THE METHOD OF LOWER AND UPPER SOLUTIONS FOR n th-ORDER PERIODIC BOUNDARY VALUE PROBLEMS^{1,2}

ALBERTO CABADA

*Universidade de Santiago de Compostela
Departamento de Análise Matemática
Facultade de Matemáticas, SPAIN*

ABSTRACT

In this paper we develop the monotone method in the presence of lower and upper solutions for the problem

$$u^{(n)}(t) = f(t, u(t)); u^{(i)}(a) - u^{(i)}(b) = \lambda_i \in \mathbb{R}, i = 0, \dots, n-1,$$

where f is a Carathéodory function. We obtain sufficient conditions for f to guarantee the existence and approximation of solutions between a lower solution α and an upper solution β for $n \geq 3$ with either $\alpha \leq \beta$ or $\alpha \geq \beta$.

For this, we study some maximum principles for the operator $Lu \equiv u^{(n)} + Mu$. Furthermore, we obtain a generalization of the method of mixed monotonicity considering f and u as vectorial functions.

Key words: Periodic boundary value problem, lower and upper solutions, monotone method.

AMS (MOS) subject classifications: 34B15, 34C25.

1. INTRODUCTION

In this paper we study the following class of boundary value problems for the ordinary differential equations:

$$u^{(n)}(t) = f(t, u(t)) \text{ for a.e. } t \in I = [a, b] \quad (1.1)$$

$$u^{(i)}(a) - u^{(i)}(b) = \lambda_i \in \mathbb{R}; \quad i = 0, 1, \dots, n-1, \quad (1.2)$$

for $n \geq 3$ where f is a Carathéodory function.

Definition 1.1: We say that $f: I \times \mathbb{R}^l \rightarrow \mathbb{R}^m$ is a *Carathéodory function*, if $f \equiv (f_1, \dots, f_m)$ satisfies the following properties:

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1. $f_i(\cdot, x)$ is measurable for all $x \in \mathbb{R}^l$ and $i \in \{1, \dots, m\}$.
2. $f_i(t, \cdot)$ is continuous for a.e. $t \in I$.
3. For every $R > 0$ and $i \in \{1, \dots, m\}$, there exists $h_{i,R} \in L^1(I)$ such that:

$$|f_i(t, x)| \leq h_{i,R}(t) \text{ for a.e. } t \in I,$$

with $\|x\| \leq R$.

To develop the monotone method we use the concept of lower and upper solutions:

Definition 1.2: Let $\alpha \in W^{n,1}(I)$, we say that α is a *lower solution* for the problem (1.1)-(1.2) if α satisfies

$$\begin{aligned} \alpha^{(n)}(t) &\geq f(t, \alpha(t)) \text{ for a.e. } t \in I \\ \alpha^{(i)}(a) - \alpha^{(i)}(b) &= \lambda_i; \quad i = 0, 1, \dots, n-2 \\ \alpha^{(n-1)}(a) - \alpha^{(n-1)}(b) &\geq \lambda_{n-1}. \end{aligned}$$

Definition 1.3: Let $\beta \in W^{n,1}(I)$, we say that β is an *upper solution* for the problem (1.1)-(1.2) if β satisfies

$$\begin{aligned} \beta^{(n)}(t) &\leq f(t, \beta(t)) \text{ for a.e. } t \in I \\ \beta^{(i)}(a) - \beta^{(i)}(b) &= \lambda_i; \quad i = 0, 1, \dots, n-2 \\ \beta^{(n-1)}(a) - \beta^{(n-1)}(b) &\leq \lambda_{n-1}. \end{aligned}$$

We suppose that f satisfies one of the following conditions, depending on various circumstances:

- (H_1) $f(t, x) - f(t, y) \leq M(x - y)$ for a.e. $t \in I$ with $\alpha(t) \leq y \leq x \leq \beta(t)$ and $M > 0$.
- (H_2) $f(t, x) - f(t, y) \geq M(x - y)$ for a.e. $t \in I$ with $\beta(t) \leq y \leq x \leq \alpha(t)$ and $M < 0$.

This problem has been studied by different authors for second order equations when $\alpha \leq \beta$ ([1]-[4], [6], [8], [10], [11]). If $\alpha \geq \beta$ the monotone method is not valid if f satisfies the condition (H_2) for some $M < 0$ ([2], [7], [12], [14]).

For $n \geq 3$ the method of lower and upper solutions has been little studied ([2], [9], [13]). In [2] the author obtains the best value on the constant M for

$n = 2$, $n = 3$ and $n = 4$ (in this last case, if $M < 0$) for which the conditions (H_1) or (H_2) imply that the monotone method is valid.

To prove the validity of the monotone method to more general cases, we present some maximum principles for the operator

$$L_n: F_{a,b}^n \rightarrow L^1(I),$$

defined by $L_n u = u^{(n)} + Mu$. Where M is a real constant different from zero, and

$$F_{a,b}^n = \left\{ u \in W^{n,1}(I); u^{(i)}(a) = u^{(i)}(b), i = 0, \dots, n-2; u^{(n-1)}(a) \geq u^{(n-1)}(b) \right\}.$$

We say that an operator L is **inverse positive** in $F_{a,b}^n$ if $Lu \geq 0$ implies $u \geq 0$ for all $u \in F_{a,b}^n$ and that L is **inverse negative** in $F_{a,b}^n$ if $Lu \geq 0$ implies $u \leq 0$ for all $u \in F_{a,b}^n$.

In Section 2, we obtain a new maximum principle for the operator L_n , using that this operator is given by the composition of the operators of first and second order.

