 \leq s < t < \infty$ for all k .
- (ii) $|A(t)| \leq \beta$ for $t \geq 0$.
- (iii) $|(I + B_k)| \leq \gamma$ where I is the identity matrix.
- (iv) $|K(t, s)| \leq M e^{\sigma(t-s)}$ where $M > 0, \sigma > 0$ for $0 \leq s \leq t < \infty$
- (v) There exists a number $\tilde{h} > 0$ such that

$$\sup_{\substack{0 \leq s \leq 1 \\ |x| \leq \tilde{h}}} \left\langle \frac{\partial \tau_k}{\partial x} (x + s I_k(x)) \right\rangle \leq 0, k=1, 2, \dots$$

$$\text{and } \sup_{|x| < \tilde{h}} \left| \frac{\partial \tau_k(x)}{\partial x} \right| \leq N, k=1, 2, \dots$$

$$(vii) \quad \left(\beta + \frac{M}{\sigma} \right) \rho N < 1$$

Then there exists a number $\rho \leq \tilde{h}$ such that if $x(t)$ is a solution of (2.1), which

lies in the ball $\{x \in \mathbb{R}^n: |x| \leq \rho\}$ for $0 \leq t \leq T, T > 0$, then the integral curve

$\{(t, x(t)): t \in [0, T]\}$ meets the hyper surface $t = \tau_k(x)$ exactly once.

Proof: Let $F(t, s) = A(t)x + \int_{t_0}^t K(t, s)x(s)ds$

If $|x| \leq \rho$ then from (2.1) and (i), (ii), (iii), and (iv) we get

$$\begin{aligned} |F(t, s)| &\leq |A(t)x| + \int_{t_0}^t |K(t, s)| |x(s)| ds \\ &\leq \beta |x| + M \int_{t_0}^t \bar{e}^{\sigma(t-s)} \sup_{0 \leq s \leq T} |x(s)| ds \\ &\leq \beta |x| + M\rho \int_{t_0}^t \bar{e}^{\sigma(t-s)} ds \\ &< \left(\beta + \frac{M}{\sigma} \right) \rho \end{aligned}$$

Now assume that some solution $x(t)$ of (2.1) under the above assumptions meets some surface $t = \tau_k(x)$ more than once.

Let $t = t_j$ be the point at which the solution first meets the surface $t = \tau_k(x)$ for some j and again another closest hit at $t = t^*$ such that $t^* - t_j > 0$. Then we have

$$t_j = \tau_k(x(t)) \text{ and } t^* = \tau_k(x(t^*)) \text{ where } t_0 < t_j < t^*$$

Then the solution satisfies the integral equation

$$x(t) = x_j + I_k(x_j) + \int_{t_j}^t F(s, x(s)) ds$$

Let
$$h = \int_{t_j}^{t^*} F(s, x(s)) ds.$$

Define the function $x(s) = \tau_k(x_j + I_k(x_j) + sh) + \tau_k(x_j + sI_k(x_j))$

for $s \in [0, 1]$. Then by mean value theorem

$$\begin{aligned} x(1) - x(0) &= \int_0^1 x'(s) ds \\ t^* - t_j &= \frac{\partial \tau_k}{\partial x}(x_j + I_k(x_j) + h) - \tau_k(x_j) \\ &= \int_0^1 \left\langle \frac{\partial \tau_k}{\partial x}(x_j + I_k(x_j) + sh), (I_k(x_j) + h) \right\rangle ds \\ &= \int_0^1 \left\langle \frac{\partial \tau_k}{\partial x}(x_j + I_k(x_j) + sh), h \right\rangle ds \\ &\quad + \int_0^1 \left\langle \frac{\partial \tau_k}{\partial x}(x_j + sI_k(x_j)), I_k(x_j) \right\rangle ds \end{aligned} \quad (2.4)$$

Since we have $\left| \frac{\partial \tau_k(x)}{\partial x} \right| \leq N$ and $|F(s, x(s))| < \left(\beta + \frac{M}{\sigma} \right) \rho$

By Cauchy-Schwartz inequality the first integral on the right hand side of (2.4) satisfies

$$\int_0^1 \left\langle \frac{\partial \tau_k}{\partial x}(x_j + I_k(x_j) + sh), h \right\rangle ds \leq N \left(\beta + \frac{M}{\sigma} \right) \rho (t^* - t_j)$$

hence we have

$$\left[1 - N \left(\beta + \frac{M}{\sigma} \right) \rho \right] (t^* - t_j) \leq \int_0^1 \left\langle \frac{\partial \tau_k}{\partial x}(x_j + sI_k(x_j)), I_k(x_j) \right\rangle ds$$

Since $\left(\beta + \frac{M}{\sigma} \right) \rho N < 1$, in view of hypothesis (v) this leads to contradiction which completes the proof of the lemma.

