PROPERTIES OF SOLUTION SET OF STOCHASTIC INCLUSIONS¹

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ABSTRACT

The properties of the solution set of stochastic inclusions $x_t - x_s \in cl_{L^2}(\int_s^t F_\tau(x_\tau) d\tau + \int_s^t G_\tau(x_\tau) dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,z}(x_\tau) \tilde{\nu}(d\tau, dz))$ are investigated. They are equivalent to properties of fixed points sets of appropriately defined set-valued mappings.

Key words: Stochastic inclusions, existence solutions, solution set, weak compactness.

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1. INTRODUCTION

There is a large number of papers (see for example [1], [4] and [5]) dealing with the existence of optimal controls of stochastic dynamical systems described by integral stochastic equations. Such problems can be described (see [10]) by stochastic inclusions $(SI(F, G, H))$ of the form

$$x_t - x_s \in cl_{L^2} \left(\int_s^t F_\tau(x_\tau) d\tau + \int_s^t G_\tau(x_\tau) dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,z}(x_\tau) \tilde{\nu}(d\tau, dz) \right),$$

where the stochastic integrals are defined by Aumann's procedure (see [7], [9]).

The results of the paper are concerned with properties of the set of all solutions to $SI(F, G, H)$. To begin with, we recall the basic definitions dealing with set-valued stochastic integrals and stochastic inclusions presented in [10]. We assume, as given, a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, where a family $(\mathcal{F}_t)_{t \geq 0}$, of σ -algebras $\mathcal{F}_t \subset \mathcal{F}$ is assumed to be increasing: $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$. We set $\mathbb{R}_+ = [0, \infty)$, and $\mathcal{B}_+$ will denote the Borel σ -algebra on

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$\mathbb{R}_+$. We consider set-valued stochastic processes $(F_t)_{t \geq 0}, (\mathcal{G}_t)_{t \geq 0}$ and $(\mathcal{R}_{t,z})_{t \geq 0, z \in \mathbb{R}^n}$, taking on values from the space $Comp(\mathbb{R}^n)$ of all nonempty compact subsets of n -dimensional Euclidean space $\mathbb{R}^n$. They are assumed to be predictable and such that $E \int_0^\infty \|\mathcal{F}_t\|^p dt < \infty$, $p \geq 1$, $E \int_0^\infty \|\mathcal{G}_t\|^2 dt < \infty$ and $E \int_0^\infty \int_{\mathbb{R}^n} \|\mathcal{R}_{t,z}\|^2 dt q(dz) < \infty$, where q is a measure on the Borel σ -algebra $\mathcal{B}^n$ of $\mathbb{R}^n$ and $\|A\| := \sup\{|a| : a \in A\}$, $A \in Comp(\mathbb{R}^n)$. The space $Comp(\mathbb{R}^n)$ is considered with the Hausdorff metric h defined in the usual way, i.e., $h(A, B) = \max\{\bar{h}(A, B), \bar{h}(B, A)\}$, for $A, B \in Comp(\mathbb{R}^n)$, where $\bar{h}(A, B) = \{dist(a, B) : a \in A\}$ and $\bar{h}(B, A) = \{dist(b, A) : b \in B\}$. Although the classical theory of stochastic integrals (see [3], [8], [12]) usually deals with measurable and $\mathcal{F}_t$ -adapted processes, it can be finally reduced (see [4], pp. 60-62) to predictable ones.

2. BASIC DEFINITIONS AND NOTATIONS

Throughout the paper we shall assume that a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfies the following usual hypotheses: (i) $\mathcal{F}_0$ contains all the P -null sets of $\mathcal{F}$, (ii) $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$ and (iii) $\mathcal{F}_t = \bigcap_{u > t} \mathcal{F}_u$, for all $t, 0 \leq t < \infty$. As usual, we consider a set $\mathbb{R}_+ \times \Omega$ as a measurable space with the product σ -algebra $\mathcal{B}_+ \otimes \mathcal{F}$. Moreover, we introduce on $\mathbb{R}_+ \times \Omega$ the predictable σ -algebra $\mathcal{P}$ generated by a semiring $\mathcal{K}$ of all predictable rectangles in $\mathbb{R}_+ \times \Omega$ of the form $\{0\} \times A_0$ and $(s, t] \times A_s$, where $A_0 \in \mathcal{F}_0$ and $A_s \in \mathcal{F}_s$ for $s < t$ in $\mathbb{R}_+$. Similarly, besides the usual product σ -algebra on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^n$, we also introduce the predictable σ -algebra $\mathcal{P}^n$ generated by a semiring $\mathcal{K}^n$ of all sets of the form $\{0\} \times A_0 \times D$ and $(s, t] \times A_s \times D$, with $A_0 \in \mathcal{F}_0$, $A_s \in \mathcal{F}_s$ for $s < t$ in $\mathbb{R}_+$ and $D \in \mathcal{B}_0^n$, where $\mathcal{B}_0^n$ consists of all Borel sets $D \subset \mathbb{R}^n$ such that their closure does not contain the point 0.

An n -dimensional stochastic process x , understood as a function $x: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^n$ with $\mathcal{F}$ -measurable sections x_t , each $t \geq 0$, is denoted by $(x_t)_{t \geq 0}$. It is measurable (predictable) if x is $\mathcal{B}_+ \otimes \mathcal{F}$ ($\mathcal{P}$, resp.)-measurable. The process $(x_t)_{t \geq 0}$ is $\mathcal{F}_t$ -adapted if x_t is $\mathcal{F}_t$ -measurable for $t \geq 0$. It is clear (see [3], [8], [11]) that every predictable process is measurable and $\mathcal{F}_t$ -adapted. In what follows the Banach space $L^p(\mathbb{R}_+ \times \Omega, \mathcal{P}, dt \times P, \mathbb{R}^n)$, $p \geq 1$, with the norm $\|\cdot\|_{\mathcal{L}_n^p}$ defined in the usual way, will be denoted by $\mathcal{L}_n^p$. Similarly, the Banach spaces

$L^p(\Omega, \mathcal{F}_t, P, \mathbb{R}^n)$ and $L^p(\Omega, \mathcal{F}, P, \mathbb{R}^n)$ with the usual norm $\|\cdot\|_{L^p_n}$ are denoted by $L^p_n(\mathcal{F}_t)$ and $L^p_n(\mathcal{F})$, respectively.

Throughout the paper, by $(w_t)_{t \geq 0}$, we mean a one-dimensional $\mathcal{F}_t$ -Brownian motion starting at 0, i.e., such that $P(w_0 = 0) = 1$. By $\nu(t, A)$ we denote a $\mathcal{F}_t$ -Poisson measure on $\mathbb{R}_+ \times \mathbb{B}^n$, and then define a $\mathcal{F}_t$ -centered Poisson measure $\tilde{\nu}(t, A)$, $t \geq 0$, $A \in \mathbb{B}^n$, by taking $\tilde{\nu}(t, A) = \nu(t, A) - tq(A)$, $t \geq 0$, $A \in \mathbb{B}^n$, where q is a measure on $\mathbb{B}^n$ such that $E\nu(t, B) = tq(B)$ and $q(B) < \infty$ for $B \in \mathbb{B}^n_0$.

For a given $\mathcal{F}_t$ -centered Poisson measure $\tilde{\nu}(t, A)$, $t \geq 0$, $A \in \mathbb{B}^n$, $\mathcal{W}^2_n$ denotes the space $L^2(\mathbb{R}_+ \times \Omega \times \mathbb{R}^n, \mathcal{P}^n, dt \times P \times q)$, with the norm $\|\cdot\|_{\mathcal{W}^2_n}$ defined in the usual way. We shall also consider the Banach spaces $L^p(\mathbb{R}_+, \mathbb{B}_+, dt, \mathbb{R}_+)$, $p \geq 1$ and $L^2(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{B}_+ \otimes \mathbb{B}^n, dt \times q, \mathbb{R}_+)$, with the usual norms by $|\cdot|_p$ and $\|\cdot\|_2$, respectively. They will be denoted by $L^p(\mathbb{B}_+)$ and $L^2(\mathbb{B}_+ \times \mathbb{B}^n)$, respectively. Finally, by $\mathcal{M}^p_n(\mathcal{P})$, $p \geq 1$ and $\mathcal{M}^2_n(\mathcal{P}^n, q)$ we shall denote the families of all $\mathcal{P}$ -measurable and $\mathcal{P}^n$ -measurable functions $f: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^n$ and $h: \mathbb{R}_+ \times \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, respectively, such that $\int_0^\infty |f_t|^p dt < \infty$ and $\int_0^\infty \int_{\mathbb{R}^n} |h_{t,z}|^2 dt q(dz) < \infty$, a.s. Elements of $\mathcal{M}^p_n(\mathcal{P})$, $p \geq 1$ and $\mathcal{M}^2_n(\mathcal{P}^n, q)$ will be denoted by $f = (f_t)_{t \geq 0}$ and $h = (h_{t,z})_{t \geq 0, z \in \mathbb{R}^n}$, respectively. We have

$$\mathcal{L}^p_n = \{f \in \mathcal{M}^p_n(\mathcal{P}): E \int_0^\infty |f_t|^p dt < \infty\}, p \geq 1,$$

$$\text{and } \mathcal{W}^2_n = \{h \in \mathcal{M}^2_n(\mathcal{P}^n, q): E \int_0^\infty \int_{\mathbb{R}^n} |h_{t,z}|^2 dt q(dz) < \infty\}.$$

Given $g \in \mathcal{M}^2(\mathcal{P})$ and $h \in \mathcal{M}^2(\mathcal{P}^n, q)$, by $(\int_0^t g_\tau dw_\tau)_{t \geq 0}$ and $(\int_0^t \int_{\mathbb{R}^n} h_{\tau,z} \tilde{\nu}(d\tau, dz))_{t \geq 0}$, we denote their stochastic integrals with respect to a $\mathcal{F}_t$ -Brownian motion $(w_t)_{t \geq 0}$ and a $\mathcal{F}_t$ -centered Poisson measure $\tilde{\nu}(t, A)$, $t \geq 0$, $A \in \mathbb{B}^n$, respectively. These integrals, understood as n -dimensional stochastic processes, have quite similar properties (see [6]).

