INITIAL AND BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, we study initial and boundary value problems for functional integro-differential equations, by using the Leray-Schauder Alternative.

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1. Introduction

The purpose of this paper is to study the existence of solutions for initial and boundary value problem (IVP and BVP, for short) for functional integro-differential equations. The paper is divided into two parts.

In Section 2 we consider the following IVP for nonlinear Volterra type integro-differential equations

$$x'(t) = A(t, x_t) + \int_0^t k(t, s)f(s, x_s)ds, \quad t \in [0, T] \quad (1.1)$$

$$x_0 = \phi, \quad (1.2)$$

where $A, f: [0, T] \times C \rightarrow R^n$ are continuous functions, and for $t \in [0, T]$, $A(t, \cdot)$ is a bounded linear operator from C to R^n , and k is a measurable for $t \geq s \geq 0$ real valued function. Here $C = C([-r, 0], R^n)$ is the Banach space of all continuous functions $\phi: [-r, 0] \rightarrow R^n$ endowed with the sup-norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

Also, for $x \in C([-r, T], R^n)$ we have $x_t \in C$ for $t \in [0, T]$, $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$ and $\phi \in C$.

The results of this section generalize recent results of Ntouyas and Tsamatos [5] when the following degenerate case

$$x'(t) = A(t)x(t) + \int_0^t k(t,s)f(s, x_s)ds, \quad t \in [0, T] \quad (1.1)'$$

$$x_0 = \phi \quad (1.2)'$$

is studied. In Section 3 we study the following BVP for nonlinear Volterra integro-differential equations

$$x'(t) = A(t, x_t) + \int_0^t k(t,s)f(s, x_s)ds, \quad t \in [0, T] \quad (1.3)$$

$$Lx = h, \quad (1.4)$$

where A, f and k are as above and L is a bounded linear operator from a Banach space $C([-r, T], R^n)$ into R^n and $h \in \text{Im}L$, the image of L . The results of this section extend previous results on BVP for functional differential equations [2], [3], [4], and [7] to functional integro-differential equations.

2. IVP for Volterra Functional Integro-Differential Equations

In this section we consider the following initial value problem

$$x'(t) = A(t, x_t) + \int_0^t k(t,s)f(t, x_t)ds, \quad 0 \leq t \leq T \quad (2.1)$$

$$x_0 = \phi. \quad (2.2)$$

Before stating our basic existence theorems, we need the following lemma which is an immediate consequence of the Topological Transversality Theorem of Granas [1], known as “**Leray-Schauder alternative**”.

Lemma 2.1: *Let S be a convex subset of a normed linear space E and assume $0 \in S$. Let $F: S \rightarrow S$ be a completely continuous operator, i.e., it is continuous and the image of any bounded set is included in a compact set, and let*

$$E(F) = \{x \in S: x = \lambda Fx \text{ for some } 0 < \lambda < 1\}.$$

Then, either $E(F)$ is unbounded or F has a fixed point.

For the IVP (2.1)-(2.2) we have the following existence theorem.

Theorem 2.2: *Let $f: [0, T] \times C \rightarrow R^n$ be a completely continuous function (i.e., it is continuous and takes closed bounded sets of $[0, T] \times C$ into bounded sets of R^n). Suppose that:*

(HA) *There exists a nonnegative integrable function p on $[0, T]$ such that $|A(t, \phi)| \leq p(t) \|\phi\|$, $(t, \phi) \in [0, T] \times C$.*

(Hk) *There exists a constant M such that $|k(t, s)| \leq M$, $t \geq s \geq 0$.*

Also we assume that there exists a constant K such that

$$\|x\|_1 \leq K,$$

for each solution x of

$$x'(t) = \lambda A(t, x_t) + \lambda \int_0^t k(t, s) f(t, x_t) ds, \quad 0 \leq t \leq T \quad (2.1)_\lambda$$

$$x_0 = \phi \quad (2.2)$$

for any $\lambda \in (0, 1)$.

