Journal of Applied Mathematics and Stochastic Analysis
7, Number 4, 1994, 509-523

STABILITY AND ASYMPTOTIC STABILITY IN IMPULSIVE SEMIDYNAMICAL SYSTEMS

SAROOP K. KAUL
University of Regina
Department of Mathematics and Statistics
Regina, Sask. CANADA S4S OA2

(Received October, 1993; revised March, 1994)

ABSTRACT

In this paper we generalize two results of Lasalle's, the invariance theorem and asymptotic stability theorem of discrete and continuous semidynamical systems, to impulsive semidynamical systems.

Key words: Stability, Asymptotic Stability, Lyapunov Functions, Impulsive Semidynamical System, Dynamical System.

AMS (MOS) subject classifications: 54H20, 54H15.

1. Introduction

Let (X,π,R) be a dynamical system, where $X=(X,d)$ is a metric space with metric d, and

$$\pi: X\times R\rightarrow X$$

is a continuous function satisfying:

(i) $\pi(x,0)=x$ for all $x\in X$ and $0\in R$, and
(ii) $\pi(\pi(x,t),x)=\pi(x,s+t)$ for all $x\in X$ and $t,s\in R$.

Then the function

$$\pi_t: X\rightarrow X$$

defined, for any fixed $t\in R$, by $\pi_t(x)=\pi(x,t)$, $x\in X$, is a homeomorphism of X onto itself.

The function

$$\pi_x: R\rightarrow X$$

defined for any fixed $x\in X$, by $\pi_x(t)=\pi(x,t)$, $x\in R$, is continuous and is called the trajectory of x.

Let $\Omega\subset X$ be an open set in X, and denote the boundary of Ω in X by M. Assume that $M\neq\phi$. Let $I: M\rightarrow\Omega$ be a continuous function and $I(M)=N$. We shall denote $I(x)$ by x^+ and say that x jumps to x^+.

Given $x\in\Omega$ we now define an "impulsive trajectory", $\widetilde{\pi}_x$, for x, over a subset of the nonnegative reals R^+ and also, simultaneously, a function Φ: $\Omega\rightarrow R^+\cup\{\infty\}=R^*$, where R^* is the space of extended positive reals.

If $\pi(x,t) \notin M$ for any $t \in R^+$, we set $\tilde{\pi}_x(t) = \pi(x,t)$ for all $t \in R^+$, and $\Phi(x) = \infty$. Otherwise, let

$$\Phi(x) = t_1,$$

where $t_1 \in R^+$ is the smallest number for which $x_1 = \pi(x, t_1) \in M$, and set

$$\tilde{\pi}_x(t) = \begin{cases} \pi(x,t), & 0 \leq t < t_1 \\ x_1^+, & t = t_1. \end{cases}$$

Since $t_1 < \infty$, we continue the process, now starting with x_1^+, repeating the above two steps. Thus if $\pi(x_1^+, t) \notin M$ for any $t \in R^+$, we set,

$$\tilde{\pi}_x(t) = \pi(x_1^+, t - t_1) \text{ for } t \geq t_1$$

and

$$\Phi(x_1^+) = \infty,$$

otherwise

$$\tilde{\pi}_x(t) = \begin{cases} \pi(x_1^+, t - t_1), & t_1 \leq t < t_2, \\ x_2^+, & t = t_2, \end{cases}$$

for some $t_2, x_2 = \pi(x_1^+, t_2) \in M$ and $\pi(x_1^+, t) \notin M$ for any $t, t_1 \leq t < t_2$.

Continuing inductively, it is clear that an impulsive trajectory $\tilde{\pi}_x$ has either no jumps, only a finite number of jumps at points $x_1, \ldots, x_n$, the jump at x_k occurring at $t = \sum_{i=1}^{k} t_i$ and for $t \geq \sum_{i=1}^{n} t_i$, $\tilde{\pi}_x(t) = \pi(x,t)$, or lastly, it has an infinite number of jumps at $\{x_n\}_{n=1}^{\infty}$ and $\tilde{\pi}_x$ is defined over the interval $[0, T(x))$, where $T(x) = \sum_{i=1}^{\infty} t_i$. Note that in the other two cases above $T(x) = \infty$. We now define $\tilde{\pi}(x,t) = \tilde{\pi}_x(t)$ for any t in $[0, T(x))$. We shall often use the fact that for any $t \in R$, such that $0 \leq t < T(x)$, there exists a $k = 0, 1, 2, \ldots$, such that, $t = \sum_{i=0}^{k} \Phi(x_i^+) + t'$, where $x_0^+ = x$, and consequently $\tilde{\pi}(x,t) = \pi(x_k^+, t')$, $0 \leq t' < \Phi(x_k^+)$.

Furthermore, the following two properties are easy to check:

(*i*) $\tilde{\pi}(x, 0) = x$ for any $x \in \Omega$ and

(*ii*) $\tilde{\pi}(\tilde{\pi}(x,t), x) = \tilde{\pi}(x, t+s)$, for $x \in \Omega$ and t and $s \in [0, T(x))$, such that $t + s \in [0, T(x))$.

Thus from here on we assume that Ω, M, I and N are as considered above in a dynamical system (X, π), and call $(\Omega, \tilde{\pi})$, with $\tilde{\pi}$ as defined above, an impulsive semidynamical system associated with (X, π) (see [6]).

Let $\Omega_0 = \{x \in \Omega : \Phi(x) < \infty\}$. Then there exists a function $g: \Omega_0 \to N$, defined by $g(x) = \tilde{\pi}(\pi, \Phi(x))$ for any $x \in \Omega_0$.

Assumption I: We assume throughout the paper that Φ is a continuous function on Ω.

It follows then that for any $x \in \Omega_0$, $g(x) = I(\pi(x, \Phi(x)))$ is a continuous function on Ω_0.

Let $\widehat{N} = \{x \in N: \Phi(x_n^+) < \infty$ for $n = 0, 1, 2, \ldots\}$. Then g maps $\widehat{N}$ into $\widehat{N}$. Consequently, $(\widehat{N}, g)$ defines a discrete semidynamical system [5] associated with the impulsive semidynamical system $(\Omega, \widetilde{\pi})$.

In this paper, as in [3] and [4], we continue to study dynamical notions defined in $(\Omega, \widetilde{\pi})$ by relating them to similar notions in $(\widehat{N}, g)$. In particular, we study stability and asymptoticstability in $(\Omega, \widetilde{\pi})$ by relating them to the corresponding notions in $(\widehat{N}, g)$. The main results obtained are (1) characterization of stability (Theorems 3.6 and 3.11) similar to Theorem 1.12 [1, p. 61], and (2) asymptotic stability (Theorems 5.4 and 5.6) in terms of Lyapunov functions which generalizes a similar theorem of Lasalle [6, Theorem 7.9, p. 9] and also Theorem 2.2 [1, p. 66].

The following notations are used throughout. For any $A \subset \Omega$ and $\epsilon > 0$, $U(A, \epsilon) = \{x \in \Omega: d(x, A) < \epsilon\}$, where $d(x, A) = inf\{d(x, y): y \in A\}$, cl denotes the closure operator in Ω; $\widehat{A} = A \cap \widehat{N}$, and for any $x \in \Omega$, $C_0(x) = \{\pi(x, t) = \widetilde{\pi}(x, t): 0 \leq t < \Phi(x)\}$.

2. Limit Sets, Prolongation Limit Sets and Prolongation Sets

Definition 2.1: We extend the known definitions of these sets in a natural way to $(\Omega, \widetilde{\pi})$. Let $x \in \Omega$.

(2.1.1) The limit set of x in $(\Omega, \widetilde{\pi})$ is defined by

$$\widetilde{L}(x) = \{y \in \Omega: \widetilde{\pi}(x, t_n) \rightarrow y \text{ for some } t_n \rightarrow T(x) \text{ and } t_n < T(x)\}.$$

(2.1.2) The prolongation limit set of x in $(\Omega, \widetilde{x})$ is defined by

$$\widetilde{J}(x) = \{y \in \Omega: \widetilde{\pi}(x_n, t_n) \rightarrow y \text{ for } x_n \rightarrow x \text{ and } t_n \rightarrow T(x),\ t_n < T(x)\}.$$

(2.1.3) The prolongation set of x in $(\Omega, \widetilde{\pi})$ is defined by

$$\widetilde{D}(x) = \{y \in \Omega: \widetilde{\pi}(x_n, t_n) \rightarrow y \text{ for } x_n \rightarrow x \text{ and } t_n \in [0, T(x))\}.$$

The definition of these sets in a discrete semidynamical system are well known. In $(\widehat{N}, g)$ we shall denote them by $\widehat{L}(x)$, $\widehat{J}(x)$ and $\widehat{D}(x)$ respectively for $x \in \widehat{N}$.