Let us denote by D the family of all n -dimensional $\mathcal{F}_t$ -adapted càdlàg processes $(x_t)_{t \geq 0}$ such that

$$E \sup_{t \geq 0} |x_t|^2 < \infty$$

$$\text{and } \lim_{\delta \rightarrow 0} \sup_{t \geq 0} \sup_{t \leq s \leq t+\delta} E |x_t - x_s|^2 = 0.$$

Recall that an n -dimensional stochastic process is said to be a càdlàg process if it has almost all sample paths right continuous with finite left limits. The space D is considered as a normed space with the norm $\|\cdot\|_\ell$ defined by

$\|\xi\|_\ell = \|\sup_{t \geq 0} |\xi_t|\|_{L_1^2}$ for $\xi = (\xi_t)_{t \geq 0} \in D$. It can be verified that $(D, \|\cdot\|_\ell)$ is a Banach space.

Given $0 \leq \alpha < \beta < \infty$ and $(x_t)_{t \geq 0} \in D$ let $x^{\alpha, \beta} = (x_t^{\alpha, \beta})_{t \geq 0}$ be defined by $x_t^{\alpha, \beta} = x_\alpha$ and $x_t^{\alpha, \beta} = x_\beta$ for $0 \leq t \leq \alpha$ and $t \geq \beta$, respectively, and $x_t^{\alpha, \beta} = x_t$ for $\alpha \leq t \leq \beta$. It is clear that $D^{\alpha, \beta} = \{x^{\alpha, \beta}: x \in D\}$ is a linear subspace of D , closed in the $\|\cdot\|_\ell$ -norm topology. Then $(D^{\alpha, \beta}, \|\cdot\|_\ell)$ is also a Banach space. Finally, as usual, by $\sigma(D, D^*)$ we shall denote a weak topology on D .

In what follows we shall deal with upper and lower semicontinuous set-valued mappings. Recall that a set-valued mapping $\mathfrak{R}$ with nonempty values in a topological space $(Y, \mathcal{T}_Y)$ is said to be upper (lower) semicontinuous [u.s.c. (l.s.c.)] on a topological space $(X, \mathcal{T}_X)$ if $\mathfrak{R}^-(C) = \{x \in X: \mathfrak{R}(x) \cap C \neq \emptyset\}$ ($\mathfrak{R}_-(C) = \{x \in X: \mathfrak{R}(x) \subset C\}$) is a closed subset of X for every closed set $C \subset Y$. In particular, for $\mathfrak{R}$ defined on a metric space $(\mathfrak{X}, d)$ with values in $Comp(\mathbb{R}^n)$, it is equivalent (see [9]) to $\lim_{n \rightarrow \infty} \bar{h}(\mathfrak{R}(x_n), \mathfrak{R}(x)) = 0$ ($\lim_{n \rightarrow \infty} \bar{h}(\mathfrak{R}(x), \mathfrak{R}(x_n)) = 0$) for every $x \in \mathfrak{X}$ and every sequence (x_n) of $\mathfrak{X}$ converging to x . If, moreover, $\mathfrak{R}$ takes convex values then it is equivalent to upper (lower) semicontinuity of a real-valued function $s(p, \mathfrak{R}(\cdot))$ on $\mathbb{R}^n$ for every $p \in \mathbb{R}^n$, where $s(\cdot, A)$ denotes a support function of a set $A \in Comp(\mathbb{R}^n)$. In what follows, we shall need the follow well-known (see [9]) fixed point and continuous selection theorems.

Theorem (Schauder, Tikhonov): *Let $(X, \mathcal{T}_X)$ be a locally convex topological Hausdorff space, $\mathfrak{K}$ a nonempty compact convex subset of X and f a continuous mapping of $\mathfrak{K}$ into itself. Then f has a fixed point in $\mathfrak{K}$.*

Theorem (Covitz, Nadler): *Let $(\mathfrak{X}, d)$ be a complete metric space and $\mathfrak{R}: \mathfrak{X} \rightarrow Cl(\mathfrak{X})$ a set-valued contraction mapping, i.e., such that $H(\mathfrak{R}(x), \mathfrak{R}(y)) \leq \lambda d(x, y)$ for $x, y \in \mathfrak{X}$ with $\lambda \in [0, 1)$, where H is the Hausdorff metric induced by the metric d on the space $Cl(\mathfrak{X})$ of all nonempty closed bounded subsets of $\mathfrak{X}$. Then there exists $x \in \mathfrak{X}$ such that $x \in \mathfrak{R}(x)$.*

Theorem (Kakutani, Fan): *Let $(X, \mathcal{T}_X)$ be a locally convex topological Hausdorff space, $\mathfrak{K}$ a nonempty compact convex subset of X and $CCl(\mathfrak{K})$ a family of all nonempty closed convex subsets of $\mathfrak{K}$. If $\mathfrak{R}: \mathfrak{K} \rightarrow CCl(\mathfrak{K})$ is u.s.c. on $\mathfrak{K}$ then there exists $x \in \mathfrak{K}$ such that $x \in \mathfrak{R}(x)$.*

Theorem (Michael): *Let $(X, \mathcal{T}_X)$ be a paracompact space and let $\mathcal{R}$ be a set-valued mapping from X to a Banach space $(Y, \|\cdot\|)$ whose values are closed and convex. Suppose, further $\mathcal{R}$ is l.s.c. on X . Then there is a continuous function $f: X \rightarrow Y$ such that $f(x) \in \mathcal{R}(x)$, for each $x \in X$.*

3. SET-VALUED STOCHASTIC INTEGRALS

Let $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ be a set-valued stochastic process with values in $Comp(\mathbb{R}^n)$, i.e. a family of $\mathcal{F}$ -measurable set-valued mappings $\mathcal{G}_t: \Omega \rightarrow Comp(\mathbb{R}^n)$, $t \geq 0$. We call $\mathcal{G}$ measurable (predictable) if it is $\mathcal{B}_+ \otimes \mathcal{F}$ ($\mathcal{P}$, resp.)-measurable. Similarly, $\mathcal{G}$ is said to be $\mathcal{F}_t$ -adapted if $\mathcal{G}_t$ is $\mathcal{F}_t$ -measurable for each $t \geq 0$. It is clear that every predictable set-valued stochastic process is measurable and $\mathcal{F}_t$ -adapted. It follows from the Kuratowski and Ryll-Nardzewski measurable selection theorem (see [9]) that every measurable (predictable) set-valued process with nonempty compact values possesses a measurable (predictable) selector. We shall also consider $\mathcal{B}_+ \otimes \mathcal{F} \otimes \mathcal{B}^n$ and $\mathcal{P}^n$ -measurable set-valued mappings $\mathcal{R}: \mathbb{R}_+ \times \Omega \times \mathbb{R}^n \rightarrow Cl(\mathbb{R}^n)$. They will be denoted as families $(\mathcal{R}_{t,z})_{t \geq 0, z \in \mathbb{R}^n}$ and called measurable and predictable, respectively set-valued stochastic processes depending on a parameter $z \in \mathbb{R}^n$. The process $\mathcal{R} = (\mathcal{R}_{t,z})_{t \geq 0, z \in \mathbb{R}^n}$ is said to be $\mathcal{F}_t$ -adapted if $\mathcal{R}_{t,z}$ is $\mathcal{F}_t$ -measurable for each $t \geq 0$ and $z \in \mathbb{R}^n$.

Denote by $\mathcal{M}_{s-v}^p(\mathcal{P})$, $p \geq 1$, and $\mathcal{M}_{s-v}^2(\mathcal{P}^n, q)$ the families of all set-valued predictable processes $F = (F_t)_{t \geq 0}$ and $\mathcal{R} = (\mathcal{R}_{t,z})_{t \geq 0, z \in \mathbb{R}^n}$, respectively, such that $E \int_0^\infty \|F_t\|^p dt < \infty$ and $E \int_0^\infty \int_{\mathbb{R}^n} \|\mathcal{R}_{t,z}\|^2 dt q(z) < \infty$. Immediately from the Kuratowski and Ryll-Nardzewski measurable selection theorem it follows that for every $F \in \mathcal{M}_{s-v}^p(\mathcal{P})$, $p \geq 1$, and $\mathcal{R} \in \mathcal{M}_{s-v}^2(\mathcal{P}^n, q)$ the sets

$$\mathcal{I}^p(F) = \{f \in \mathcal{L}_n^p: f_t(\omega) \in F_t(\omega), dt \times P - a.e.\}$$

and

$$\mathcal{I}_q^2(\mathcal{R}) = \{h \in \mathcal{W}_n^2: h_{t,z}(\omega) \in \mathcal{R}_{t,z}(\omega), dt \times P \times q - a.e.\}$$

are nonempty.