Then the initial value problem (2.1)-(2.2) has at least one solution on $[-r, T]$.

Proof: We will rewrite (2.1) as follows. For $\phi \in C$ define $\tilde{\phi} \in B$, $B = C([-r, T], R^n)$ by

$$\tilde{\phi}(t) = \begin{cases} \phi(t), & -r \leq t \leq 0 \\ \phi(0), & 0 \leq t \leq T. \end{cases}$$

If $x(t) = y(t) + \tilde{\phi}(t)$, $t \in [-r, T]$ it is easy to verify that y satisfies

$$y_0 = 0,$$

$$y(t) = \int_0^t A(s, y_s + \tilde{\phi}) ds + \int_0^t \int_0^\tau k(t, s) f(s, y_s + \tilde{\phi}_s) ds d\tau, \quad 0 \leq t \leq T$$

if and only if x satisfies

$$x(t) = \phi(0) + \int_0^t A(s, x_s) ds + \int_0^t \int_0^\tau k(t, s) f(s, x_s) ds d\tau, \quad 0 \leq t \leq T$$

and $x_0 = \phi$.

Define $N: B_0 \rightarrow B_0$, $B_0 = \{y \in B: y_0 = 0\}$ by

$$Ny(t) = \begin{cases} 0, & -h \leq t \leq 0 \\ \int_0^t A(s, y_s + \tilde{\phi}_s) ds + \int_0^t \int_0^\tau k(t, s) f(s, y_s + \tilde{\phi}_s) ds d\tau, & 0 \leq t \leq T. \end{cases}$$

N is clearly continuous. We shall prove that N is completely continuous.

Let $\{h_\nu\}$ be a bounded sequence in B_0 , i.e.,

$$\|h_\nu\| \leq b, \text{ for all } \nu,$$

where b is a positive constant. We obviously have $\|h_{\nu t}\| \leq b$, $t \in [0, T]$, for all ν . Hence we obtain

$$\|Nh_\nu\| \leq p_0(b + \|\phi\|) + MM_0m_0,$$

where

$$p_0 = \int_0^T p(t) dt,$$

$$M_0 = \sup\{|f(t, u)| : t \in [0, T], \|u\| \leq b + \|\phi\|\}.$$

and

$$m_0 = \int_0^T \int_0^\tau p(t) dt d\tau.$$

This means that $\{Nh_\nu\}$ is uniformly bounded.

Moreover, the sequence $\{Nh_\nu\}$ is equicontinuous, since for $t_1, t_2 \in [-r, T]$ we have

$$|Nh_\nu(t_1) - Nh_\nu(t_2)| \leq [p_0(b + \|\phi\|) + MM_0m_0] |t_1 - t_2|.$$

Thus, by the Arzela-Ascoli theorem, the operator N is completely continuous.

Finally, the set $E(N) = \{y \in B_0 : y = \lambda Ny, \lambda \in (0, 1)\}$ is bounded by assumption, since $\|x\|_1 \leq K$ implies

$$\|y\|_1 \leq K + \|\phi\|.$$

Consequently, by Lemma 2.1, the operator N has a fixed point y^* in B_0 . Then $x^* = y^* + \tilde{\phi}$ is a solution of the IVP (2.1)-(2.2). This proves the theorem.

The applicability of Theorem 2.1 depends upon the existence of a priori bounds for the solutions of the initial value problem (2.1) $_{\lambda}$ -(2.2), which are independent of λ . Conditions on f which imply the desired a priori bounds are given in the following:

Theorem 2.3: Assume that (HA) and (Hk) hold. Also assume that

(Hf) There exists a continuous function m such that $|f(t, \phi)| \leq m(t)\Omega(\|\phi\|)$, $0 \leq t \leq T, \phi \in C$ where Ω is a continuous nondecreasing function defined on $[0, \infty)$ and positive on $(0, \infty)$.