Furthermore, we denote the orbit of a point $x \in \Omega$ in $(\Omega, \widetilde{\pi})$ by $\widetilde{C}(x) = \{\widetilde{\pi}(x, t): t \in [0, T(x))\}$ and its closure in Ω by $\widetilde{K}(x)$. These sets in $(\widehat{N}, g)$ are denoted by $\widehat{C}(x)$ and $\widehat{K}(x)$ for $x \in \widehat{N}$.

Generally one uses a + sign with the above symbols for these sets for flows in positive direction, but as we are dealing with flow only in positive direction we have dispensed with it for convenience.

Definition 2.2: A subset A of Ω is said to be positively invariant if for any $x \in A$, $\widetilde{C}(x) \subset A$, and it is said to be invariant if, it is positively invariant and, furthermore, given $x \in A$ and $t \in [0, T(x))$ there exists a $y \in A$ such that $\widetilde{\pi}(y, t) = x$.

A subset $\widehat{A}$ of $\widehat{N}$ is said to be positively invariant if $g(\widehat{A}) \subset \widehat{A}$ and is said to be invariant if $\widehat{A} = g(\widehat{A})$.

It is clear that $\widetilde{\pi}$ is not a continuous function. However, the following lemma holds.

Lemma 2.3: *Suppose $\{x_n\}$ is a sequence in Ω, convergent to a point $y \in \Omega$. Then for any $t \in [0, T(y))$, there exists a sequence of real numbers, $\{\epsilon_n\}$, $\epsilon_n \to 0$, such that, $t + \epsilon_n < T(x_n)$ and $\tilde{\pi}(x_n, t + \epsilon_n) \to |\tilde{\pi}(y,t)|$. In particular, if $\tilde{\pi}(y,t) = y_1^+$, then $\tilde{\pi}(x_n, t+\epsilon_n)$ can be chosen to be $x_{n,1}^+$.*

Proof: First suppose that $\Phi(y) < \infty$. Then clearly, we may assume, since Φ is continuous, that $\Phi(x_n) < \infty$.

Case 1: If $0 \le t < \Phi(y)$, let $\epsilon < \Phi(y) - t$. Then from the continuity of Φ we may conclude that $\Phi(y) - \epsilon < \Phi(x_n)$ for all n, so that $t < \Phi(x_n)$ and $\tilde{\pi}(x_n, t) = \pi(x_n, t)$, and the continuity of π implies the result for $\epsilon_n = 0$.

Case 2: If $t = \Phi(y)$, then $\tilde{\pi}(y,t) = y_1^+$. If $\Phi(x_n) > \Phi(y)$, $\tilde{\pi}(x_n, \Phi(x_n)) = \tilde{\pi}(x_n, t + \epsilon_n) = x_{n,1}^+ \to y_1^+$ because $x_{n,1} = \pi(x_n, \Phi(x_n) \to y_1$, and I is continuous.

Case 3: Now suppose $t > \Phi(y)$, so that $t = \sum_{i=1}^{m-1} \Phi(y_i^+) + t'$, $0 \le t' < \Phi(y_m^+)$, where $y_{i+1}^+ = \tilde{\pi}(y_i^+, \Phi(y_i^+))$, $i = 0, \ldots, m-1$. Define $\{x_{n,i}^+\}$, inductively by $x_{n,i+1}^+ = \tilde{\pi}(x_{n,i}^+, \Phi(x_{n,i}^+))$. Clearly, if $t_n = \sum_{i=1}^{m-1} \Phi(x_{n,i})$, then $\tilde{\pi}(x_n, t) \to y_m^+$. Now set $\epsilon_n = t_n + t' - t$. Then

$$\tilde{\pi}(x_n, t + \epsilon_n) \to \tilde{\pi}(y_n^+, t') = \tilde{\pi}(y,t)$$

completes the proof.

Now suppose $\Phi(x) = \infty$. Then continuity of Φ implies that for any $t \in [0, \infty)$ there exists an n_0 such that for $n \ge n_0$, $\Phi(x_n) > t$. Consequently, for $n \ge n_0$, $\tilde{\pi}(x_n, t) = \pi(x_n, t)$ and the result follows from the continuity of π.

Lemma 2.4: *Suppose $A \subset \Omega$ is positively invariant in $(\Omega, \tilde{\pi})$, then clA is positively invariant in $(\Omega, \tilde{\pi})$.*

Proof: Suppose $x \in clA$ and $t \in [0, T(x))$. Then there exists a sequence $\{x_n\}$ in A with $x_n \to x$. Hence by Lemma 2.3, $\tilde{\pi}(x_n, t + \epsilon_n) \to \tilde{\pi}(x,t)$ for some sequence $\epsilon_n \to 0$ such that $t + \epsilon_n \in [0, T(x_n))$. Since A is invariant, $\tilde{\pi}(x_n, t + \epsilon_n) \in A$, hence $\tilde{\pi}(x,t) \in clA$. This completes the proof.

Lemma 2.5: *Suppose $p \in \hat{N}$, and $\hat{K}(p)$ is compact, then $T = \sum_{n=0}^{\infty} \Phi(p_n^+)$ is infinite, where $p = p_0^+$.*

Proof: Suppose T is finite. Then $\Phi(p_n^+) \to 0$. Assuming that $p_{n_k}^+ \to q \in \Omega$, $\pi(p_{n_k}^+, \Phi(p_{n_k}^+)) \to \pi(q, 0) = q$. But, $\pi(p_{n_k}, \Phi(p_{n_k})) \in M$, which is a closed subset of X. Hence $q \in M$ implying $M \cap \Omega \ne \Phi$. But this is a contradiction since Ω is open in X and has no point in common with its boundary M in X.

Assumption II: From now on we assume that if $x \in \hat{N}$, then $T(x) = \infty$; so that if for some $x \in \Omega$, $\tilde{\pi}(x, \Phi(x)) \in \hat{N}$, then $T(x)$ is also infinite. Note that if $\tilde{\pi}(x, \Phi(x)) \notin \hat{N}$ then $T(x) = \infty$ anyway. Thus from here on $T(x)$ will be assumed to be infinity for all $x \in \Omega$, and $[0, T(x))$ will be replaced by R^+.

Lemma 2.6: *For any $x \in \Omega$, $\tilde{L}(x)$ is closed and positively invariant in $(\Omega, \tilde{\pi})$.*

Proof: To show that it is closed is trivial. To prove that $\tilde{L}(x)$ is positively invariant, let $y \in \tilde{L}(x)$. Then there exists a sequence $\{t_n\}$ in R^+, $t_n \to \infty$, such that $\tilde{\pi}(x, t_n) = x_n \to y$. Now, given $t \in R^+$, there exists a sequence $\{\epsilon_n\}$ as in Lemma 2.3, such that,

$$\tilde{\pi}(x_n, t + \epsilon_n) \to \tilde{\pi}(y, t).$$

Consequently,

$$\tilde{\pi}\,[\tilde{\pi}\,(x,t_n),t+\epsilon_n] = \tilde{\pi}\,(x,t_n+t+\epsilon_n) \to \tilde{\pi}\,(y,t)$$

and since $t_n + t + \epsilon_n \to \infty$, $\tilde{\pi}\,(y,t) \in \tilde{L}\,(x)$.

Lemma 2.7: *For any $x \in \Omega$, and $t \in R^+$, if $z = \tilde{\pi}\,(x,t)$ then $\tilde{L}\,(x) \subseteq \tilde{L}\,(z)$.*

Proof: Let $y \in \tilde{L}\,(x)$, and $\tilde{\pi}\,(x,t_n) \to y$ for $t_n \to \infty$. Since $t_n \geq t$ for all $n \geq n_0$ for some n_0, we have $\tilde{\pi}\,(x,t_n) = \tilde{\pi}\,(\tilde{\pi}\,(x,t),t_n - t) \to y$ completes the proof since $t_n - t \to \infty$.

Lemma 2.8: *For any $x \in \Omega$, $\tilde{J}\,(x)$ is closed and positively invariant in $(\Omega, \tilde{\pi}\,)$.*

Proof: It is easy to see that $\tilde{J}\,(x)$ is closed. To prove positive invariance, let $y \in \tilde{J}\,(x)$ and $\tilde{\pi}\,(x_n,t_n) \to y$ for $x_n \to x$ and $t_n \to \infty$. Given any $t \in R^+$, $\tilde{\pi}\,(\tilde{\pi}\,(x_n,t_n),t+\epsilon_n) \to \tilde{\pi}\,(y,t)$ as in Lemma 2.3. Hence, $\tilde{\pi}\,(x_n,t_n+t+\epsilon_n) \to \tilde{\pi}\,(y,t) \in \tilde{J}\,(x)$.

Lemma 2.9: *For any $x \in \Omega$ and any $t \in R^+$, if $\tilde{\pi}\,(x,t) = z$, then $\tilde{J}\,(x) \subset \tilde{J}\,(x)$.*

Proof: Similar to the proof of Lemma 2.7.