Given set-valued processes $F = (F_t)_{t \geq 0} \in \mathcal{M}_{s-v}^p(\mathcal{P})$, $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0} \in \mathcal{M}_{s-v}^2(\mathcal{P})$ and $\mathcal{R} = (\mathcal{R}_{t,z})_{t \geq 0, z \in \mathbb{R}^n} \in \mathcal{M}_{s-v}^2(\mathcal{P}^n, q)$ by their stochastic integrals $\mathcal{I}F$, $\mathcal{I}\mathcal{G}$ and $\mathcal{I}\mathcal{R}$ we mean families $\mathcal{I}F = (\mathcal{I}_t F)_{t \geq 0}$, $\mathcal{I}\mathcal{G} = (\mathcal{I}_t \mathcal{G})_{t \geq 0}$, and $\mathcal{I}\mathcal{R} = (\mathcal{I}_t \mathcal{R})_{t \geq 0}$ subsets of $L_n^p(\mathcal{F}_t)$, $p \geq 1$ and $L_n^2(\mathcal{F}_t)$, respectively, defined by

$\mathfrak{I}_t F = \{\mathfrak{I}_t f : f \in \mathcal{F}^p(F)\}$, $\mathfrak{I}_t \mathfrak{G} = \{\mathfrak{I}_t g : g \in \mathcal{F}^2(\mathfrak{G})\}$ and $\mathcal{T}_t \mathcal{R} = \{\mathcal{T}_t h : h \in \mathcal{F}_q^2(\mathcal{R})\}$, where $\mathfrak{I}_t f = \int_0^t f_s ds$, $\mathfrak{I}_t g = \int_0^t g_s dw_s$ and $\mathcal{T}_t h = \int_0^t \int_{\mathbb{R}^n} h_{s,z} \tilde{\nu}(ds, dz)$. Given $0 \leq \alpha < \beta < \infty$, we also define $\int_\alpha^\beta F_s ds = \{\int_\alpha^\beta f_s ds : f \in \mathcal{F}^p(F)\}$, $\int_\alpha^\beta \mathfrak{G}_s dw_s = \{\int_\alpha^\beta g_s dw_s : g \in \mathcal{F}^2(\mathfrak{G})\}$ and $\int_\alpha^\beta \int_{\mathbb{R}^n} \mathcal{R}_{s,z} \tilde{\nu}(ds, dz) = \{\int_\alpha^\beta \int_{\mathbb{R}^n} h_{s,z} \tilde{\nu}(ds, dz) : h \in \mathcal{F}^2(\mathcal{R})\}$. The following properties of set-valued stochastic integrals are given in [10].

Proposition 1: *Let $F \in \mathcal{M}_{s-v}^p(\mathcal{P})$, $p \geq 1$, $\mathfrak{G} \in \mathcal{M}_{s-v}^2(\mathcal{P})$ and $\mathcal{R} \in \mathcal{M}_{s-v}^2(\mathcal{P}^n, q)$. Then*

- (i) $\mathfrak{I}_t \mathfrak{G}$ and $\mathcal{T}_t \mathcal{R}$ are closed subsets of $L_n^2(\mathcal{F}_t)$ for each $t \geq 0$.
- (ii) If, moreover, $F, \mathfrak{G}$ and $\mathcal{R}$ take on convex values then $\mathfrak{I}_t F$, $\mathfrak{I}_t \mathfrak{G}$ and $\mathcal{T}_t \mathcal{R}$ are convex and weakly compact in $L_n^p(\mathcal{F}_t)$ and $L_n^2(\mathcal{F}_t)$, respectively, for each $t \geq 0$.

Proposition 2: *Let $F \in \mathcal{M}_{s-v}^2(\mathcal{P})$, $\mathfrak{G} \in \mathcal{M}_{s-v}^2(\mathcal{P})$ and $\mathcal{R} \in \mathcal{M}_{s-v}^2(\mathcal{P}^n, q)$. Assume $(x_t)_{t \geq 0} \in D$ is such that*

$$x_t - x_s \in cl_{L^2} \left(\int_s^t F_\tau d\tau + \int_s^t \mathfrak{G}_\tau dw_\tau + \int_s^t \int_{\mathbb{R}^n} \mathcal{R}_{\tau,z} \tilde{\nu}(d\tau, dz) \right)$$

for every $0 \leq s < t < \infty$. Then for every $\epsilon > 0$ there are $f^\epsilon \in \mathcal{F}^p(F)$, $g^\epsilon \in \mathcal{F}^2(\mathfrak{G})$ and $h^\epsilon \in \mathcal{F}_q^2(\mathcal{R})$ such that

$$\sup_{t \geq 0} \| (x_t - x_0) - \left(\int_0^t f_\tau^\epsilon d\tau + \int_0^t g_\tau^\epsilon dw_\tau + \int_0^t \int_{\mathbb{R}^n} h_{\tau,z}^\epsilon \tilde{\nu}(d\tau, dz) \right) \|_{L^2} \leq \epsilon.$$

Proposition 3: *Assume $F \in \mathcal{M}_{s-v}^2(\mathcal{P})$, $\mathfrak{G} \in \mathcal{M}_{s-v}^2(\mathcal{P})$ and $\mathcal{R} \in \mathcal{M}_{s-v}^2(\mathcal{P}^n, q)$ take on convex values and let $(x_t)_{t \geq 0} \in D$. Then*

$$x_t - x_s \in \int_s^t F_\tau d\tau + \int_s^t \mathfrak{G}_\tau dw_\tau + \int_s^t \int_{\mathbb{R}^n} \mathcal{R}_{\tau,z} \tilde{\nu}(d\tau, dz)$$

for $0 \leq s < t < \infty$ if and only if there are $f \in \mathcal{F}^2(F)$, $g \in \mathcal{F}^2(\mathfrak{G})$ and $h \in \mathcal{F}_q^2(\mathcal{R})$ such that

$$x_t = x_0 + \int_0^t f_\tau d\tau + \int_0^t g_\tau dw_\tau + \int_0^t \int_{\mathbb{R}^n} h_{\tau,z} \tilde{\nu}(d\tau, dz), \text{ a.s. for each } t \geq 0.$$

4. STOCHASTIC INCLUSIONS

Let $F = \{(F_t(x))_{t \geq 0} : x \in \mathbb{R}^n\}$, $G = \{(G_t(x))_{t \geq 0} : x \in \mathbb{R}^n\}$ and $H = \{(H_{t,z}(x))_{t \geq 0, z \in \mathbb{R}^n} : x \in \mathbb{R}^n\}$. Assume F, G and H are such that $(F_t(x))_{t \geq 0} \in \mathcal{M}_{s-v}^p(\mathcal{P})$, $(G_t(x))_{t \geq 0} \in \mathcal{M}_{s-v}^2(\mathcal{P})$ and $(H_{t,z}(x))_{t \geq 0, z \in \mathbb{R}^n} \in \mathcal{M}_{s-v}^2(\mathcal{P}^n, q)$ for each $x \in \mathbb{R}^n$.

By a stochastic inclusion, denoted by $SI(F, G, H)$, corresponding to F, G and H given above, we mean the relation

$$x_t - x_s \in cl_{L^2} \left(\int_s^t F_\tau(x_\tau) d\tau + \int_s^t G_\tau(x_\tau) dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,z}(x_\tau) \tilde{\nu}(d\tau, dz) \right)$$

that is to be satisfied for every $0 \leq s < t < \infty$ by a stochastic process $x = (x_t)_{t \geq 0} \in D$ such that $F \circ mx \in \mathcal{M}_{s-v}^p(\mathcal{P})$, $G \circ mx \in \mathcal{M}_{s-v}^2(\mathcal{P})$ and $H \circ mx \in \mathcal{M}_{s-v}^2(\mathcal{P}^n, q)$, where $F \circ mx = (F_t(x_t))_{t \geq 0}$, $G \circ mx = (G_t(x_t))_{t \geq 0}$ and $H \circ mx = (H_{t,z}(x_t))_{t \geq 0, z \in \mathbb{R}^n}$. Every stochastic process $(x_t)_{t \geq 0} \in D$, satisfying the conditions mentioned above, is said to be global solution to $SI(F, G, H)$.

Corollary 1: *If F, G and H take on convex values then $SI(F, G, H)$ has a form*

$$x_t - x_s \in \int_s^t F_\tau(x_\tau) d\tau + \int_s^t G_\tau(x_\tau) dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,z}(x_\tau) \tilde{\nu}(d\tau, dz)$$

and $(x_t)_{t \geq 0} \in D$ is a global solution to $SI(F, G, H)$ if and only if there are $f \in \mathcal{P}^2(F \circ mx)$, $g \in \mathcal{P}^2(G \circ mx)$ and $h \in \mathcal{P}_q^2(H \circ mx)$ such that

$$x_t = x_0 + \int_0^t f_\tau d\tau + \int_0^t g_\tau dw_\tau + \int_0^t \int_{\mathbb{R}^n} h_{\tau,z} \tilde{\nu}(d\tau, dz), \text{ a.s. for each } t \geq 0.$$

Given $0 \leq \alpha < \beta < \infty$, a stochastic process $(x_t)_{t \geq 0} \in D$ is said to be a local solution to $SI(F, G, H)$ on $[\alpha, \beta]$ if

$$x_t - x_s \in cl_{L^2} \left(\int_s^t F_\tau(x_\tau) d\tau + \int_s^t G_\tau(x_\tau) dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,z}(x_\tau) \tilde{\nu}(d\tau, dz) \right)$$

for $\alpha \leq s < t \leq \beta$.