Then, the initial value problem (2.1)-(2.2) has a solution on $[-r, T]$ provided

$$\int_0^T m_1(s) ds < \int_0^\infty \frac{ds}{s + \Omega(s)}, \quad m_1(t) = \sup\{1, p(t), Mm(t)\}.$$

Proof: To prove the existence of a solution of the IVP (2.1)-(2.2), we apply Theorem 2.1. In order to apply this theorem, we must establish the a priori bounds for the solutions of the IVP (2.1) $_{\lambda}$ -(2.2). Let x be a solution of (2.1) $_{\lambda}$. From

$$x(t) = \phi(0) + \lambda \int_0^t A(s, x_s) ds + \lambda \int_0^t \int_0^\tau k(t, s) f(s, x_s) ds d\tau, \quad 0 \leq t \leq T$$

we have

$$|x(t)| \leq |\phi(0)| + \lambda \int_0^t |A(s, x_s)| ds + \int_0^t \int_0^\tau |k(t, s)| |f(s, x_s)| ds d\tau, \quad 0 \leq t \leq T,$$

from which, by (HA), (Hf), and (Hk), we get

$$|x(t)| \leq \|\phi\| + \int_0^t p(s) \|x_s\| ds + M \int_0^t \int_0^\tau m(s) \Omega(\|x_s\|) ds d\tau.$$

We consider the function μ given by

$$\mu(t) = \sup\{|x(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq T.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in [0, t]$, by the previous inequality we have

$$\mu(t) \leq \|\phi\| + \int_0^t p(s)\mu(s)ds + M \int_0^t \int_0^\tau m(s)\Omega(\mu(s))dsd\tau, \quad 0 \leq t \leq T \quad (2.3)$$

If $t^* \in [-r, 0]$ then $\mu(t) = \|\phi\|$ and (2.3) obviously holds.

Denoting by $u(t)$ the right-hand side of (2.3) we have

$$\mu(t) \leq u(t), \quad 0 \leq t \leq T,$$

$$u(0) = \|\phi\|,$$

and

$$\begin{aligned} u'(t) &= p(t)\mu(t) + M \int_0^t m(s)\Omega(\mu(s))ds \\ &\leq p(t)u(t) + M \int_0^t m(s)\Omega(u(s))ds \\ &\leq m_1(t)[u(t) + \int_0^t \Omega(u(s))ds], \quad 0 \leq t \leq T. \end{aligned}$$

Let

$$v(t) = u(t) + \int_0^t \Omega(u(s))ds, \quad 0 \leq t \leq T.$$

Then

$$v(0) = u(0), \quad u(t) \leq v(t), \quad u'(t) \leq m_1(t)v(t), \quad 0 \leq t \leq T$$

and

$$\begin{aligned} v'(t) &= u'(t) + \Omega(u(t)) \\ &\leq m_1(t)v(t) + \Omega(v(t)) \\ &\leq m_1(t)[v(t) + \Omega(v(t))], \quad 0 \leq t \leq T \end{aligned}$$

or

$$\frac{v'(t)}{v(t) + \Omega(v(t))} \leq m_1(t), \quad 0 \leq t \leq T.$$

This implies

$$\int_{v(0)}^{v(t)} \frac{ds}{s + \Omega(s)} \leq \int_0^T m_1(t)dt < \int_{v(0)}^{\infty} \frac{ds}{s + \Omega(s)}, \quad 0 \leq t \leq T.$$

This inequality implies that there is a constant K such that $u(t) \leq K$, $t \in [0, T]$, and hence $\mu(t) \leq K$, $t \in [0, T]$. Therefore,

$$\|x\|_1 \leq K, \quad (2.4)$$

and the proof of the theorem is complete.

By applying Theorem 2.3, we have the following result which concerns the global existence of solutions for the IVP (1.1)-(1.2). The proof is omitted since it is similar to that of Theorem 2.3 of [5].