Lemma 2.10: *For any $x \in \Omega$, $\tilde{K}(x) = \tilde{C}\,(x) \cup \tilde{L}\,(x)$. Consequently $\tilde{K}(x)$ is positively invariant in $(\Omega, \tilde{\pi}\,)$.*

Proof: Trivial.

Corollary 2.10: *For any $A \subset \Omega$, $\tilde{K}(A) = \cup\{\tilde{K}(x): x \in A\}$ is positively invariant.*

Lemma 2.11: *For any $x \in \Omega, \tilde{D}(x) = \tilde{K}(x) \cup \tilde{J}\,(x)$ is closed and positively invariant.*

Proof: Both $\tilde{K}(x)$ and $\tilde{J}\,(x)$ are positively invariant and closed as shown above. To prove equality is straight forward. Hence the result.

Lemma 2.12: *Let $\hat{A} \subset \hat{N}$ be a compact and positively invariant set in $(\hat{N},g)$. Then the closure of $\tilde{K}(\hat{A})$ in X is compact. Furthermore, $\tilde{K}(\hat{A}) = \tilde{C}\,(\hat{A})$ is closed and positively invariant in $(\Omega, \tilde{\pi}\,)$.*

Proof: Let $p \in \hat{A}$. Then by positive invariance of $\hat{A}$ under g, it is clear that $p_n^+ \in \hat{A}$ for all n. Hence $\tilde{C}\,(p) = \cup\{C_0(p_n^+): n = 0,1,2,\ldots\}$, $p_0^+ = p$. Since $\hat{A}$ is compact, $cl\{p_n^+: n = 0,1,2,\ldots\} = Q$ is compact, and clearly $\tilde{K}(p) = \cup\{C_0(q)\colon\ q \in Q\}$. Hence $\tilde{K}(\hat{A}) = \cup\{C_0(p)\colon\ p \in \hat{A}\}$. Let H be the closure of $\hat{k}(\hat{A})$ in X. Let $\{y_n\}$ be a sequence in H. Then there exists a sequence $\{y'_n\}$ in $\hat{k}(\hat{A})$, such that, $d(y_n,y'_n) < \frac{1}{n}$. Consequently, there exists a $x_n \in \hat{A}$, such that $y'_n \in C_0(x_n)$, that is, $y'_n = \pi(x_n,t_n)$, $0 \leq t_n < \Phi(x_n)$. Assuming that $x_n \to x \in \hat{A}$, clearly $\pi(x_n,t_n) \to \pi(x,t) = y$, where $0 \leq t \leq \Phi(x)$, since both π and Φ are continuous. Consequently, $y_n \to y$, and $y \in H$. Thus H is compact.

Note that $\hat{k}(\hat{A}) = \tilde{C}\,(\hat{A})$ as shown above. Hence if for any sequence $\{y_n\}$ in $\tilde{C}\,(\hat{A})$, $y_n \to y \in \Omega$, then, as in the above proof, $0 \leq t < \Phi(x)$ and $y \in C_0(x) \subset \tilde{C}\,(\hat{A})$. Therefore $\tilde{C}\,(\hat{A})$ is closed. That it is positively invariant follows form the fact that $\hat{A}$ is positively invariant in $\hat{N}$.

Lemma 2.13: *Let $x \in \Omega$ and $\tilde{\pi}\,(x,\Phi(x)) = p \in \hat{N}$. If $\hat{C}(p)$ has compact closure in $\hat{N}$, then $\tilde{L}\,(x) = C_0(\hat{L}(p))$.*

Proof: Let $y \in \tilde{L}\,(x)$ and $\tilde{\pi}\,(x,t_k) \to y$ for $t_k \to \infty$. Assume $t_k > \Phi(x)$ for all k. Then $\tilde{\pi}\,(p,t_k - \Phi(x)) = \pi(p_{n_k}^+, t'_k)$, $0 \leq t'_k < \Phi(p_{n_k}^+)$. Since $\hat{C}(p)$ has compact closure in $\hat{N}$, we may assume that $p_{n_k}^+ \to q \in \hat{N}$. Since $t_k \to \infty$, clearly $n_k \to \infty$, and $q \in \hat{L}(p)$. By Lemma 2.12, since $y \in \Omega$, $y = \pi(q,t)$, $0 \leq t < \Phi(q)$. Hence, $y \in C_0(q) \subset C_0(\hat{L}(p))$.

Conversely, suppose $y \in C_0(\hat{L}(p))$. Then $y = \pi(q,t)$, $0 \leq t < \Phi(q)$ for some $q \in \hat{L}(p)$. Now let

$p_{n_k}^+ \to q$ as $n_k \to \infty$, where $p_{n_k}^+ = g^{n_k}(p)$. Then $\widetilde{\pi}\,(p_{n_k}^+, t+\epsilon_n) \to \pi(q,t) = y$ by Lemma 2.3. If $t_k = \sum_{i=0}^{n_k} \Phi(p_i^+) + \Phi(x)$, then $\widetilde{\pi}\,(x, t_k + t + \epsilon_n) \to y$, and $y \in \widetilde{L}\,(x)$. This completes the proof.

Lemma 2.14: *Let* $\widehat{A} \subset \widehat{N}$ *be compact and* $\widehat{A} = g(\widehat{A})$. *Then for any* $t \in R^+$, $C_0(\widehat{A}) \subset \widetilde{\pi}\,(C_0(\widehat{A}), t)$.

Proof: Let $y \in C_0(\widehat{A})$. Then there exists a $q \in \widehat{A}$ such that $y \in C_0(q)$. Since $\widehat{A}$ is compact, and $\Phi(x) > \infty$ for each $x \in \widehat{A}$, $inf\{\Phi(x): x \in \widehat{A}\} = T > 0$. Hence given q, there exists a sequence $q_1, \ldots, q_{m+1} = q$, of points in $\widehat{A}$, such that, $g(q_i) = q_{i+1}$, $i = 1, 2, \ldots, m$ and

$$\sum_{i=1}^{m-1} \Phi(q_i) \leq t < \sum_{i=1}^{m} \Phi(q_i).$$

Set $t' = t - \sum_{i=1}^{m-1} \Phi(q_i)$. Then $0 \leq t' < \Phi(q_m)$. Suppose $y = \pi(q,s)$, $0 \leq s < \Phi(q)$. Then $t_1 = (\Phi(q_m) - t') + s \geq 0$. Let $z = \widetilde{\pi}\,(q_1, t_1)$. Then $z \in C_0(\widehat{A})$, and

$$\widetilde{\pi}\,(z,t) = \widetilde{\pi}\,(\widetilde{\pi}\,(q_1,t_1), \sum_{i=1}^{m-1} \Phi(q_i) + t') = \widetilde{\pi}\,(q_1, t_1 + \sum_{i=1}^{m-1} \Phi(q_i) + t')$$

$$= \widetilde{\pi}\,(q_1, \Phi(q_m) - t' + s + \sum_{i=1}^{m-1} \Phi(q_i) + t')$$

$$= \widetilde{\pi}\,(q_1, \sum_{i=1}^{m} \Phi(q_i) + s) = \widetilde{\pi}\,(q_{m+1}, s) = y.$$

This completes the proof.

Corollary 2.15: Let $x \in \Omega$ and $\widetilde{\pi}\,(x, \Phi(x)) = p \in \widehat{N}$. If $\widehat{C}(p)$ has compact closure in $\widehat{N}$, the $\widetilde{L}\,(x)$ is invariant.

Proof: By Lemma 2.13, $\widetilde{L}\,(x) = C_0(\widehat{L}(p))$. But $\widehat{L}(p)$ is invariant [5, Theorem 5.2, p. 4]. By Lemma 2.6, $\widetilde{L}\,(x)$ is positively invariant, hence Lemma 2.14 implies that $\widetilde{L}\,(x)$ is invariant.

Lastly, we note a trivial result for future reference.

Lemma 2.16: *If* $A \subset \Omega$ *and* $\widetilde{D}(A) = A$ *in* $(\Omega, \widetilde{\pi}\,)$, *then* A *is positively invariant in* $(\Omega, \widetilde{\pi}\,)$.

Proof: By definition for any $x \in A$, $\widetilde{C}\,(x) \subset \widetilde{K}\,(x) \subset \widetilde{D}\,(x) \subset A$. Hence the result.

3. Stability

Definition 3.1: A subset A of Ω is called a *cylindrical* set if for any $x \in A$, $C_0(x) \subset A$, where $C_0(x) = \{\pi(x,t): 0 \leq t < \Phi(x)\}$.

Note: For any $x \in \Omega$, $C_0(x)$ is a cylindrical set.

The following is trivial.

Lemma 3.2: *If* $A \subset \Omega$ *is a cylindrical set, then so is* clA.