Corollary 2: *A stochastic process $(x_t)_{t \geq 0} \in D$ is a local solution to $SI(F, G, H)$ on $[\alpha, \beta]$ if and only if $x^{\alpha, \beta}$ is a global solution to $SI(F^{\alpha, \beta}, G^{\alpha, \beta}, H^{\alpha, \beta})$, where $F^{\alpha, \beta} = \mathbb{I}_{[\alpha, \beta]} F$, $G^{\alpha, \beta} = \mathbb{I}_{[\alpha, \beta]} G$ and $H^{\alpha, \beta} = \mathbb{I}_{[\alpha, \beta]} H$.*

A stochastic process $(x_t)_{t \geq 0} \in D$ is called a global (local on $[\alpha, \beta]$, resp.) solution to an initial value problem for stochastic inclusion $SI(F, G, H)$ with an initial condition $y \in L^2(\Omega, \mathcal{F}_0, \mathbb{R}^n)$ ($y \in L^2(\Omega, \mathcal{F}_\alpha, \mathbb{R}^n)$, resp.) if $(x_t)_{t \geq 0}$ is a global (local on $[\alpha, \beta]$, resp.) solution to $SI(F, G, H)$ and $x_0 = y$ ($x_\alpha = y$, resp.). An initial-value problem for $SI(F, G, H)$ mentioned above will be denoted by $SI_y(F, G, H)$ ($SI_y^{\alpha, \beta}(F, G, H)$, resp.). In what follows, we denote a set of all global (local on $[\alpha, \beta]$, resp.) solutions to $SI_y(F, G, H)$ by $\Lambda_y(F, G, H)$ ($\Lambda_y^{\alpha, \beta}(F, G, H)$, resp.).

Suppose F, G and H satisfy the following conditions $(\mathcal{A}_1)$:

- (i) $F = \{(F_t(x))_{t \geq 0} : x \in \mathbb{R}^n\}$, $G = \{(G_t(x))_{t \geq 0} : x \in \mathbb{R}^n\}$ and $H = \{(H_{t,z}(x))_{t \geq 0, z \in \mathbb{R}^n} : x \in \mathbb{R}^n\}$ are such that mappings $\mathbb{R}^+ \times \Omega \times \mathbb{R}^n \ni (t, \omega, x) \rightarrow F_t(x)(\omega) \in Cl(\mathbb{R}^n)$, $\mathbb{R}_t \times \Omega \times \mathbb{R}^n \ni (t, \omega, x) \rightarrow G_t(x)(\omega) \in Cl(\mathbb{R}^n)$ and $\mathbb{R}_t \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \ni (t, \omega, z, x) \rightarrow H_{t,z}(x)(\omega) \in Cl(\mathbb{R}^n)$ are $\mathcal{P} \otimes \mathcal{B}^n$ and $\mathcal{P}^n \otimes \mathcal{B}^n$ -measurable, respectively.
- (ii) $(F_t(x))_{t \geq 0}$, $(G_t(x))_{t \geq 0}$, $(H_{t,z}(x))_{t \geq 0, z \in \mathbb{R}^n}$ are uniformly p - and square-integrable bounded, respectively, i.e.,

$$\begin{aligned} (\sup_{x \in \mathbb{R}^n} \|F_t(x)\|)_{t \geq 0} &\in \mathcal{L}_1^p, (\sup_{x \in \mathbb{R}^n} \|G_t(x)\|)_{t \geq 0} \in \mathcal{L}_1^2 \quad \text{and} \\ (\sup_{x \in \mathbb{R}^n} \|H_{t,z}(x)\|)_{t \geq 0, z \in \mathbb{R}^n} &\in \mathcal{W}_1^2. \end{aligned}$$

Corollary 3: For every $(x_t)_{t \geq 0} \in D$ and F, G, H satisfying $(\mathcal{A}_1)$ one has $F \circ mx \in \mathcal{M}_{x-v}^p(\mathcal{P})$, $G \circ mx \in \mathcal{M}_{s-v}^2(\mathcal{P})$ and $H \circ mx \in \mathcal{M}_{s-v}^2(\mathcal{P}^n, q)$.

Now define a linear continuous mapping Φ on $\mathcal{L}_n^p \times \mathcal{L}_n^2 \times \mathcal{W}_n^2$ by taking $\Phi(f, g, h) = (\mathcal{I}_t f + \mathcal{J}_t g + \mathcal{T}_t h)_{t \geq 0}$ to each $(f, g, h) \in \mathcal{L}_n^p \times \mathcal{L}_n^2 \times \mathcal{W}_n^2$. It is clear that Φ maps $\mathcal{L}_n^p \times \mathcal{L}_n^2 \times \mathcal{W}_n^2$ into D . Given above F, G and H satisfying $(\mathcal{A}_1)$, define a set-valued mapping $\mathcal{H}$ on D by setting

$$\mathcal{H}(x) = cl_\rho(\Phi(\mathcal{I}^p(F \circ mx) \times \mathcal{I}^2(G \circ mx) \times \mathcal{I}_q^2(H \circ mx))) \quad (1)$$

for $x = (x_t)_{t \geq 0} \in D$, where the closure is taken in the norm topology in $(D, \|\cdot\|_\rho)$. Similarly, for given $0 \leq \alpha < \beta < \infty$, we define a set-valued mapping $\mathcal{H}^{\alpha, \beta}$ on D by taking

$$\mathcal{H}^{\alpha, \beta}(x) = cl_\rho(\Phi(\mathcal{I}^p(F^{\alpha\beta} \circ mx) \times \mathcal{I}^2(G^{\alpha\beta} \circ mx) \times \mathcal{I}_q^2(H^{\alpha\beta} \circ mx))) \quad (2)$$

where $F^{\alpha\beta}$, $G^{\alpha\beta}$ and $H^{\alpha\beta}$ are as above.

Corollary 4: For every F, G and H taking on convex values and

satisfying $(\mathcal{A}_1)$, one has $\mathfrak{H}(x) = \Phi(\mathcal{Y}^p(F \circ mx) \times \mathcal{Y}^2(G \circ mx) \times \mathcal{Y}_q^2(H \circ mx))$ and $\mathfrak{H}^{\alpha, \beta}(y) = \Phi(\mathcal{Y}^p(F^{\alpha\beta} \circ mx) \times \mathcal{Y}^2(G^{\alpha\beta} \circ mx) \times \mathcal{Y}_q^2(H^{\alpha\beta} \circ mx))$ for $x \in D$.

Let $S(F, G, H)$ and $S^{\alpha, \beta}(F, G, H)$ denote the set of all fixed points of $\mathfrak{H}$ and $\mathfrak{H}^{\alpha, \beta}$, respectively. It will be shown below that $S^{\alpha, \beta}(F, G, H) \subset D^{\alpha, \beta}$. Immediately from Proposition 2 (see [10]) the following result follows.

Proposition 4: Assume F, G and H satisfy $(\mathcal{A}_1)$ and take on convex values. Then $\Lambda_0(F, G, H) = S(F, G, H)$ and $\Lambda_0^{\alpha, \beta}(F, G, H) = S^{\alpha, \beta}(F, G, H)$ for every $0 \leq \alpha < \beta < \infty$, respectively.

Proposition 5: Assume F, G and H satisfy $(\mathcal{A}_1)$ and let $0 \leq \alpha < \beta < \infty$. Then $x \in S^{\alpha, \beta}(F, G, H)$ if and only if

- (i) $x_t = 0$ a.s. for $t \in [0, \alpha]$,
- (ii) $x_t = x_\beta$ a.s. for $t \geq \beta$,
- (iii) for every $\epsilon > 0$ there is $(f^\epsilon, g^\epsilon, h^\epsilon) \in \mathcal{Y}^p(F^{\alpha\beta} \circ mx) \times \mathcal{Y}^2(G^{\alpha\beta} \circ mx) \times \mathcal{Y}_q^2(H^{\alpha\beta} \circ mx)$ such that $\| \sup_{\alpha \leq t \leq \beta} |x_t - \Phi_t(f^\epsilon, g^\epsilon, h^\epsilon)| \|_{L_1^2} < \epsilon$.