Theorem 2.4: Assume that (HA) and (Hk) hold. Also assume that

$(Hf)'$ There exists a continuous function m such that $|f(t, \phi)| \leq m(t)\Omega(\|\phi\|)$, $0 \leq t < \infty$, $\phi \in C$, where Ω is a continuous nondecreasing function defined on $[0, \infty)$ and positive on $(0, \infty)$,

and

$$\int_0^\infty \frac{ds}{s + \Omega(s)} = +\infty.$$

Then the initial value problem

$$x'(t) = A(t, x_t) + \int_0^t k(t, s)f(t, x_t)ds, \quad t \geq 0 \quad (2.1)'$$

$$x_0 = \phi \quad (2.2)'$$

has a solution defined on $[0, \infty)$.

Consider now the following special case of initial value problem (2.1)-(2.2), i.e.,

$$x'(t) = A(t)x(t) + \int_0^t k(t, s)f(t, x_t)ds, \quad 0 \leq t \leq T \quad (2.5)$$

$$x_0 = \phi, \quad (2.6)$$

where $A(t)$ is an $n \times n$ continuous matrix for $t \in [0, T]$ and f is a continuous mapping from $[0, T] \times C$ to R^n .

Any solution of this problem may be represented as follows:

$$x(t) = \Phi(t)\Phi^{-1}(t)\phi(0) + \int_0^t \Phi(t)\Phi^{-1}(t) \int_0^\tau k(t, s)f(s, x_s)dsd\tau, \quad 0 \leq t \leq T,$$

where $\Phi(t)$ is the fundamental matrix of solutions of the homogeneous system $x'(t) = A(t)x(t)$, $0 \leq t \leq T$. $\Phi(t)$ is extended to $[-r, 0]$ by I , the identity matrix.

Let $M_1 = \max\{\sup|\Phi(t)\Phi^{-1}(t)| : t, s \in [0, T], 1\}$. Using this formula, we obtain the following theorem proved earlier in [5].

Theorem 2.5: If (Hf) and (Hk) hold, then the initial value problem (2.5)-(2.6) has at least one solution on $[-r, T]$, provided that

$$MM_1 \int_0^T \int_0^t m(s) ds dt < \int_{M_1 \|\phi\|}^{\infty} \frac{ds}{\Omega(s)}.$$

3. BVP For Volterra Functional Integro-Differential Equations

Consider in this section the following BVP for nonlinear Volterra type integro-differential equations

$$x'(t) = A(t, x_t) + \int_0^t k(t, s) f(s, x_s) ds, \quad t \in [0, T] \quad (3.1)$$

$$Lx = h, \quad (3.2)$$

where A, f and k are as in the previous section and L is a bounded linear operator from a Banach space $C([-r, T], R^n)$ into R^n and $h \in \text{Im} L$, is the image of L .

We will now introduce some necessary preliminaries. Consider a linear nonhomogeneous system of differential equations

$$x'(t) = A(t, x_t) + g(t) \quad (3.3)$$

$$x_0 = \phi \quad (3.4)$$

for which we assume that (HA) holds.

For any initial function $\phi \in C$ we denote by $x(\phi, g)(t)$, the solution of (3.3) satisfying $x(\phi, g) = \phi$. For each $\phi \in C$ and g as above, the initial value problem (3.3)-(3.4) has a unique solution $x(\phi, g)$ defined on $[-r, T]$ such that

$$x(\phi, g)(t) = x(\phi, 0)(t) + \int_0^t U(t, s) g(s) ds, \quad t \in [0, T], \quad (3.5)$$

where $U(t, s)$ is the fundamental matrix of $x'(t) = A(t, x_t)$. Denote by $|U(t, s)|$, the operator norm of the matrix $U(t, s)$ and set

$$P = \sup\{|U(t, s)| : 0 \leq s, t \leq T\}.$$

Set $S: C \rightarrow C([-r, T], R^n)$ be the solution mapping defined by

$$S\phi = x(\phi, 0).$$

Then S is a bounded linear operator and hence the composite mapping $L_S = LS$ is a bounded linear operator from C into R^n . We assume that

(HL) There exists a bounded linear operator $L_S^*: R^n \rightarrow C$ such that $L_S L_S^* L_S = L_S$.