Lemma 3.3: *Let* V *be an open set in* Ω, *then* $C_0(V) = \cup\{C_0(x);\ x \in V\} = W$ *is a cylindrical open set in* Ω.

Proof: W is clearly a cylindrical set. Let $z \in W$. Then there is a $y \in V$ such that

$z = \pi(y,t)$, $0 \leq t < \Phi(y)$. Let $0 < \epsilon < \Phi(y) - t$. By continuity of Φ there exists an open set U containing y and $U \subset V$, such that, if $x \in U$, then $|\Phi(x) - \Phi(y)| < \epsilon$. Hence $t < \Phi(y) - \epsilon < \Phi(x)$, and $\pi_t(x) = \pi(x,t) \in C_0(x) \subset W$. Since π_t is a homeomorphism, $y \in \pi_t(U) \subset W$ implies that W is open.

Definition 3.4: A subset A of Ω is said to be stable in $(\Omega, \tilde{\pi})$ if given any cylindrical open set U containing A there exists an open set V, such that $A \subset V \subset U$, and for any $x \in V$, $\tilde{C}(x) \subset U$.

Theorem 3.5: *Let Ω be locally compact. Let A be a cylindrical subset of Ω and for each $x \in A$, $\Phi(x) < \infty$. Then for any $\epsilon > 0$ there exists an open set $U \supset A$, such that, $C_0(U) \subset U(A,\epsilon)$.*

Proof: Let $x \in A$. Then, since π is continuous and Ω is locally compact, there exists a compact neighborhood $U(x)$ of x and a real number $\delta(x) > 0$, such that $\delta(x)$ is less than both ϵ and $\Phi(x)$, and for any $y \in \Omega$ such that $d(x,y) < \delta(x)$, and any t, t' in $[0, \Phi(x) + \epsilon]$, satisfying $|t - t'| < \delta(x)$, we have $d(\pi(x,t), \pi(y,t')) < \epsilon$. And, since Φ is continuous, there exists a $\nu(x)$ such that for any $y \in \Omega$ for which $d(x,y) < \nu(x)$, $|\Phi(x) - \Phi(y)| < \delta(x)/2$ and $U(x, \nu(x)) \subset U(x)$. We claim that for any y, such that, $d(x,y) < \nu(x)$, if $z \in C_0(y)$ then $d(z, C_0(x)) < \epsilon$. Let $z = \pi(y,t)$, $0 \leq t < \Phi(y)$. Since $\Phi(y) < \Phi(x) + \delta(x)/2$, $t - \delta(x)/2 = t' < \Phi(x)$, and $|t' - t| < \delta(x)$. Hence, $d(\pi(x, t', \pi(y,t)) < \epsilon$. This proves the claim. Hence if $U = \cup\{U(x, \nu(x)) : x \in A\}$, then U is open, contains A and $C_0(U) \subseteq U(A,\epsilon)$.

Example 1: The following example shows that if a set $A \subset \Omega$ does not satisfy the condition that $\Phi(x) < \infty$ for each $x \in A$, then the above theorem is note true.

Consider the dynamical system (R^2, π) shown in the Figure 1 below:

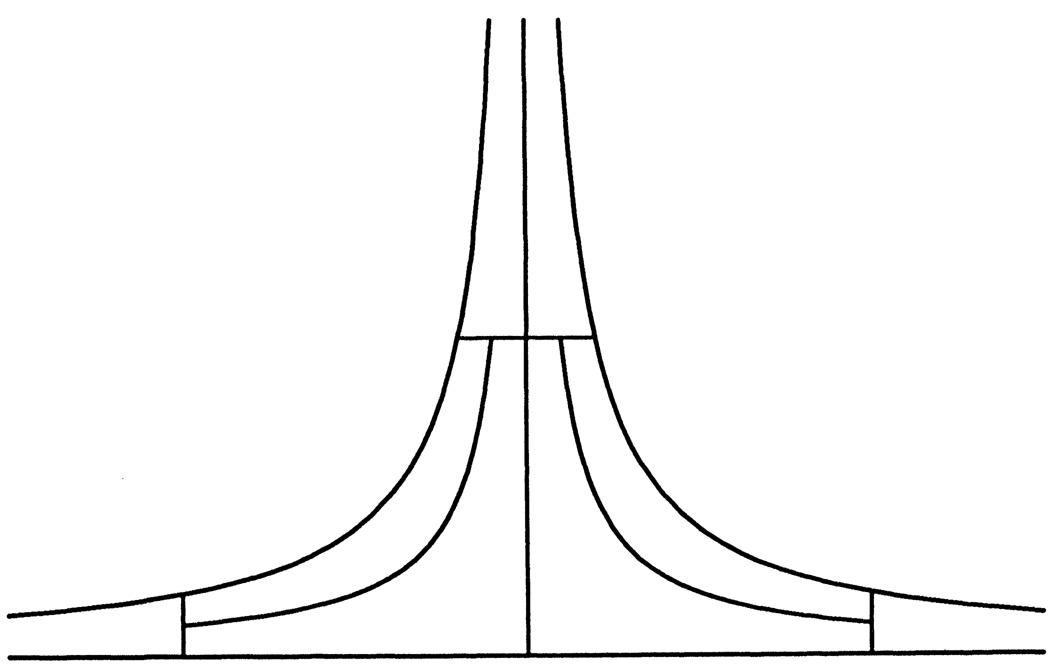

Figure 1

Let X be the open set in R^2 obtained by deleting the set $\{(x,y): xy = \pm k, y > 0\}$ for some $k > 1$. The two curves, $xy = \pm k$ define trajectories in the original dynamical system. Hence (X,π) is a dynamical system, with metric space x. Let $M = \{(x,y):\ 0 \leq y < 1,\ x = \pm k\}$, and $N = \{(x,y):\ -1 < x < 1, \text{ and } y = k\}$. Let $\Omega = X - M$, and define $I: M \rightarrow N$ by

$$I(x,y) = (y^2, k) \text{ if } x = k, \text{ and } I(x,y) = (-y^2, k) \text{ if } x = -k.$$

Notice that deleting the orbits as we did from the original dynamical system to get X, makes Φ continuous on Ω.

Now suppose $A = \{(0,y): 0 < y \leq k\}$. Then A is a cylindrical set. Ω is locally compact. But for each $x \in A$, $\Phi(x) = \infty$. Note also that for this A the above theorem fails, because any cylindrical open set containing A has finite diameter of $2k$.

Lemma 3.6: *Let $A \subset \Omega$ be positively invariant in $(\Omega, \widetilde{\pi})$ and for each $x \in A$, let $\Phi(x) < \infty$. Then $p = \widetilde{\pi}(x, \Phi(x)) \in \widehat{A}$. Consequently, $A \cap N = A \cap \widehat{N}$.*

Proof: Since $\Phi(x) < \infty$, $p \in N$. Since A is positively invariant $p \in A$ also. Repeating the above argument we get $p_1^+ \in N$ and $p_1^+ \in A$. Thus, inductively, $p_n^+ \in A \cap N$ for $n = 1, 2, \ldots$. Consequently, $p \in \widehat{N}$ and therefore $p \in \widehat{A} = A \cap \widehat{N}$.

If, furthermore, $x \in N$, then clearly, $x \in \widehat{N}$, and the last result follows.

Lemma 3.7: *Let $A \subset \Omega_0$ and $\widetilde{D}(A) = A$. If N has compact closure in Ω, then there exists an open set V containing $\widehat{A}$ such that $V \cap N = V \cap \widehat{N}$.*

Proof: Suppose no such open set exists. Then there exists a sequence $\{x_n\}$ in $N - \widehat{N}$, $x_n \rightarrow x \in \widehat{A}$. Since $x_n \in N - \widehat{N}$, there exists a $t_n \in R^+$, such that, $\widetilde{\pi}(x_n, t_n) = q_n \in N$, and $\Phi(q_n) = \infty$. Since N has compact closure in Ω, we may assume that $q_n \rightarrow q \in \Omega$. However, $q \in \widetilde{D}(A) = A$, and, therefore, $\Phi(q) < \infty$. But this contradicts the continuity of Φ at q, and completes the proof.