Proof: $(\Rightarrow)$ Let $x \in S^{\alpha, \beta}(F, G, H)$. By the definition of $\mathfrak{H}^{\alpha, \beta}$, for every $\epsilon > 0$, there is $(f^\epsilon, g^\epsilon, h^\epsilon) \in \mathcal{Y}^p(F^{\alpha\beta} \circ mx) \times \mathcal{Y}^2(G^{\alpha\beta} \circ mx) \times \mathcal{Y}_q^2(H^{\alpha\beta} \circ mx)$ such that $\|x - \Phi(f^\epsilon, g^\epsilon, h^\epsilon)\|_\ell < \epsilon$. We have of course $\Phi_t(f^\epsilon, g^\epsilon, h^\epsilon) = 0$ and $\Phi_t(f^\epsilon, g^\epsilon, h^\epsilon) = \Phi_\beta(f^\epsilon, g^\epsilon, h^\epsilon)$, a.s. for $0 \leq t \leq \alpha$ and $t \geq \beta$, respectively. Then

$$\begin{aligned} \left\| \sup_{0 \leq t \leq \alpha} |x_t| \right\|_{L_1^2} &= \left\| \sup_{0 \leq t \leq \alpha} |x_t - \Phi_t(f^\epsilon, g^\epsilon, h^\epsilon)| \right\|_{L_1^2} \\ &\leq \|x - \Phi(f^\epsilon, g^\epsilon, h^\epsilon)\|_\ell < \epsilon. \end{aligned}$$

and

$$\begin{aligned} \left\| \sup_{t \geq \beta} |x_t - x_\beta| \right\|_{L_1^2} &= \left\| \sup_{t \geq \beta} |x_t - \Phi_t(f^\epsilon, g^\epsilon, h^\epsilon)| \right\|_{L_1^2} \\ &+ \left\| \sup_{t \geq \beta} |x_\beta - \Phi_\beta(f^\epsilon, g^\epsilon, h^\epsilon)| \right\|_{L_1^2} < 2\epsilon. \end{aligned}$$

Therefore, $\sup_{0 \leq t \leq \alpha} |x_t| = 0$ and $\sup_{t \geq \beta} |x_t - x_\beta| = 0$ a.s.

By the properties of $\Phi(f^\epsilon, g^\epsilon, h^\epsilon)$, (i) and (ii), (iii) easily follow.

$(\Leftarrow)$ Conditions (i) – (iii) imply

$$\|x - \Phi(f^\epsilon, g^\epsilon, h^\epsilon)\|_\ell = \left\| \sup_{\alpha \leq t \leq \beta} |x_t - \Phi_t(f^\epsilon, g^\epsilon, h^\epsilon)| \right\|_{L_1^2} < \epsilon.$$

Therefore, $x \in cl_{\ell} \Phi(\mathcal{Y}^p(F^{\alpha\beta} \circ mx) \times \mathcal{Y}^2(G^{\alpha\beta} \circ mx) \times \mathcal{Y}_q^2(H^{\alpha\beta} \circ mx))$. $\square$

Proposition 6: Assume F, G and H satisfy $(\mathcal{A}_1)$ and let $(\tau_n)_{n=1}^{\infty}$ be a sequence of positive numbers increasing to $+\infty$. If $x^1 \in S^{0, \tau_1}(F, G, H)$ and $x^{n+1} \in x_{\tau_n}^n + S^{\tau_n, \tau_{n+1}}(F, G, H)$ for $n = 1, 2, \dots$, then $x = \sum_{n=1}^{\infty} \mathbb{I}_{[\tau_{n-1}, \tau_n)}(x^n - x_{\tau_{n-1}}^{n-1})$ belongs to $S(F, G, H)$, where $x_0^0 = 0$.

Proof: For every $n = 1, 2, \dots$ one has $x^n - x_{\tau_{n-1}}^{n-1} \in S^{\tau_{n-1}, \tau_n}(F, G, H)$. Then, by Proposition 5, for every $n = 1, 2, \dots$ and $\epsilon > 0$ there is $(f^n, g^n, h^n) \in \mathcal{Y}^p(F^{\tau_{n-1}, \tau_n} \circ mx^n) \times \mathcal{Y}^2(G^{\tau_{n-1}, \tau_n} \circ mx^n) \times \mathcal{Y}_q^2(H^{\tau_{n-1}, \tau_n} \circ mx^n)$ such that

$$\| \sup_{\tau_{n-1} \leq t \leq \tau_n} |(x_t^n - x_{\tau_{n-1}}^{n-1}) - \Phi_t(f^n, g^n, h^n)| \|_{L_1^2} < \epsilon/2^n.$$

Put $f^\epsilon = \sum_{n=1}^{\infty} \mathbb{I}_{[\tau_{n-1}, \tau_n)} f^n$, $g^\epsilon = \sum_{n=1}^{\infty} \mathbb{I}_{[\tau_{n-1}, \tau_n)} g^n$ and $h^\epsilon = \sum_{n=1}^{\infty} \mathbb{I}_{[\tau_{n-1}, \tau_n)} h^n$. By the decomposability (see [9], [10]) of $\mathcal{Y}^2(F \circ mx)$, $\mathcal{Y}^2(G \circ mx)$ and $\mathcal{Y}_q^2(H \circ mx)$, we get $f^\epsilon \in \mathcal{Y}^2(F \circ mx)$, $g^\epsilon \in \mathcal{Y}^2(G \circ mx)$ and $h^\epsilon \in \mathcal{Y}_q^2(H \circ mx)$. Moreover

$$\begin{aligned} & \|x - \Phi(f^\epsilon, g^\epsilon, h^\epsilon)\|_{\ell} \\ & \leq \left\| \sum_{n=1}^{\infty} \sup_{\tau_{n-1} \leq t \leq \tau_n} |(x_t^n - x_{\tau_{n-1}}^{n-1}) - \Phi_t(f^n, g^n, h^n)| \right\|_{L_1^2} \\ & \leq \sum_{n=1}^{\infty} \left\| \sup_{\tau_{n-1} \leq t \leq \tau_n} |(x_t^n - x_{\tau_{n-1}}^{n-1}) - \Phi_t(f^n, g^n, h^n)| \right\|_{L_1^2} < \epsilon. \end{aligned}$$

Therefore, $x \in cl_{\ell} \Phi(\mathcal{Y}^2(F \circ mx) \times \mathcal{Y}^2(G \circ mx) \times \mathcal{Y}_q^2(H \circ mx))$. $\square$

In what follows we shall deal with $F = \{(F_t(x))_{t \geq 0} : x \in \mathbb{R}^n\}$, $G = \{(G_t(x))_{t \geq 0} : x \in \mathbb{R}^n\}$ and $H = \{(H_{t,z}(x))_{t \geq 0, z \in \mathbb{R}^n} : x \in \mathbb{R}^n\}$ satisfying conditions $(\mathcal{A}_1)$ and any one of the following conditions.

$(\mathcal{A}_2)$ F, G and H are such that set-valued functions $D \ni x \rightarrow (F \circ mx)_t(\omega) \subset \mathbb{R}^n$, $D \ni x \rightarrow (G \circ mx)_t(\omega) \subset \mathbb{R}^n$ and $D \ni x \rightarrow (H \circ mx)_{t,z}(\omega) \subset \mathbb{R}^n$ are w.-w.s.u.s.c. on D , i.e., for every $x \in D$ and every sequence (x_n) of $(D, \|\cdot\|_{\ell})$ converging weakly to x , one has $\bar{h}(\int \int_A (F \circ mx_n)_t dtdP, \int \int_A (F \circ mx)_t dtdP) \rightarrow 0$, $\bar{h}(\int \int_A (G \circ mx_n)_t dtdP, \int \int_A (G \circ mx)_t dtdP) \rightarrow 0$ and $\bar{h}(\int \int_B \int (H \circ mx_n)_{t,z} dtq(dz)dP, \int \int_B \int (H \circ mx)_{t,z} dtq(dz) dP) \rightarrow 0$.

- ($\mathcal{A}_3$) F, G and H are such that set-valued functions $D \ni x \rightarrow (F \circ mx)_t(\omega) \subset \mathbb{R}^n$, $D \ni x \rightarrow (G \circ mx)_t(\omega) \subset \mathbb{R}^n$ and $D \ni x \rightarrow (H \circ mx)_{t,z}(\omega) \subset \mathbb{R}^n$ are *s.-w.s.l.s.c.* on D , i.e., for every $x \in D$ and every sequence (x_n) of $(D, \|\cdot\|_\ell)$ converging weakly to x , one has $\bar{h}((F \circ mx)_t(\omega), (F \circ mx^n)_t(\omega)) \rightarrow 0$, $\bar{h}((G \circ mx)_t(\omega), (G \circ mx^n)_t(\omega)) \rightarrow 0$ and $\bar{h}((H \circ mx)_{t,z}(\omega), (H \circ mx^n)_{t,z}(\omega)) \rightarrow 0$ *a.e.*
- ($\mathcal{A}_4$): There are $k, \ell \in \mathcal{L}_1^2$ and $m \in \mathcal{W}_1^2$ such that $\|\int_0^\infty h[(F \circ mx)_t, (F \circ my)_t] dt\|_{\mathcal{L}_1^2} \leq E \int_0^\infty k_t |x_t - y_t| dt$, $\|h(G \circ mx, G \circ my)\|_{\mathcal{L}_1^2} \leq E \int_0^\infty \ell_t |x_t - y_t| dt$ and $\|h(H \circ mx, H \circ my)\|_{\mathcal{W}_1^2} \leq E \int_0^\infty \int_{\mathbb{R}^n} m_{t,z} |x_t - y_t| dt q(dz)$ for $x, y \in D$.
- ($\mathcal{A}'_4$) There are $k, \ell \in L^2(\mathbb{B}_+)$ and $m \in L^2(\mathbb{B}_+ \times \mathbb{B}^n)$ such that $h(F_t(x_2)(\omega), F_t(x_1)(\omega)) \leq k(t) |x_1 - x_2|$, $h(G_t(x_2)(\omega), G_t(x_1)(\omega)) \leq \ell(t) |x_1 - x_2|$ and $h(H_{t,z}(x_2)(\omega), H_{t,z}(x_1)(\omega)) \leq m(t, z) |x_1 - x_2|$ *a.e.*, each $t \geq 0$ and $x_1, x_2 \in \mathbb{R}^n$.