Therefore L_S^* is the generalized inverse of L_S . Then any solution to the BVP (3.1)-(3.2) is a fixed point of the operator F with

$$Fx = F_1 x + F_2 x,$$

where

$$(F_1x)(t) = SL_S^*(h - LF_2x)(t), \quad -r \leq t \leq T, \quad (3.6)$$

and

$$F_2x(t) = \begin{cases} 0, & -r \leq t \leq 0 \\ \int_0^t \int_0^\tau U(t, s)k(t, s)f(s, x_s)dsd\tau, & 0 \leq t \leq T. \end{cases} \quad (3.7)$$

For a proof of this fact, the reader is referred to Kaminogo [4].

Now, we present our main result on the existence of solutions of the BVP (3.1)-(3.2).

Theorem 3.1: Assume that (HA) , (Hk) , (Hf) and (HL) hold, then, if

$$MP(|SL_S^*| |L| + 1) \int_0^T \int_0^\tau m(s)dsd\tau < \int_c^\infty \frac{ds}{\Omega(s)}, \quad c = \max\{|SL_S^*| |h|, \|\phi\|\}$$

the BVP (3.1)-(3.2) has at least one solution on $[-r, T]$.

Proof: To prove the existence of a solution of the BVP (3.1)-(3.2), we apply Lemma 2.1. In order to apply this lemma, we must establish the a priori bounds for the BVP (3.1) $_{\lambda}$ -(3.2) $_{\lambda}$. Let x be a solution of the BVP (3.1) $_{\lambda}$ -(3.2) $_{\lambda}$. Then,

$$x(t) = \lambda\{SL_S^*(h - LF_2x)(t) + (F_2x)(t)\}, \quad t \in [0, T]$$

where $F_2(t)$ is given by (3.6). From this, we get

$$\begin{aligned} |x(t)| &\leq |SL_S^*|(|h| + |L|P \int_0^t \int_0^\tau m(s)\Omega(\|x_s\|)dsd\tau) \\ &\quad + P \int_0^t \int_0^\tau m(s)\Omega(\|x_s\|)dsd\tau \\ &\leq |SL_S^*| |h| + P(|SL_S^*| |L| + 1) \int_0^t \int_0^\tau m(s)\Omega(\|x_s\|)dsd\tau, \quad 0 \leq t \leq T. \end{aligned}$$

As in Theorem 2.3, we consider the function μ given by

$$\mu(t) = \sup\{|x(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq T.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in [0, t]$, by the previous inequality we have

$$\begin{aligned} \mu(t) = |x(t^*)| &\leq |SL_S^*| |h| + P(|SL_S^*| |L| + 1) \int_0^t \int_0^\tau m(s)\Omega(\mu(s))dsd\tau \\ &\leq c + P(|SL_S^*| |L| + 1) \int_0^t \int_0^\tau m(s)\Omega(\mu(s))dsd\tau, \end{aligned}$$

where $c = \max\{|SL_S^*| |h|, \|\phi\|\}$.

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|$ and the previous inequality obviously holds true.

Denoting by $u(t)$ the right-hand side of the above inequality, we have

$$\mu(t) \leq u(t), \quad 0 \leq t \leq T,$$

$$u(0) = c,$$

and

$$\begin{aligned} u'(t) &= P(|SL_S^*| |L| + 1) \int_0^t m(s) \Omega(\mu(s)) ds \\ &\leq P(|SL_S^*| |L| + 1) \int_0^t m(s) \Omega(u(s)) ds \\ &\leq P(|SL_S^*| |L| + 1) \Omega(u(t)) \int_0^t m(s) ds, \quad 0 \leq t \leq T. \end{aligned}$$

or

$$\frac{v'(t)}{\Omega(u(t))} \leq P(|SL_S^*| |L| + 1) \int_0^t m(s) ds, \quad 0 \leq t \leq T.$$