Lemma 3.8: *Suppose $A \subset \Omega$ is positively invariant and for each $x \in A$, $\Phi(x) < \infty$. Suppose V is an open set containing $\widehat{A}$ such that $\widehat{V} = V \cap N = V \cap \widehat{N}$. Then there exists a cylindrical open set $U \supset A$ such that for any $y \in U$, $\Phi(y) < \infty$ and $\widetilde{\pi}(y, \Phi(y)) \in \widehat{V}$.*

Proof: Let $x \in A$. Then $\Phi(x) < \infty$ implies $\pi(x, \Phi(x)) = x_1 \in M$. Since A is positively invariant, by Lemma 3.7, $\widetilde{\pi}(x, \Phi(x)) = x_1^+ \in \widehat{A}$. Now I being continuous, there exists an open set $U(x_1)$ in M, containing x_1, such that $I(U(x_1)) \subset \widehat{V}$. Again, by the continuity of π and Φ, there exists an open set $U(x)$ in Ω containing x, such that, for any $y \in U(x)$, $\pi(y, \Phi(y)) \in U(x_1)$ and $\Phi(y) < \infty$. Furthermore, we may assume that $U(x)$ lies in V. Hence $U = C_0[\cup\{U(x): x \in A\}]$ is a cylindrical open set (Lemma 3.3), and satisfies the requirements of the lemma.

Theorem 3.9: *Let Ω be locally compact. Let $A \subset \Omega$ be a closed cylindrical set and for each $x \in A$, let $\Phi(x) < \infty$. If A is stable in $(\Omega, \widetilde{\pi})$, then*

1. *$\widetilde{D}(A) = A$, and*
2. *there exists an open set U in Ω containing $\widehat{A} = A \cap \widehat{N}$, such that, $U \cap N = U \cap \widehat{N}$.*

Proof: 1. Follows from the usual argument using Theorem 3.5.

2. From continuity of Φ, each $x \in A$ is contained in an open set V_x such that for each $y \in V_x$, $\Phi(y) < \infty$. Set $V = \cup\{C_0(V_x): x \in A\}$. Then V is a cylindrical open set (Lemma 3.3) containing A. Hence, by stability of A there exists an open set U, $A \subset U \subset V$ such that $\widetilde{C}(x) \subset V$ for each $x \in U$. Set $W = \cup\{\widetilde{C}(x): x \in U\}$. Then clearly W is positively invariant and since for each $x \in V$, $\Phi(x) < \infty$, by Lemma 3.6, $W \cap N = W \cap \widehat{N}$. Hence, $U \cap N = U \cap \widehat{N}$. This completes the proof.

The converse is proved in Theorem 3.13.

Example 2: Note that in Example 1 given earlier, A is stable but $\widetilde{D}(A) \neq A$.

Lemma 3.10: *Suppose $\widehat{A} \subset \widehat{N}$ is compact and stable in $(\widehat{N}, g)$ and admits a compact neighborhood V in Ω such that $V \cap N = V \cap \widehat{N} = \widehat{V}$. Then there exists an open set W containing $A = C_0(\widehat{A})$ such that for any $x \in W$, $\Phi(x) < \infty$ and $\widetilde{J}(x) \subset C_0(\widehat{J}(p))$ where $p = \widetilde{\pi}(x, \Phi(x))$.*

Proof: By stability of $\widehat{A}$ there exists an open set $\widehat{U}$ in $\widehat{N}$, such that $\widehat{A} \subset \widehat{U} \subset \widehat{V}$, and for any $p \in \widehat{U}, p_n^+ \in \widehat{V}$ for all $n = 0, 1, 2, \ldots$. By Lemma 2.12, A is closed and positively invariant in $(\Omega, \widetilde{\pi})$. Consequently, by Lemma 3.8 there exists a cylindrical open set W containing A such that for any $x \in W, \Phi(x) < \infty$ and $p = \widetilde{\pi}(x, \Phi(x)) \in \widehat{U}$. To prove the result, pick a $y \in \widetilde{J}(x)$. Then $\widetilde{\pi}(x_k, t_k) \to y$ for some $x_k \to x$ and $t_k \to \infty$, where x_k's may be assumed to lie in W. By lemma 2.3, $p_k = \widetilde{\pi}(x_k, t + \epsilon_k) \to p$ and ϵ_k are so chosen that $p_k \in \widehat{U}$, where $t = \Phi(x)$. Assuming that $t_k - t - \epsilon_k \geq 0$, we have $\widetilde{\pi}(x_k, t_k) = \widetilde{\pi}(p_k, t_k - t - \epsilon_k) = \pi(p^+_{k,n_k}, t'_k) \to y$, where $0 \leq t'_k < \Phi(p^+_{k,n_k})$, $k = 1, 2, \ldots$. Since $p_k \in \widehat{U}$, $p^+_{k,n_k} \in \widehat{V}; \widehat{V}$ being compact we may assume $p^+_{k,n_k} \to q$. Then clearly $q \in \widehat{J}(p)$. By continuity of Φ, $t'_k \to t'$, and $y = \widetilde{\pi}(q, t')$ for $0 \leq t' \leq \Phi(q)$. But $y \in \Omega$, hence $t' < \Phi(q)$ and therefore $y \in C_0(q)$. Thus, $\widetilde{J}(x) \subset C_0(\widehat{J}(p))$, and the lemma is proved.

Theorem 3.11: *Under the hypotheses of the above lemma, $\widetilde{D}(A) = A$.*

Proof: A is clearly cylindrical. By Lemma 2.12, since A is positively invariant and closed and for any $x \in A$, $\widetilde{K}(x) \subset A$. To complete the proof, it is enough to note that for any $x \in A$, $\widetilde{J}(x) \subset A$. Since $\widehat{A}$ is stable, for any $p \in \widehat{A}$, $\widehat{J}(p) \subset \widehat{A}$, hence by Lemma 2.10, $\widetilde{J}(x) \subset A$. This completes the proof by Lemma 2.11.

Theorem 3.12: *If $\widetilde{D}(A) = A$ and $\widehat{A} = A \cap \widehat{N}$, then in $(\widehat{N}, g)$, $\widehat{D}(\widehat{A}) = \widehat{A}$. If, furthermore, there exists a compact neighborhood $\widehat{V}$ of $\widehat{A}$ in $\widehat{N}$, then $\widehat{A}$ is stable.*

Proof: To prove that $\widehat{D}(\widehat{A}) = \widehat{A}$ it is enough to show that $\widehat{D}(\widehat{A}) \subset \widehat{A}$. So let $q \in \widehat{D}(\widehat{A})$. Then there exists a $\{p_k\}$ in $\widehat{N}$, $p_k \to p \in \widehat{A}$, and a sequence of positive integers $\{n_k\}$, such that, $p^+_{kn_k} \to q$ and $q \in \widehat{N}$. But, clearly, $q \in \widetilde{D}(A) = A$, hence $q \in \widehat{N} \cap A = \widehat{A}$. This proves the inclusion. The last statement follows from Lemma 7.4 (*a*) [5, p. 8].

Theorem 3.13: *Let $A \subset \Omega$ be closed, let $\Phi(x) < \infty$ for each $x \in A$ and let $\widetilde{D}(A) = A$. Let N be closed and let there exist a compact neighborhood V of $\widehat{A} = A \cap \widehat{N}$, such that, $V \cap N = V \cap \widehat{N} = \widehat{V}$. Then A is stable.*

Proof: Since $\widetilde{D}(A) = A$, A is positively invariant in $(\Omega, \widetilde{\pi})$. Since $\widehat{V}$ is compact in $\widehat{N}$ and $\widehat{A} = A \cap \widehat{N} = A \cap N$ is closed in N, $\widehat{A}$ is compact. Hence $\widehat{A}$ is stable in $(\widehat{N}, g)$ (Theorem 3.12). Let W be a cylindrical open set containing A. We may assume without loss of generality that $\widehat{V} \subset W$. By stability of $\widehat{A}$, there exists an open set $\widehat{U}$ in $\widehat{N}$ such that $\widehat{A} \subset \widehat{U} \subset \widehat{V}$ and for any $p \in \widehat{U}$, $p_n^+ \in \widehat{V}$ for all $n = 1, 2, \ldots$. Let G be an open set containing A as in Lemma 3.9 so that for any $x \in G$, $\Phi(x) < \infty$ and $p = \widetilde{\pi}(x, \Phi(x)) \in \widehat{U}$. Furthermore, let $G \subset W$. Then for any $x \in G$, $\widetilde{C}(x) = \cup \{C_0(p_n^+) : n = 0, 1, 2, \ldots\} \cup C_0(x) \subset C_0(x) \cup C_0(\widehat{V}) \subset W$. This completes the proof.

Theorem 3.14: *Suppose $A \subset \Omega$ is closed, $\Phi(x) < \infty$ for each $x \in A$. Let $\widehat{A} = A \cap \widehat{N}$ be compact and stable in $(\widehat{N}, g)$, and suppose there exists an open set V in Ω such that $V \cap N = V \cap \widehat{N} = \widehat{V}$ and clV is compact. Then A is stable in $(\Omega, \widetilde{\pi})$.*

Proof: By Theorem 3.11, $\widetilde{D}(A) = A$, hence by Theorem 3.13, A is stable. This completes the proof.