It is clear that the upper (lower) semicontinuity of F, G and H does not imply their weak (strong) - weak sequential upper (lower) semicontinuity presented above. We shall show that in some special cases, i.e., for concave (convex, resp.), set-valued mappings such implication holds true. Recall a set-valued mapping $\mathcal{R}$, defined on a locally convex topological space $(X, \mathcal{T}_X)$ with values in a normed space is said to be concave (convex) if $\mathcal{R}(\alpha x_1 + \beta x_2) \subset \alpha \mathcal{R}(x_1) + \beta \mathcal{R}(x_2)$ ($\alpha \mathcal{R}(x_1) + \beta \mathcal{R}(x_2) \subset \mathcal{R}(\alpha x_1 + \beta x_2)$), for every $x_1, x_2 \in X$ and $\alpha, \beta \in [0, 1]$ satisfying $\alpha + \beta = 1$.

Lemma 1: Suppose F, G and H satisfy ($\mathcal{A}_1$) with $p = 1$, take on convex values and are concave (convex) with respect to $x \in \mathbb{R}^n$. If moreover F, G and H are *u.s.c.* (*l.s.c.*) with respect to $x \in \mathbb{R}^n$ then they are *w.-w.s.u.s.c.* (*s.-w.s.l.s.c.*).

Proof: Let $x \in D$ be fixed and let (x^n) be a sequence of D weakly converging to x . Denote $K_p(t, \omega, y) := -s(p, F_t(y_t)(\omega))$ for $p \in \mathbb{R}^n$, $y \in D$, $t \geq 0$ and $\omega \in \Omega$. We shall show that for every $A \in \mathcal{P}$ and every $p \in \mathbb{R}^n$ one has

$$\int_A \int K_p(t, \omega, x) dt dP \leq \liminf_{n \rightarrow \infty} \int_A \int K_p(t, \omega, x^n) dt dP,$$

which is equivalent to the weak-weak sequential upper semicontinuity of F at $x \in D$ in the sense defined in ($\mathcal{A}_2$). Similarly, the weak-weak sequential upper

semicontinuity of G and H can be verified.

Let $A \in \mathcal{P}$, $p \in \mathbb{R}^n$ be given. Denote $j_n = \int \int_A K_p(t, \omega, x^n) dt dP$ for $n = 1, 2, \dots$ and put $i := \liminf_{n \rightarrow \infty} \int \int_A K_p(t, \omega, x^n) dt dP$. By taking a suitable subsequence, say (n_k) of (n) we may well assume that $j_{n_k} \rightarrow i$ as $k \rightarrow \infty$. By the Banach and Mazur theorem (see [2]) for every $s = 1, 2, \dots$ there are numbers $\alpha_k^s \geq 0$ with $k = 1, 2, \dots, N$ and $N = 1, 2, \dots$ satisfying $\sum_{k=1}^N \alpha_k^s = 1$ and such that $\|z_N^s - x\|_{\ell} \rightarrow 0$ as $N \rightarrow \infty$, where $z_N^s(t, \omega) = \sum_{k=1}^N \alpha_k^s x_t^{n_k+s}(\omega)$. By the definition of the norm $\|\cdot\|_{\ell}$ there is a subsequence, say again (z_N^s) , of (z_N) such that $\sup_{t \geq 0} |z_N^s(t, \omega) - x_t(\omega)| \rightarrow 0$ a.s. for $s = 1, 2, \dots$. Put

$$\eta_N^s = \sum_{k=1}^N \alpha_k^s K_p(\cdot, \cdot, x^{n_k+s}),$$

$$j_k^s = \int \int_A K_p(t, \omega, x^{n_k+s}) dt dP$$

and let $\delta_s = \max_{N \geq s+1} \max_{1 \leq k \leq N} |j_k^s - i|$ for $s = 1, 2, \dots$. We have $\delta_s \rightarrow 0$ as $s \rightarrow \infty$. By the uniform square boundedness of F there is $m_F \in \mathcal{L}_1^2$ such that $\eta_N^s \geq -m_F$ a.e. for $N, s = 1, 2, \dots$. Therefore, $\liminf_{N \rightarrow \infty} \eta_N^s \geq -m_F$ a.e. for $s = 1, 2, \dots$. Then by Fatou's lemma one obtains

$$\int \int_A \liminf_{N \rightarrow \infty} \eta_N^s dt dP \leq \liminf_{N \rightarrow \infty} \int \int_A \eta_N^s dt dP \leq i + \delta_s$$

for $s = 1, 2, \dots$, because for every $s = 1, 2, \dots$, we have $i - \delta_s \leq \int \int_A \eta_N^s dt dP \leq i + \delta_s$.

Taking $\eta = \liminf_{s \rightarrow \infty} [\liminf_{N \rightarrow \infty} \eta_N^s]$ a.e., we get $\eta \geq -m_F$ a.e. and $\int \int_A \eta dt dP \leq i$. We shall verify that we also have $K(t, \omega, x) \leq \eta(t, \omega)$ for a.e. $(t, \omega) \in \mathbb{R}_+ \times \Omega$. Indeed, by upper semicontinuity of F with respect to $x \in \mathbb{R}^n$, a real valued function $x \rightarrow -s(p, F_t(x))$ is lower semicontinuous on $\mathbb{R}^n$, a.s. for every $t \geq 0$ and $p \in \mathbb{R}^n$. Therefore for every $m, s = 1, 2, \dots$ there is $M \geq 1$ such that

$$-s(p, F_t(x_t)) - \frac{1}{m} < -s(p, F_t(\sum_{k=1}^N \alpha_k^s x_t^{n_k+s}))$$

a.s. for every $t \geq 0$ and $N \geq M$. Hence, by the properties of F , it follows

$$-s(p, F_t(x_t)) - \frac{1}{m} < \sum_{k=1}^N \alpha_k^s [-s(p, F_t(x_t^{n_k+s}))] =: \eta_N^s(t, \cdot)$$

a.s. for $t \geq 0$, $s, m = 1, 2, \dots$ and $N \geq M$. Therefore, for $m = 1, 2, \dots$ almost everywhere, one gets

$$K_p(\cdot, \cdot, x) - \frac{1}{m} \leq \liminf_{i \rightarrow \infty} f[\liminf_{N \rightarrow \infty} f \eta_N^*] = \eta.$$

Finally, we get

$$\int \int_A K_p(t, \omega, x) dt dP \leq \int \int_A \eta(t, \omega) dt dP \leq i. \quad \square$$

5. PROPERTIES OF SOLUTION SET

We shall prove here the existence theorems for $SI(F, G, H)$. We show first that conditions $(\mathcal{A}_1)$ and anyone of conditions $(\mathcal{A}_2)$ – $(\mathcal{A}_4)$ or $(\mathcal{A}'_4)$ imply the existence of fixed points for the set-valued mappings $\mathcal{H}$ and $\mathcal{H}^{\alpha, \beta}$ defined above. Hence, by Propositions 4 and 5, the existence theorems for $SI(F, G, H)$ will follow. We begin with the following lemmas.

Lemma 2: *Assume F, G and H take on convex values, satisfy $(\mathcal{A}_1)$ with $p = 2$ and $(\mathcal{A}_2)$. Then a set-valued mapping $\mathcal{H}$ is u.s.c. as a multifunction defined on a locally convex topological Hausdorff space $(D, \sigma(D, D^*))$ with nonempty values in $(D, \sigma(D, D^*))$.*

Proof: Let C be a nonempty weakly closed subset of D and select a sequence (x^n) of $\mathcal{H}^-(C)$ weakly converging to $x \in D$. There is a sequence (y^n) of C such that $y^n \in \mathcal{H}(x^n)$ for $n = 1, 2, \dots$. By the uniform square-integrable boundedness of F, G and H , there is a convex weakly compact subset $\mathfrak{B} \subset \mathcal{L}_n^2 \times \mathcal{L}_n^2 \times \mathcal{W}_n^2$ such that $\mathcal{H}(x^n) \subset \Phi(\mathfrak{B})$. Therefore, $y^n \in \Phi(\mathfrak{B})$, for $n = 1, 2, \dots$ which, by the weak compactness of $\Phi(\mathfrak{B})$, implies the existence of a subsequence, say for simplicity (y^k) , of (y^n) weakly converging to $y \in \Phi(\mathfrak{B})$. We have $y^k \in \mathcal{H}(x^k)$ for $k = 1, 2, \dots$. Let $(f^k, g^k, h^k) \in \mathcal{Y}^2(F \circ mx^k) \times \mathcal{Y}^2(G \circ mx^k) \times \mathcal{Y}_q^2(H \circ mx^k)$ be such that $\Phi(f^k, g^k, h^k) = y^k$, for each $k = 1, 2, \dots$. We have of course $(f^k, g^k, h^k) \in \mathfrak{B}$. Therefore, there is a subsequence, say again $\{(f^k, g^k, h^k)\}$ of $\{(f^k, g^k, h^k)\}$ weakly converging in $\mathcal{L}_n^2 \times \mathcal{L}_n^2 \times \mathcal{W}_n^2$ to $(f, g, h) \in \mathfrak{B}$. Now, for every $A \in \mathcal{P}$ one obtains

$$\begin{aligned} & \text{dist} \left(\int \int_A f_t dt dP, \int \int_A F_t(x) dt dP \right) \leq \\ & \leq \left| \int \int_A [f_t - f_t^k] dt dP \right| + \text{dist} \left(\int \int_A f_t^k dt dP, \int \int_A F_t(x_k) dt dP \right) \end{aligned}$$

$$+ \bar{h} \left(\int \int_A F_t(x^k) dt dP, \int \int_A F_t(x) dt dP \right).$$

Therefore (see [8], Lemma 4.4) $f \in \mathcal{F}^2(F \circ mx)$. Quite similarly, we also get $t \in \mathcal{F}^2(G \circ mx)$ and $h \in \mathcal{F}_q^2(H \circ mx)$. Thus, $\Phi(f, g, h) \in \mathcal{H}(x)$, which implies $y \in \mathcal{H}(x)$. On the other hand we also have $y \in C$, because C is weakly closed. Therefore, $x \in \mathcal{H}^-(C)$. Now the result follows immediately from Eberlein and Šmulian's theorem. $\square$

Lemma 3: Assume F, G and H take on convex values, satisfy $(\mathcal{A}_1)$ with $p = 2$ and $(\mathcal{A}_3)$. Then a set-valued mapping $\mathcal{H}$ is l.s.c. as a multifunction defined on a locally convex topological Hausdorff space $(D, \sigma(D, D^*))$ with nonempty values in $(D, \sigma(D, D^*))$.