Define the matrix $G(t)$ as

$$G(t) = \int_t^\infty \psi^\tau(s, t) \psi(s, t) ds \text{ where } \psi^\tau \text{ is the transpose of } \psi.$$

Clearly $G(t)$ is symmetric. And define $W(t, x) = \langle G(t)x, x \rangle^{1/2}$ and

$\forall x \in V_0$ for $(t, x) \in (t_{k-1}, t_k) \times R^n$ as

$$V(t, x) = W(t, x) + \beta \int_{t_0}^t \int_t^\infty \|K(u, s)\| du \|x(s)\| ds.$$

Theorem 2.1: Assume the following conditions hold.

$$(i) \quad L\|x\| \leq \langle G(t)x, x \rangle^{\frac{1}{2}} \leq \frac{1}{2\hat{M}}\|x\|$$

$$(ii) \quad \|G(t)x\| \leq \hat{K} \langle G(t)x, x \rangle^{\frac{1}{2}}$$

$$(iii) \quad -\hat{M} + \beta \int_t^\infty \|K(u, t)\| du \leq 0, \beta \geq \hat{K}$$

$$(iv) \quad \|x\| > \|x + I_k(x)\| \text{ and}$$

$$\langle G(t)x, x \rangle^{\frac{1}{2}} > \langle G(t)(x + I_k(x)), (x + I_k(x)) \rangle^{\frac{1}{2}} \quad \text{where}$$

$L, \hat{M}, \hat{K}$, and β are positive real numbers

Then the zero solution of (2.1) is uniformly stable.

Proof: Let $W(t, x) = \langle G(t)x, x \rangle^{\frac{1}{2}}$

$$W'(t, x) = \frac{\langle G'(t)x, x \rangle}{2\langle G(t)x, x \rangle^{\frac{1}{2}}} + \frac{\langle 2G(t)x, x \rangle}{2\langle G(t)x, x \rangle^{\frac{1}{2}}}$$

$$\begin{aligned} \text{we have} \quad \frac{\partial \psi(s, t)}{\partial t} &= -\psi(s, t) A(t) \\ \frac{\partial \psi^T(s, t)}{\partial t} &= -A^T(t) \psi^T(s, t) \end{aligned}$$

$$\text{Hence} \quad G'(t) = -I - \int_t^\infty \left[\frac{\partial \psi^T(s, t)}{\partial t} \psi(s, t) \psi^T(s, t) \frac{\partial \psi(s, t)}{\partial t} \right]$$

$$\text{Which implies} \quad G'(t) = -I - A^T(t) G(t) - G(t) A(t)$$

Hence

$$w'(t, x) = \frac{-\langle x, x \rangle}{2\langle G(t)x, x \rangle^{\frac{1}{2}}} + \frac{\langle G(t)x, \int_{t_0}^t K(t, s)x(s) ds \rangle}{\langle G(t)x, x \rangle^{\frac{1}{2}}} \quad (2.1)$$

for $t \neq I_k, (t, x) \in \bar{UG}_k$

Now

$$v'(t, x) = \frac{-\langle x, x \rangle}{2\langle G(t)x, x \rangle^{\frac{1}{2}}} + \frac{\langle G(t)x, \int_{t_0}^t K(t, s)x(s) ds \rangle}{\langle G(t)x, x \rangle^{\frac{1}{2}}} \quad (2.1)$$

$$+ \beta \int_t^\infty \|K(u, t)\| du \|x(t)\| - \beta \int_t^\infty \|K(t, s)\| du \|x(s)\| ds \text{ for } t \neq \tau_k, (t, x) \in \bar{UG}_k$$

by (i) and (ii) we get

$$v'(t, x) \leq -\hat{M}\|x\| + \hat{K} \int_{t_0}^t \|K(t, s)\| \|x(s)\| ds \quad (2.1)$$

$$+ \beta \int_t^\infty \|K(u, t)\| du \|x(t)\| - \beta \int_t^\infty \|K(t, s)\| du \|x(s)\| ds$$

Hence in view of assumption (iii) it follows that

$$v'(t, x(s)) \leq 0 \text{ for } t \neq \tau_k, (t, x) \in \bar{UG}_k \quad (2.1)$$

This implies for $t \neq \tau_k$ that by hypothesis (iv)

$$L\|x(t)\| \leq v(t, x) \leq v(t_0, x_0) \leq w(t_0, x_0) \leq \frac{1}{2\hat{M}}\|x_0\|$$

this gives the uniform stability of (2.1)

Remark 2.1: In the above theorem it is not assumed the descrent property on V .