The results of this section can be combined as:

Theorem 3.15: *Let Ω be locally compact, $A \subset \Omega$ be a closed, cylindrical set and $\Phi(x) < \infty$ for each $x \in A$. Let N be closed and let there be a compact neighborhood V of $\widehat{A} = A \cap \widehat{N}$ in Ω,* *such that,* $V \cap N = V \cap \hat{N}$. *Then the following are equivalent:*

(a) A *is stable in* $(\Omega, \tilde{\pi})$.
(b) $\tilde{D}(A) = A$ *in* $(\Omega, \tilde{\pi})$.
(c) $\hat{D}(\hat{A}) = \hat{A}$ *in* $(\hat{N}, g)$.
(d) $\hat{A}$ *is stable in* $(\hat{N}, g)$.

In view of Lemma 3.8, Theorem 3.15 has another version as follows:

Theorem 3.16: *Let* Ω *be locally compact,* $A \subset \Omega$ *be a closed, cylindrical set and* $\Phi(x) < \infty$ *for each* $x \in A$. *Let* N *be compact. Then the following are equivalent:*

(a) A *is stable in* $(\Omega, \tilde{\pi})$.
(b) $\tilde{D}(A) = A$ *in* $(\Omega, \tilde{\pi})$.
(c) $\hat{D}(\hat{A}) = \hat{A}$ *in* $(\hat{N}, g)$.
(d) $\hat{A}$ *is stable in* $(\hat{N}, g)$.

4. Asymptotic Stability

For any subset A of Ω we set $\tilde{P}(A) = \{x \in \Omega: \tilde{L}(x) \neq \emptyset$ and $\tilde{L}(x) \subset A\}$, and $\tilde{Q}(A) = \{x \in \Omega: \tilde{J}(x) \neq \emptyset$ and $\tilde{J}(x) \subset A\}$.

$\hat{P}(\hat{A})$ and $\hat{Q}(\hat{A})$ and defined similarly in $(\hat{N}, g)$ for $\hat{A} \subset \hat{N}$.

Definition 4.1: We say A is an *attractor* if $\tilde{P}(A)$ is a neighborhood of A, and a *uniform attractor* if $\tilde{Q}(A)$ is a neighborhood of A. Furthermore, A is said to be *asymptotically stable* in $(\Omega, \tilde{\pi})$ if A is both stable and an attractor.

For $\hat{A} \subset \hat{N}$, attractor and uniform attractor are similarly defined if $\hat{P}(\hat{A})$ and $\hat{Q}(\hat{A})$ are neighborhoods of $\hat{A}$ in $\hat{N}$ respectively and $\hat{A}$ is asymptotically stable if $\hat{A}$ is stable and an attractor in $(\hat{N}, g)$.

Lemma 4.2: *Let* $x \in \hat{N}$. *If* $w \in \hat{L}(x)$ *then* $\hat{J}(x) \subset \hat{J}(w)$.

Proof: $\hat{J}(x)$ can be characterized as follows: (*i*) $y \in \hat{J}(x)$ if and only if (*ii*) given any $\epsilon > 0$, $\delta > 0$ and a positive integer n_0, there exists a $z \in U(x, \delta)$ such that $g^n(z) \in U(y, \epsilon)$ for some $n \geq n_0$.

Suppose $y \in \tilde{J}(x)$. Let $\epsilon > 0$, $\delta > 0$ and n_0 be given. Since $w \in \hat{L}(x)$, $g^{n_k}(x) \to w$ for some $n_k \to \infty$, hence there exists a k_0 such that for $k \geq k_0$, $g^{n_k}(x) \in U(w, \delta)$. From the continuity of g^n, there exists δ_1, such that $g^n(U(x, \delta_1)) \subset U(w, \delta)$, for $n = n_{k_0}$. Hence by (*ii*) there exists a $z \in U(x, \delta_1)$ and a positive integer m such that, $\ell = m - n \geq n_0$ and $g^m(z) \in U(y, \epsilon)$. Hence $g^n(z) \in U(w, \delta)$ and $g^{\ell}(g^n(z)) \in U(y, \epsilon)$ implies by (*ii*) that $y \in \hat{J}(w)$. This completes the proof.

Lemma 4.3: *Let* $Y \subset \hat{N}$ *be closed. If* Y *is positively invariant and a uniform attractor in* $(\hat{N}, g)$, *then* $\hat{D}(Y) = Y$. *If, furthermore,* $Y \subset G$ *where* G *is a compact neighborhood of* Y, *then* Y *is asymptotically stable.*

Proof: To prove $\hat{D}(Y) = Y$, it is enough to show that $\hat{D}(Y) \subset Y$. But $\hat{D}(Y) = \hat{K}(Y) \cup \hat{J}(Y)$. Since Y is a uniform attractor $\hat{J}(Y) \subset Y$, and since Y is closed and invariant then $\hat{K}(Y) \subset Y$. Hence $\hat{D}(Y) \subset Y$.

To prove the second part, note that Y is stable by [5, Lemma 7.4]. Since Y is stable, there exists an open set U in $\hat{N}$, $Y \subset U \subset G$, so that if $p \in U$, $g^n(p) \in G$ for all $n = 1, 2, \ldots$. Consequently, for any $p \in U$, $\hat{L}(p) \neq \emptyset$, and since $\hat{L}(p) \subset \hat{K}(p)$, $Y \subset U \cap \hat{Q}(Y) \subset \hat{P}(Y)$ and Y is an attractor.

Lemma 4.4: *Let* Y *be asymptotically stable in* $(\hat{N}, g)$, *then* Y *is a uniform attractor in*

$(\widehat{N}, g)$.

Proof: It is enough to show that $\widehat{P}(Y) \subset \widehat{Q}(Y)$; so let $x \in \widehat{P}(Y)$. Then $\widehat{L}(x) \neq \emptyset$ and $\widehat{L}(x) \subset Y$. Hence $\widehat{J}(x) \neq \emptyset$, and by Lemma 4.2 if $w \in \widehat{L}(x)$, then $\widehat{J}(x) \subset \widehat{J}(w)$. Since Y is stable and $w \in Y$, $\widehat{J}(w) \subset Y$. Hence $\widehat{J}(x) \subset Y$, and $x \in \widehat{Q}(Y)$. This completes the proof.

Lemma 4.5: *Let $A \subset \Omega$ be an attractor in $(\Omega, \widetilde{\pi})$. If for each $p \in A \cap \widehat{N} = \widehat{A}$, $\widehat{K}(p)$ is compact, then $\widehat{A}$ is an attractor in$(\widehat{N}, g)$.*

Proof: Let $p \in \widetilde{P}(A) \cap \widehat{N}$. Then $\widetilde{L}(p) \neq \Phi$ and $\widetilde{L}(p) \subset A$. By Lemma 2.13, $\widetilde{L}(p) = C_0(\widehat{L}(p))$. Hence $\widehat{L}(p) \neq \emptyset$ and $C_0(\widehat{L}(p)) \cap \widehat{N} \subset A \cap \widehat{N} = \widehat{A}$. But $C_0(\widehat{L}(p)) \cap \widehat{N} = \widehat{L}(p)$, because of the definition of C_0 and because $\widehat{L}(p) \subset \widehat{N}$. Hence $\widehat{A} \subset \widetilde{P}(A) \cap \widehat{N} \subset \widehat{P}(\widehat{A})$ and $\widehat{A}$ is an attractor, since $\widetilde{P}(A)$ is a neighborhood of A.

Theorem 4.6: *Let Ω be locally compact, $A \subset \Omega$ be a closed cylindrical set and $\Phi(x) < \infty$ for each $x \in A$. Let N be closed and let there be a compact neighborhood V of $\widehat{A} = A \cap \widehat{N}$ in Ω such that $V \cap N = V \cap \widehat{N} = \widehat{V}$. Then the following statements are true.*

(*a*) *If $\widehat{A}$ is asymptotically stable in (N, g), then A is a uniform attractor in $(\Omega, \widetilde{\pi})$.*

(*b*) *If A is asymptotically stable in $(\Omega, \widetilde{\pi})$, then A is a uniform attractor in $(\Omega, \widetilde{\pi})$.*

Proof: (*a*) Since A is asymptotically stable by Lemma 4.4, $\widehat{A}$ is a uniform attractor. Since $\widehat{A}$ is stable in $(\widehat{N}, g)$, there exists an open set $\widehat{U}$ in $\widehat{N}$ containing $\widehat{A}$ and contained in $\widehat{Q}(\widehat{A})$, such that, for any $p \in \widehat{U}$, $\widehat{K}(p) \subset \widehat{V}$. By Lemma 3.8 there exists an open set W containing A such that for any $x \in W$, $\Phi(x) < \infty$ and $p = \widetilde{\pi}(x, \Phi(x)) \in \widehat{U}$. Then $x \in W$ satisfies all conditions of Lemma 3.10, hence $\widetilde{J}(x) \subset C_0(\widehat{J}(p))$. But $p \in \widehat{Q}(\widehat{A})$ implies $\widehat{J}(p) \subset \widehat{A}$, hence $\widetilde{J}(x) \subset C_0(\widehat{A}) \subset A$, since A is cylindrical. To see that $\widetilde{J}(x) \neq \Phi$, note that $\widetilde{J}(x) \supset \widetilde{L}(x)$ and by Lemma 2.13, $\widetilde{L}(x) = C_0(\widehat{L}(p))$ and $\widehat{L}(p) \neq \Phi$, hence $\widetilde{L}(x) \neq \Phi$. This proves that $W \subset \widetilde{Q}(A)$, hence A is a uniform attractor.