Proof: Let C be a nonempty weakly closed subset of D and (x^n) a sequence of $\mathcal{H}_-(C)$ weakly converging to $x \in D$. Select arbitrarily $y \in \mathcal{H}(x)$ and suppose $(f, g, h) \in \mathcal{F}^2(F \circ mx) \times \mathcal{F}^2(G \circ mx) \times \mathcal{F}_q^2(H \circ mx)$ is such that $y = \Phi(f, g, h)$. Let $(f^n, g^n, h^n) \in \mathcal{F}^2(F \circ mx^n) \times \mathcal{F}^2(G \circ mx^n) \times \mathcal{F}_q^2(H \circ mx^n)$ be such that

$$|f_t(\omega) - f_t^n(\omega)| = \text{dist}(f_t(\omega), (F \circ mx^n)_t(\omega)),$$

$$|g_t(\omega) - g_t^n(\omega)| = \text{dist}(g_t(\omega), (G \circ mx^n)_t(\omega)) \quad \text{and}$$

$|h_{t,z}(\omega) - g_{t,z}^n(\omega)| = \text{dist}(h_{t,z}(\omega), (H \circ mx^n)_{t,z}(\omega))$ on $\mathbb{R}_+ \times \Omega$ and $\mathbb{R}_+ \times \Omega \times \mathbb{R}^n$, respectively, for each $n = 1, 2, \dots$. By virtue of $(\mathcal{A}_3)$ one gets $|f_t(\omega) - f_t^n(\omega)| \rightarrow 0$, $|g_t(\omega) - g_t^n(\omega)| \rightarrow 0$ and $|h_{t,z}(\omega) - h_{t,z}^n(\omega)| \rightarrow 0$ a.e., as $n \rightarrow \infty$. Hence, by $(\mathcal{A}_1)$ we can easily see that a sequence (y_n) , defined by $y^n = \Phi(f^n, g^n, h^n)$, weakly converges to y . But $y^n \in \mathcal{H}(x^n) \subset C$ for $n = 1, 2, \dots$ and C is weakly closed. Then $y \in C$ which implies $\mathcal{H}(x) \subset C$. Thus $x \in \mathcal{H}_-(C)$. $\square$

Lemma 4. Suppose F, G and H satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_4)$ or $(\mathcal{A}'_4)$. Then $H(\mathcal{H}(x), \mathcal{H}(y)) \leq L \|x - y\|_\ell$ or $H(\mathcal{H}(x), \mathcal{H}(y)) \leq L' \|x - y\|_\ell$, respectively, for every $x, y \in D$, where H is the Hausdorff metric induced by the norm $\|\cdot\|_\ell$, $L = \left\| \int_0^\infty k_t dt \right\|_{L_1^2} + 2 \left\| \int_0^\infty \ell_t dt \right\|_{L_1^2} + 2 \left\| \int_0^\infty \int_{\mathbb{R}^n} m_{\tau,z} d\tau q(dz) \right\|_{L_1^2}$ and $L' = \|k\|_1 + 2\|\ell\|_2 + 2\|m\|_2$.

Proof: Let $x, y \in D$ be given and let $u \in \mathcal{H}(x)$. For every $\epsilon > 0$, there is $(f^\epsilon, g^\epsilon, h^\epsilon) \in \mathcal{F}^2(F \circ mx) \times \mathcal{F}^2(G \circ mx) \times \mathcal{F}_q^2(H \circ mx)$ such that $\|u - \Phi(f^\epsilon, g^\epsilon, h^\epsilon)\|_\ell < \epsilon$. Select now $(\tilde{f}^\epsilon, \tilde{g}^\epsilon, \tilde{h}^\epsilon) \in \mathcal{F}^2(F \circ my) \times \mathcal{F}^2(G \circ my) \times \mathcal{F}_q^2(H \circ my)$ such that

$|f_t^\epsilon(\omega) - \tilde{f}_t^\epsilon(\omega)| = \text{dist}(f_t^\epsilon(\omega), (F \circ my)_t(\omega)),$
 $|g_t^\epsilon(\omega) - \tilde{g}_t^\epsilon(\omega)| = \text{dist}(g_t^\epsilon(\omega), (G \circ my)_t(\omega))$ and
 $|h_{t,z}^\epsilon(\omega) - \tilde{h}_{t,z}^\epsilon(\omega)| = \text{dist}(h_{t,z}^\epsilon(\omega), (H \circ my)_{t,z}(\omega))$ on $\mathbb{R}_+ \times \Omega$ and $\mathbb{R}_+ \times \Omega \times \mathbb{R}^n$,
 respectively. Now, by $(\mathcal{A}_4)$ it follows

$$\begin{aligned}
 & E \left[\sup_{t \geq 0} \left| \int_0^t (f_\tau^\epsilon - \tilde{f}_\tau^\epsilon) d\tau \right| \right]^2 \leq E \left[\int_0^\infty |f_\tau^\epsilon - \tilde{f}_\tau^\epsilon| d\tau \right]^2 \\
 & \leq E \left[\int_0^t \bar{h}((F \circ mx)_\tau, (F \circ my)_\tau) d\tau \right]^2 \leq \left(E \int_0^t k_\tau |x_\tau - y_\tau| d\tau \right)^2 \\
 & \leq \left[E \left(\sup_{t \geq 0} |x_t - y_t| \cdot \int_0^\infty k_\tau d\tau \right) \right]^2 \leq E \left(\int_0^\infty k_\tau d\tau \right)^2 \cdot \|x - y\|_\ell^2.
 \end{aligned}$$

Similarly, by Doob's inequality, we obtain

$$\begin{aligned}
 & E \left[\sup_{t \geq 0} \left| \int_0^t (g_\tau^\epsilon - \tilde{g}_\tau^\epsilon) dw_\tau \right| \right]^2 \leq 4E \int_0^\infty |g_\tau^\epsilon - \tilde{g}_\tau^\epsilon|^2 d\tau \\
 & \leq 4E \int_0^\infty [\bar{h}((G \circ mx)_\tau, (G \circ my)_\tau)]^2 d\tau \leq 4 \left(E \int_0^\infty \ell_\tau |x_\tau - y_\tau| d\tau \right)^2 \\
 & \leq 4 \left[E \left(\sup_{t \geq 0} |x_t - y_t| \cdot \int_0^\infty \ell_\tau d\tau \right) \right]^2 \leq 4E \left(\int_0^\infty \ell_\tau d\tau \right)^2 \|x - y\|_\ell^2.
 \end{aligned}$$

Quite similarly, we also get

$$\begin{aligned}
 & E \left[\sup_{t \geq 0} \left| \int_0^t \int_{\mathbb{R}^n} h_\tau^\epsilon - \tilde{h}_{\tau,z}^\epsilon \tilde{\nu}(d\tau, dz) \right| \right]^2 \\
 & \leq 4E \left(\int_0^\infty \int_{\mathbb{R}^n} m_{\tau,z} d\tau q(dz) \right)^2 \cdot \|x - y\|_\ell^2.
 \end{aligned}$$

Therefore

$$\|u - \Phi(\tilde{f}^\epsilon, \tilde{g}^\epsilon, \tilde{h}^\epsilon)\|_\ell$$

$$\leq \|u - \Phi(f^\epsilon, g^\epsilon, h^\epsilon)\|_\ell + \|\Phi(f^\epsilon, g^\epsilon, h^\epsilon) - \Phi(\tilde{f}^\epsilon, \tilde{g}^\epsilon, \tilde{h}^\epsilon)\|_\ell \leq \epsilon + L \|x - y\|_\ell,$$

where L is such as above. This implies $\bar{H}(\mathcal{H}(x), \mathcal{H}(y)) \leq L \|x - y\|_\ell$. Quite similarly we also get $\bar{H}(\mathcal{H}(y), \mathcal{H}(x)) \leq L \|x - y\|_\ell$. Therefore $H(\mathcal{H}(x), \mathcal{H}(y)) \leq L \|x - y\|_\ell$. Using conditions $(\mathcal{A}'_4)$ instead of $(\mathcal{A}_4)$ we also get $H(\mathcal{H}(y), \mathcal{H}(x)) \leq L' \|x - y\|_\ell$. $\square$