Then,

$$\int_{u(0)}^{u(t)} \frac{ds}{\Omega(s)} \leq P(|SL_S^*| |L| + 1) \int_0^T \int_0^\tau m(s) ds d\tau < \int_{u(0)}^\infty \frac{ds}{\Omega(s)}, \quad 0 \leq t \leq T.$$

This inequality implies that there is a constant K such that $u(t) \leq K$, $t \in [0, T]$, and hence $\mu(t) \leq K$, $t \in [0, T]$. Since for every $t \in [0, T]$, $\|x_t\| \leq \mu(t)$, we have

$$\|x\|_1 \leq K,$$

where K depends only on T and the functions m and Ω .

In the second step, we notice that any solution of the BVP (3.1)-(3.2) is a fixed point of the operator F with

$$Fx = SL_S^*(h - LF_2x) + F_2x$$

which is a completely continuous operator ([4]).

Finally, the set $E(F) = \{x \in B: x = \lambda Fx \text{ for some } 0 < \lambda < 1\}$ is bounded, since in the first step we have proved that $\|x\|_1 \leq K$.

Consequently, by Lemma 2.1, the BVP (3.1)-(3.2) has at least one solution, completing the proof of the theorem.

We shall now consider equation (3.1) when the linear part $A(t, x_t)$ is not a functional on C . More precisely, we shall consider the functional differential equation of the form

$$x'(t) = A(t)x(t) + \int_0^t k(t,s)f(s, x_s)ds, \quad t \in [0, T], \quad (3.8)$$

where $A(t)$ is a continuous $n \times n$ matrix for $t \in [0, T]$.

Let us assume that $\Phi(t)$ is the fundamental matrix of solutions of the homogeneous system

$$x'(t) = A(t)x(t), \quad 0 \leq t \leq T \quad (3.9)$$

with $\Phi(0) = I$, the identity matrix. $\Phi(t)$ is extended to $[-r, 0]$ by I . We denote by L_0 , the $n \times n$ matrix whose elements are the values of L on the corresponding columns of $\Phi(t)$. Assume that L_0 is nonsingular with inverse L_0^{-1} . Then it is well known (Opial [6]) that:

(I) The BVP (3.8)-(3.2) has a solution for any $h \in R^n$, if and only if, the corresponding homogeneous BVP

$$x'(t) = A(t)x(t)$$

$$Lx = 0$$

has only the trivial solution $x(t) = 0$.

(II) The solution of the BVP (3.8)-(3.2) is unique and is given by the explicit formula

$$x(t) = \Phi(t)L_0^{-1}(h - LF_2(t)) + F_2(t),$$

where

$$F_2(t) = \begin{cases} 0, & -r \leq t \leq 0 \\ \int_0^t \Phi(t) \int_0^\tau \Phi^{-1}(s)k(t,s)f(s, x_s)dsd\tau, & 0 \leq t \leq T. \end{cases}$$

Let

$$\alpha = \sup\{|\Phi(t)| : 0 \leq t \leq T\},$$

$$\beta = \sup\{|\Phi^{-1}(t)| : 0 \leq t \leq T\}.$$

Then we have:

Theorem 3.2: Assume that (Hf) and (Hk) hold. Assume also that the linear operator L is such that the operator L_0 has a bounded inverse L_0^{-1} .

Then if

$$\alpha\beta M(\alpha | L_0^{-1} | | L | + 1) \int_0^T \int_0^t m(s)ds < \int_c^\infty \frac{ds}{\Omega(s)}, \quad (3.10)$$

where $c = \max\{\alpha | L_0^{-1} | | h |, \| \phi \| \}$, the BVP (3.8)-(3.2) has at least one solution.

Proof: The proof is similar to that of the previous theorem and it is omitted.

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