(*b*) In view of (*a*), it is enough to show that $\widehat{A}$ is asymptotically stable in $(\widehat{N}, g)$. But by Theorem 3.15, $\widehat{A}$ is stable in $(\widehat{N}, g)$ and by Lemma 4.5, since A is an attractor, $\widehat{A}$ is an attractor in $(\widehat{N}, g)$. This completes the proof.

Theorem 4.7: *Under the assumptions of Theorem 4.6, the following are equivalent:*

(*a*) *A is asymptotically stable in $(\Omega, \widetilde{\pi})$.*

(*b*) *$\widehat{A}$ is asymptotically stable in $(\widehat{N}, g)$.*

Proof: Suppose (*a*): Since A is stable in $(\Omega, \widetilde{\pi})$ by Theorem 3.15, $\widehat{A}$ is stable in $(\widehat{N}, g)$, and by Lemma 4.5, $\widehat{A}$ is an attractor. Thus (*b*) is true.

Suppose (*b*): Since $\widehat{A}$ is stable in $(\widehat{N}, g)$, by Theorem 3.15, A is stable in $(\Omega, \widetilde{\pi})$, and by Theorem 4.6 (*a*), A is a uniform attractor. Now let $x \in Int\, V \cap Int\, \widetilde{Q}(A)$. Then clearly $\widetilde{L}(x) \neq \Phi$, and since $\widetilde{J}(x) \subset A$, $\widetilde{L}(x) \subset A$. Hence A is an attractor. This proves (*a*).

5. Lyapunov Function

In this section we introduce a Lyapunov function in $(\Omega, \widetilde{\pi})$ and extend the Invariance Theorem and the Asymptotic Stability Theorem [6] to $(\Omega, \widetilde{\pi})$.

Let $G \subset \Omega$ be a positively invariant closed set, and recall that $\overline{G}$ denotes the closure of G in X.

Definition 5.1: A function

$$V: \overline{G} \to R$$

is said to be a Lyapunov function, if

1. V is continuous,
2. $V(x^+) \leq V(x)$ for $x \in \overline{G} \cap M$, and
3. $\dot{V}(x) \leq 0$ for $x \in G$, where $\dot{V}(x) = \lim_{t \to 0^+} \frac{1}{t}\{V(\widetilde{\pi}(x,t)) - V(x)\}$.

We remark that it follows that V satisfies $V(\widetilde{\pi}(x,t)) \leq V(x)$ for all $x \in G$ and $t \in R^+$.

Let

$$E = \{x \in G: \dot{V}(x) = 0\}.$$

Let $A \subset E$ be the largest invariant set under $\widetilde{\pi}$.

We assume for the remainder of this section that for any $x \in G$, $\Phi(x) < \infty$, $G = C_0(\widehat{G})$, where $\widehat{G} = G \cap \widehat{N}$ is compact. By Lemma 3.6 then $G \cap \widehat{N} = G \cap N$.

Let N be closed, so that $\widehat{A} = A \cap N = A \cap \widehat{N}$ is also compact.

Lemma 5.1: *$\widehat{A}$ is invariant under g.*

Proof: Let $p \in \widehat{A}$. Then for any $t_1 \in R^+$ there exists an $x \in A$ such that $\widetilde{\pi}(x, t_1) = p$.

Suppose $t_1 > \Phi(x)$: Then $\widetilde{\pi}(x, \Phi(x)) = q \neq p$. Since A is positively invariant, $q \in A$. Consequently, $q \in A \cap N \subset G \cap N = G \cap \widehat{N} = \widehat{G}$. Hence $q \in \widehat{A}$. Since $\widehat{A}$ is compact, $S = inf\{\Phi(p): p \in \widehat{A}\} > 0$. Thus, proceeding inductively, we can show that there exists a $q_m \in \widehat{A}$ such that $\widetilde{\pi}(q_m, \Phi(q_m)) = p$. Thus $\widehat{A} \subset g(\widehat{A})$.

Suppose $t_1 \leq \Phi(x)$: Now $x \in A \subset G$ implies there exists a $q \in \widehat{G}$ such that $x \in C_0(q)$, since $C_0(\widehat{G}) = G$. That is, $x = \pi(q,s)$, $0 \leq s < \Phi(q)$. Let $T > s$. Then given x there is a $y \in A$ such that $\widetilde{\pi}(y,T) = x$.

Case 1: If $\widetilde{\pi}(y,t) = \pi(y,t)$ for $0 \leq t < T$, then $\pi(\pi(y, T-s), s) = x = \pi(q,s)$ implies that $\pi(y, T-s) = q$, since (X, π) is a dynamical system. Consequently $q \in A$ and therefore $q \in A \cap \widehat{G} = \widehat{A}$, and proceeding as above we can show that $\widehat{A} \subset g(\widehat{A})$.

Case 2: $\widetilde{\pi}(y,t) \neq \pi(y,t)$, $0 \leq t < T$. In this case $\widetilde{\pi}(y, s_1) = q_1 \in N$ for some s_1, $0 \leq s_1 < T$. Again, as shown earlier, $q_1 \in G$. Consequently $q_1 \in \widehat{A}$. Thus we have $\widehat{A} \subset g(\widehat{A})$.

The inclusion $g(\widehat{A}) \subset \widehat{A}$ is trivial, since A is positively invariant. Hence the result.

Let $\widehat{V} = V \mid \widehat{G}$. Then clearly $\widehat{V}: \widehat{G} \to R$ is a Lyapunov function with respect to g. That is, (1) $\widehat{V}$ is continuous, and (2) $\widehat{V}(q(p)) \leq V(p)$ for $p \in \widehat{G}$ [6, Def. 6.1, p. 6].

Let

$$\widehat{E} = \{p \in \widehat{G}: \widehat{V}(p) = \widehat{V}(g(p))\}.$$

Let $F \subset \widehat{E}$ be the largest invariant set under g.

Theorem 5.2: (Invariance Theorem). *There exists an $a \in R$ such that for any $x \in G$, $\widetilde{L}(x) \subset A \cap V^{-1}(a)$.*

Proof: Let $x \in G$. Then $\widetilde{\pi}(x, \Phi(x)) = p \in \widehat{G}$. By Theorem 6.3 [6, p. 6], it follows that $\widehat{L}(p) \subset F \cap \widehat{V}^{-1}(a)$ for some $a \in R$. By Lemma 2.13 $\widetilde{L}(x) = C_0(\widehat{L}(p))$ consequently for any $y \in \widetilde{L}(x)$, $V(y) = a$ and $\dot{V}(y) = 0$. Therefore, $\widetilde{L}(x) \subset E$ and since $\widetilde{L}(x)$ is invariant by Corollary 2.15, $\widetilde{L}(x) \subset A$. This completes the proof.

Theorem 5.3: (Asymptotic Stability).

(1) *If $A \subset Int G$, then A is an attractor.*

(2) *If, furthermore, $V(A) = a$ for some $a \in R$, then A is asymptotically stable.*

Proof: (1) Let $x \in G$, then $\widetilde{\pi}(x, \Phi(x)) = p \in \widehat{G}$ and $\widehat{L}(p) \neq \emptyset$ implies $\widetilde{L}(x) \neq \emptyset$. By Theorem 5.2 it follows that A is an attractor.

(2) We first show that $F = \widehat{A}$: First note that if $p \in \widehat{E}$ then $V(\pi(x,t)) = V(x)$ for $0 \leq t < \Phi(x)$. Hence $C_0(p) \subset E$. Consequently, $C_0(\widehat{E}) \subset E$. Let $p \in \widehat{A} \subset A$. Since $g(p) \in A \cap \widehat{G} = \widehat{A}$, $\dot{V}(p) = 0$ and $p \in \widehat{E}$, that is $\widehat{A} \subset \widehat{E}$. Since $\widehat{E}$ is a closed subset of $\widehat{G}$, then $\widehat{E}$ is compact. Also, $F \subset \widehat{E}$ is the largest invariant set in $\widehat{E}$ under g, F is also closed and hence compact. Therefore, by Lemmas 2.12 and 2.14, $C_0(F) \subset C_0(\widehat{G}) \subset G$ is invariant under $\widetilde{\pi}$. Thus,

$$F = C_0(F) \cap \widehat{G} \subset A \cap \widehat{G} = \widehat{A}$$

implies that $F = \widehat{A}$.