Lemma 5: *Suppose F, G and H satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_4)$ or $(\mathcal{A}'_4)$. Then for every $0 \leq \alpha < \beta < \infty$ one has $H(\mathcal{H}^{\alpha, \beta}(x), \mathcal{H}^{\alpha, \beta}(y)) \leq L_{\alpha\beta} \|x - y\|_\ell$ or $H(\mathcal{H}^{\alpha, \beta}(x), \mathcal{H}^{\alpha, \beta}(y)) \leq L'_{\alpha\beta} \|x - y\|_\ell$, respectively, for every $x, y \in D^{\alpha, \beta}$, where H is a Hausdorff metric induced by the norm $\|\cdot\|_\ell$, $L_{\alpha, \beta} = \left\| \int_0^\infty \mathbb{I}_{[\alpha, \beta]}(t) k_t dt \right\|_{L_1^2} + 2 \left\| \int_0^\infty \mathbb{I}_{[\alpha, \beta]}(t) \ell_t dt \right\|_{L_1^2} + 2 \left\| \int_0^\infty \int_{\mathbb{R}^n} \mathbb{I}_{[\alpha, \beta]}(t) m_{t, z} dq(dz) \right\|_{L_1^2}$ and $L'_{\alpha, \beta} = \|\mathbb{I}_{[\alpha, \beta]} k\|_1 + 2 \|\mathbb{I}_{[\alpha, \beta]} \ell\|_2 + 2 \|\mathbb{I}_{[\alpha, \beta]} m\|_2$.*

Proof: The proof follows immediately from Lemma 4 applied to $F^{\alpha\beta} = \mathbb{I}_{[\alpha, \beta]} F$, $G^{\alpha\beta} = \mathbb{I}_{[\alpha, \beta]} G$ and $H^{\alpha\beta} = \mathbb{I}_{[\alpha, \beta]} H$. $\square$

Immediately from Lemma 2 and the Kakutani and Fan fixed point theorem the following result follows.

Lemma 6: *If F, G and H take on convex values and satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, then $S(F, G, H) \neq \emptyset$.*

Proof: Let $\mathfrak{B} = \{(f, g, h) \in \mathcal{L}_n^2 \times \mathcal{L}_n^2 \times \mathcal{W}_n^2 : |f_t(\omega)| \leq \|F_t(\omega)\|, |g_t(\omega)| \leq \|G_t(\omega)\|, |h_{t, z}(\omega)| \leq \|H_{t, z}(\omega)\| \text{ and put } \mathfrak{K} = \Phi(\mathfrak{B})\}$. It is clear that $\mathfrak{K}$ is a nonempty convex weakly compact subset of D such that $\mathcal{H}(x) \subset \mathfrak{K}$ for $x \in D$. By (ii) of Proposition 1, $\mathcal{H}(x)$ is a convex and weakly compact subset of D , for each $x \in D$. By Lemma 2, $\mathfrak{K}$ is u.s.c. on a locally convex topological Hausdorff space $(D, \sigma(D, D^*))$. Therefore, by the Kakutani and Fan fixed point theorem, we get $S(F, G, H) \neq \emptyset$. $\square$

Lemma 7. *If F, G and H take on convex values and satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_3)$, then $S(F, G, H) \neq \emptyset$.*

Proof. Let $\mathfrak{K}$ be as in Lemma 6. By virtue of Lemma 3, $\mathfrak{K}$ is l.s.c. as a set-valued mapping from a paracompact space $\mathfrak{K}$ considered with its relative topology induced by a weak topology $\sigma(D, D^*)$ on D into a Banach space $(D, \|\cdot\|_\ell)$. By (ii) of Proposition 1, $\mathcal{H}(x)$ is a closed and convex subset of D , for

each $x \in \mathcal{K}$. Therefore, by Michael's theorem, there is a continuous selection $f: \mathcal{K} \rightarrow D$ for $\mathcal{K}$. But $\mathcal{K}(\mathcal{K}) \subset \mathcal{K}$. Then f maps $\mathcal{K}$ into itself and is continuous with respect to the relative topology on $\mathcal{K}$, defined above. Therefore, by the Schauder and Tikhonov fixed point theorem, there is $x \in \mathcal{K}$ such that $x = f(x) \in \mathcal{K}(x)$. $\square$

Lemma 8. *If F, G and H satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_4)$ or $(\mathcal{A}'_4)$ then $S(F, G, H) \neq \emptyset$.*

Proof. Let $(\tau_n)_{n=1}^\infty$ be a sequence of positive numbers increasing to $+\infty$. Select a positive number σ such that $L_{k\sigma, (k+1)\sigma} < 1$ or $L'_{k\sigma, (k+1)\sigma} < 1$, respectively, for $k = 0, 1, \dots$, where $L_{k\sigma, (k+1)\sigma}$ and $L'_{k\sigma, (k+1)\sigma}$ are as in Lemma 5. Suppose a positive integer n_1 is such that $n_1\sigma < \tau_1 \leq (n_1 + 1)\sigma$. By virtue of Lemma 5, $\mathcal{H}^{k\sigma, (k+1)\sigma}$ is a set-valued contraction for every $k = 0, 1, \dots$. Therefore, by the Covitz and Nadler fixed point theorem, there is $z^1 \in S^{0, \sigma}(F, G, H)$. By the same argument, there is $z^2 \in z^1_\sigma + S^{\sigma, 2\sigma}(F, G, H)$, because $z^1_\sigma + \mathcal{H}^{\sigma, 2\sigma}$ is again a set-valued contraction mapping. Continuing the above procedure we can finally find a $z^{n_1+1} \in z^{n_1\sigma} + S^{n_1\sigma, \tau_1}(F, G, H)$. Put

$$\begin{aligned} x^1 = & \sum_{k=0}^{n_1-1} \mathbb{I}_{[k\sigma, (k+1)\sigma)}(z^{k+1} - z^k_{k\sigma}) \\ & + \mathbb{I}_{[n_1\sigma, \tau_1)}(z^{n_1+1} - z^{n_1\sigma}_{n_1\sigma}) + \mathbb{I}_{(\tau_1, \infty)}(z^{n_1+1}_{\tau_1} - z^{n_1\sigma}_{n_1\sigma}), \end{aligned}$$

where $z^0_0 = 0$. Similarly, as in the proof of Proposition 6, we can easily verify that $x^1 \in S^{0, \tau_1}(F, G, H)$. Repeating the above procedure to the interval $[\tau_1, \tau_2]$, we can find $x^2 \in x^1_{\tau_1} + S^{\tau_1, \tau_2}(F, G, H)$. Continuing this process we can define a sequence (x^n) of D satisfying the conditions of Proposition 6. Therefore $S(f, G, H) \neq \emptyset$. $\square$

Now as a corollary of Proposition 4 and Lemmas 6-8, the following results follow.

Theorem 1. *Suppose F, G and H take on convex values, satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ or $(\mathcal{A}_3)$. Then $\Lambda_0(F, G, H) \neq \emptyset$.*

Theorem 2. *Suppose F, G and H satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_4)$ or $(\mathcal{A}'_4)$ and*

take on convex values. Then $\Lambda_0(F, G, H) \neq \emptyset$.

From the stochastic optimal control theory point of view (see [6]), it is important to know whether the set $\Lambda_0(F, G, H)$ is at least weakly compact in $(D, \|\cdot\|_\rho)$. We have the following result dealing with this topic.

Theorem 3. *Suppose F, G and H take on convex values and satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$. Then $\Lambda_0(F, G, H)$ is a nonempty weakly compact subset of $(D, \|\cdot\|_\rho)$.*

Proof. Nonemptiness of $\Lambda_0(F, G, H)$ follows immediately from Theorem 1. By virtue of Proposition 4 and the Eberlein and Šmulian theorem for the weak compactness of $\Lambda_0(F, G, H)$, it suffices only to verify that $S(F, G, H)$ is sequentially weakly compact. But $S(F, G, H) \subset \Phi(\mathfrak{B})$, where $\mathfrak{B}$ is a weakly compact subset of $\mathcal{L}_n^2 \times \mathcal{L}_n^2 \times \mathcal{W}_n^2$ defined in Lemma 6. Hence, by the properties of the linear mapping Φ , the relative sequential weak compactness of $S(F, G, H)$ follows. Suppose (x^n) is a sequence of $S(F, G, H)$ weakly converging to $x \in \Phi(\mathfrak{B})$, and let $(f^n, g^n, h^n) \in \mathcal{Y}^2(F \circ mx^n) \times \mathcal{Y}^2(G \circ mx^n) \times \mathcal{Y}_q^2(H \circ mx^n)$ be such that $x^n = \Phi(f^n, g^n, h^n)$, for $n = 1, 2, \dots$. By the weak compactness of $\mathfrak{B}$, there is a subsequence, denoted again by $\{(f^n, g^n, h^n)\}$, of $\{(f^n, g^n, h^n)\}$ weakly converging to $(f, g, h) \in \mathfrak{B}$. Similarly, as in the proof of Lemma 2, we can verify that $(f, g, h) \in \mathcal{Y}^2(F \circ mx) \times \mathcal{Y}^2(G \circ mx) \times \mathcal{Y}_q^2(H \circ mx)$. This and the weak convergence of $\{\Phi(f^n, g^n, h^n)\}$ to $\Phi(f, g, h)$ imply that $x = \Phi(f, g, h) \in \mathfrak{H}(x)$. Thus $x \in S(F, G, H) \square$

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