Theorem 2.2 Assume the following conditions hold for $\|x\| < \rho$

- (i) $L\|x\| \leq \langle G(t)x, x \rangle^{\frac{1}{2}} < +\frac{1}{2\hat{M}}\|x\|$
- (ii) $\|G(t)x\| < \hat{K}\langle G(t)x, x \rangle^{\frac{1}{2}}$
- (iii) $\hat{\gamma} \leq \hat{M} - \beta \int_t^{\infty} \|K(u, t)\| du$ for some $\hat{\gamma} > 0, \beta > \hat{K}$
- (iv) $\|x\| > \|x + I_k(x)\|$ and $\langle G(t)x, x \rangle^{\frac{1}{2}} > \langle G(t)(x + I_k(x)), (x + I_k(x)) \rangle^{\frac{1}{2}}$

where $L, \hat{M}, \hat{K}, \hat{\gamma}$, and β are positive real numbers.

Then the zero solution of (2.1) is uniformly asymptotically stable.

Proof: By Theorem 2.1 the zero solution of (2.1) is uniformly stable. Following the proof of Theorem 2.1 one obtains

$$V'(t, x) \leq -\hat{\gamma}\|x\| \text{ for } t \neq \tau_k, \|x\| < \rho \text{ and } (t, x) \in \bigcup_{k=1}^{\infty} U_k. \quad (2.1)$$

Let δ be the number corresponding to ϵ in the definition of uniform stability.

$$\text{Take } T(\epsilon) = \left\lfloor \frac{1}{2\hat{M}} \right\rfloor_{\frac{\hat{\gamma}\delta}{\rho}} \|x_0\| \text{ where } x(t_0) = x_0$$

We now claim that $\|x(t^*, t_0, x_0)\| \leq \delta$ for some $t^* \in [t_0, t_0 + \tau]$

Whenever $\|x(s)\| < \rho$ for $0 \leq s \leq t_0$.

For if $\|x(t, t_0, x_0)\| > \delta$ for all $t \in [t_0, t_0 + \tau]$, then

By hypotheses (i) and (iv)

$$0 < L\delta = L\|x(t, t_0, x_0)\| \leq V(t, x(t)) \leq V(t_0, x_0) + \int_{t_0}^t v'_{2.1}(s, x(s)) ds$$

$$\leq \left\lfloor \frac{1}{2\hat{M}} \right\rfloor \rho - \hat{\gamma} \int_{t_0}^t \|x(s)\| ds$$

put $t = t_0 + T$, then we get

$$\begin{aligned} 0 < L\delta &\leq \left[\frac{1}{2\hat{M}} \right] \rho - T\gamma\delta \\ &\leq \left[\frac{1}{2\hat{M}} \right] \rho - \left[\frac{1}{2\hat{M}} \right] \frac{\rho}{\gamma\delta} \rho\gamma\delta = 0 \end{aligned}$$

and thus we have a contradiction.

Hence there exists $t^* \in [t_0, t_0 + \tau]$ such that $\|x(t^*, t_0, x_0)\| < \delta$.

By uniform stability it follows that $\|x(t, t_0, x_0)\| < \epsilon$ for all $t > t^*$ or $t \geq t_0 + T$ which completes the proof.

REFERENCES

1. Bainov, D. D., and Dishliev, A. B., "Sufficient Conditions for Absence of Beating in System of Differential Equations with Impulses," *App. Anal.* Vol. 18: 66-73 (1984).
2. Lakshmikantham, V., and Liu, Xinzhi, "Stability Criteria for Impulsive Differential Equations in Terms of Two Measures," *Journal of mathematical Analysis and Applications*, Vol. 137: 591-604 (1989).
3. Ramamohana, Rao, M., Sathanantham, S. and Sivasundaram, S., "Asymptotic Behavior of Solutions of Impulsive Integro Differential Systems." To appear in *Journal of Applied Math and Computation*.

4. Simeonov, P. S., and Bainov, D. D., "Stability with Respect to Part of the Variables in Systems with Impulse Effect," *Journal of Mathematical Analysis and Applications*, Vol 117: 247-263 (1986).
5. Simeonov, P. S. and Bainov, D. D., "Stability of The Solutions of Integro-Differential Equation with Impulse Effect," *Math Rep Toyama Univ.*, Vol. 9: 1-24 (1986).