Now $\widehat{V}(\widehat{A}) = a$ and $A \subset Int\, G$ implies that $\widehat{A} \subset A \cap \widehat{G} \subseteq Int\, G \cap \widehat{G} \subset Int(G \cap \widehat{N}) = Int\, \widehat{G}$. Hence, by Theorem 7.9 [6, p. 9], $\widehat{A}$ is asymptotically stable in $(\widehat{N}, g)$. Consequently, by Theorem 4.7, A is asymptotically stable in $(\Omega, \widetilde{\pi})$.

In the remainder of this section we shall prove the converse of Theorem 5.3. We begin by proving:

Theorem 5.4: *Let $\widehat{A} \subset \widehat{N}$ be compact and asymptotically stable in $(\widehat{N}, g)$. Suppose there exists a compact neighborhood $\widehat{V}$ of $\widehat{A}$ in $\widehat{N}$. Then there exists a positively invariant set $\widehat{G}$ in $\widehat{N}$ containing $\widehat{A}$ and a function $\alpha: \widehat{G} \to R^+$, such that,*

(*i*) *α is continuous,*

(*ii*) *$\alpha(g(x)) < \alpha(x)$ if $x \in \widehat{G} - \widehat{A}$,*

(*iii*) *$\alpha(x) = 0$ if $x \in \widehat{A}$, and*

(*iv*) *$\widehat{A} \subset Int\, \widehat{G}$.*

Proof: By Lemma 4.4, $\widehat{A}$ is a uniform attractor in $(\widehat{N}, g)$. Hence there exists a compact neighborhood V of $\widehat{A}$, such that, $V \cap N = V \cap \widehat{N} = \widehat{V} \subset Int\, \widehat{Q}(\widehat{A})$, where $p \in \widehat{Q}(\widehat{A})$ implies that $\widehat{J}(p) \neq \emptyset$ and $\widehat{J}(p) \subset \widehat{A}$. Since $\widehat{A}$ is stable there exists a compact neighborhood $\widehat{U}$ of $\widehat{A}$, such that, for any $p \in \widehat{U}$, $\widehat{K}(p) \subset \widehat{V}$.

Now on the closure $\widehat{G}$ in $\widehat{N}$ of the positively invariant set $\widehat{W} = U\{g^n(\widehat{U}): n = 0, 1, \ldots\}$ we define a function

$$\beta: \widehat{G} \to R^+$$

as follows: For $x \in \widehat{G}$,

$$\beta(x) = sup\{d(g^n(x), \widehat{A}): n = 0, 1, 2, \ldots\}.$$

Since $\widehat{V}$ is compact and $g^n(x) \in \widehat{V}$ for each n, $\beta(x)$ is well defined, and clearly for any $x \in \widehat{G}$, $\beta(g(x)) \leq \beta(x)$. Furthermore, $\beta(x) = 0$ if and only if $x \in \widehat{A}$.

Now define $\beta_n: \widehat{G} \to R^+$ by setting:

$$\beta_n(x) = \beta(g^n(x)), \text{ for any } x \in \widehat{G}.$$

Then clearly $\beta_{n+1}(x) \leq \beta_n(x)$, $n = 0, 1, 2, \ldots$. Since $\widehat{J}(x) \subset \widehat{A}$ for any $x \in \widehat{U}$, $g^n(x) \to \widehat{A}$ as $n \to \infty$. Hence $\beta_n(x) \to 0$ for any $x \in \widehat{G}$ and $\{\beta_n(x)\}$ is monotone decreasing on $\widehat{G}$.

We claim that $\beta_n(x) \to 0$ uniformly on $\widehat{G}$ as $n \to \infty$. Note that $\widehat{G} \subset \widehat{Q}(\widehat{A})$, hence for any $p \in \widehat{G}$, $\widehat{J}(p) \subset \widehat{A}$. Thus, it is easy to see that given any $\epsilon > 0$ there exists an n_0 and a $\delta > 0$ such that if $y \in U(p, \delta) \cap \widehat{G}$ then $g^n(y) \in U(\widehat{A}, \epsilon)$ for all $n \geq n_0$. For otherwise there exists a sequence $\{p_k\}$ converging to p and $\{n_k\}$, $n_k \to \infty$, such that, $g^{n_k}(p_k) \notin U(\widehat{A}, \epsilon)$, implying, since $\widehat{V}$ is compact and contains $\{g^{n_k}(p_k)\}$, that $\widehat{J}(p) \not\subset A$, a contradiction. This completes the proof, since $\widehat{G}$ is compact.

Now define

$$\alpha: \widehat{G} \to R^+$$

by setting,

$$\alpha(x) = \sum_{n=0}^{\infty} \frac{1}{2^n}\beta_n(x)$$

for $x \in \hat{G}$. Then, by Abel's Theorem [7, (*ii*), p. 421] α is continuous on $\hat{G}$ and clearly satisfies (*ii*) and (*iii*) above. This completes the proof.

Theorem 5.5: *Let $A \subset \Omega$ be a closed cylindrical set and for each $x \in A$, $\Phi(x) < \infty$. Let N be closed, $\hat{A} = A \cap \hat{N}$ be compact and $C_0(\hat{A}) = A$. If A is asymptotically stable then there exists a positively invariant set G in Ω containing A which admits a Lyapunov function $\gamma: G \to R^+$ such that*

(*i*) $A \subset Int\, G$,
(*ii*) $\gamma(\tilde{\pi}(x,t)) \leq \gamma(x)$ *for all* $t \in R^+$, *and*
(*iii*) $\gamma(x) = 0$ *if and only if* $x \in A$.

Proof: By Theorem 4.7, $\hat{A}$ is asymptotically stable in $(\hat{N}, g)$ and there exists an open set $V \supset \hat{A}$ such that $\hat{V} = V \cap N = V \cap \hat{N}$ and $\hat{V}$ has compact closure in $\hat{N}$.

Since A is stable given the cylindrical open set $C_0(V)$ containing A, there exists an open set $U \supset A$, such that, $U \subset C_0(V)$ and for any $x \in U$, $\tilde{C}(x) \subset C_\Omega(V)$. This implies that $\hat{U} = U \cap \hat{N} \subset \hat{V}$ and for any $p \in \hat{U}$, $g^n(p) \in \hat{V}$ for $n = 0, 1, 2, \ldots$. Let $\hat{G}$ and

$$\alpha: \hat{G} \to R^+$$

be as in Theorem 5.4.

Now by Lemmas 3.8 and 3.10 there exists an open set $W \supset A$ such that $W \subset C_0(V)$ and for any $x \in W$, $\Phi(x) < \infty$, $\tilde{\pi}(x, \Phi(x)) \in \hat{U}$ and $\tilde{J}(x) \subset C_0(\hat{J}(p))$. Let $G = U\{\tilde{C}(x): x \in W\}$. We now define a function

$$\hat{\alpha}: \overline{G} \to R^+$$

as follows: Let $x \in G$:

1. If $x \in C_0(\hat{G})$, then set $\hat{\alpha}(x) = \alpha(p)$ where $x = \pi(p,t)$, $0 \leq t < \Phi(p)$.
2. If $x \notin C_0(\hat{G})$, then $\tilde{\pi}(x, \Phi(x)) = p \in \hat{U}$, and we set

$$\hat{\alpha}(x) = \Phi(x) + \alpha(p).$$

3. If $x \in \overline{G} \cap M$, then $x = \pi(y, \Phi(y))$ for some $y \in G$ and we set

$$\gamma(x) = \lim_{t \to t_0^+} \alpha(\pi(y,t)), \text{ where } t_0 = \Phi(y).$$

This limit exists because $\alpha(\pi(y,t))$ is a nonincreasing function of t.

That the three conditions (*i*), (*ii*) and (*iii*) as required are satisfied follows easily from this definition.

References

[1] Bainov, D.D., Lakshmikantham, V. and Simenov, P.S., *Theory of Impulsive Differential Equations*, World Science 1989.

[2] Bhatia, N.P. and Szego, G.P., *Stability Theory of Dynamical Systems*, Springer-Verlag 1970.

[3] Kaul, S.K., On impulsive semidynamical systems - II recursive properties, *Nonlinear Analysis, TMA* **16** (1991), 635-645.

[4] Lasalle, J.P., *The Stability of Dynamical Systems*, Society for Industrial and Applied Mathematics 1976.

[5] Lasalle, J.P. and Lefschetz, *Stability by Lyapunov's Direct Method*, Academic Press 1961.

[6] Rozko, V.F., Lyapunov stability in discontinuous dynamical systems, *Diff. Uravm.* **11** (1975), 1005, 1012.

[7] Stromberg, K.R., *An Introduction to Classical Real Analysis*, Wadsworth Int. Math Series 1981.