This result is used in Section 3 to extend to more general cases the validity of the monotone method for the problem (1.1)-(1.2) and in Section 4 it is applied to obtain a new generalization of the method of mixed monotony [5] when f and u are vectorial functions.

2. MAXIMUM PRINCIPLES

In this section we improve the following result obtained in [2], which generalizes theorem 4 in [15].

Lemma 2.1: *Let $A(n) \equiv \frac{n^n n!}{\left[\frac{n}{2}\right]^n (b-a)^n (n-1)^{n-1}}$, where $\left[\frac{n}{2}\right]$ is the integer part of $\frac{n}{2}$. Then if $M \in (0, A(n)]$ ($M \in [-A(n), 0)$), the operator L_n is inverse positive (inverse negative) on $F_{a,b}^n$.*

Furthermore, if $M \in [-[A(n)]^2, 0)$ the operator L_{2n} is inverse negative on $F_{a,b}^{2n}$.

For this, we use the following known result.

Lemma 2.2:

1. L_1 is inverse positive (inverse negative) on $F_{a,b}^1$ for all $M > 0$ ($M < 0$).
2. L_2 is inverse negative on $F_{a,b}^2$ for all $M < 0$.
3. ([13], Lemma 2.1) The operator $N_{A,B}u = u'' - 2Au' + (A^2 + B^2)u$ is inverse positive on $F_{0,2\pi}^2$ if and only if $(0 <) B \leq \frac{1}{2}$.

Now, we prove the following preliminary lemma.

Lemma 2.3: Let $Lu = u^{(n)} + \sum_{i=0}^{n-1} a_i u^{(i)}$ and $Nu = u^{(m)} + \sum_{i=0}^{m-1} b_i u^{(i)}$. Then if L is inverse positive on $F_{a,b}^n$ and N is inverse positive (inverse negative) on $F_{a,b}^m$ then $L \circ N$ is inverse positive (inverse negative) on $F_{a,b}^{n+m}$.

Proof: Since $u \in F_{a,b}^{n+m}$ it is clear that

$$(Nu)^{(i)}(a) = (Nu)^{(i)}(b), \quad i = 0, \dots, n-2$$

and

$$(Nu)^{(n-1)}(a) \geq (Nu)^{(n-1)}(b).$$

In consequence, since L is inverse positive on $F_{a,b}^n$, we have that $Nu \geq 0$. Now, using that N is inverse positive (inverse negative) on $F_{a,b}^m$, we obtain that $u \geq 0$ ($u \leq 0$). $\square$

Thus, we are in position to prove the following lemma.

Lemma 2.4: Let $M > 0$. The following properties hold:

1. Let $n = 4k$, $k \in \{1, 2, \dots\}$.
If $M \leq \left[\frac{\pi}{(b-a)\sin\left(\frac{n+2}{2n}\pi\right)} \right]^n$, then L_n is inverse positive on $F_{a,b}^n$.
2. Let $n = 2 + 4k$, $k \in \{1, 2, \dots\}$.
If $M \leq \left[\frac{\pi}{b-a} \right]^n$, then L_n is inverse positive on $F_{a,b}^n$.
3. Let n be odd.
If $M \leq \left[\frac{\pi}{(b-a)\sin\left(\frac{n+1}{2n}\pi\right)} \right]^n$, then L_n is inverse positive on $F_{a,b}^n$.

Proof: Since, if $u \in W^{n,1}(I)$ satisfies

$$L_n u(t) = \sigma(t), \quad u^{(i)}(a) = u^{(i)}(b), \quad i = 0, \dots, n-2 \quad \text{and} \quad u^{(n-1)}(a) - u^{(n-1)}(b) = \lambda,$$

then $v(t) = \left(\frac{2\pi}{b-a} \right)^{n-1} u \left(\frac{b-a}{2\pi} t + a \right)$ satisfies

$$v^{(n)}(t) + \left(\frac{b-a}{2\pi}\right)^n Mv(t) = \left(\frac{b-a}{2\pi}\right) \sigma\left(\frac{b-a}{2\pi}t + a\right),$$

with

$$v^{(i)}(0) = v^{(i)}(2\pi), \quad i = 0, \dots, n-2 \quad \text{and} \quad v^{(n-1)}(0) - v^{(n-1)}(2\pi) = \lambda.$$

It is sufficient to study the operator L_n on $F_{0,2\pi}^n$ because to obtain the estimate on the interval $[a, b]$ we multiply by $\left(\frac{2\pi}{b-a}\right)^n$ the estimate obtained on $[0, 2\pi]$.

Let $m > 0$ such that $m^n = M$.

First, we suppose that n is even.

In this case the polynomial function $p(\lambda) = \lambda^n + m^n = 0$ if and only if

$$\lambda = \lambda_l = m \left[\cos\left(\frac{2l+1}{n}\pi\right) \pm i \sin\left(\frac{2l+1}{n}\pi\right) \right] \equiv a_l \pm i\beta_l,$$

$$l = 0, 1, \dots, \frac{n-2}{2}.$$

As consequence we have that

$$\lambda^n + m^n = \prod_{l=0}^{\frac{n-2}{2}} (\lambda^2 - 2\alpha_l \lambda + m^2),$$

and

$$L_n \equiv T_0 \circ T_1 \circ \dots \circ T_{\frac{n-2}{2}}. \quad (2.3)$$

Where $T_l u = u'' - 2\alpha_l u' + m^2 u$.

If $n = 4k$ for some $k \in \{1, 2, \dots\}$, then $\beta_l \leq \beta_{\frac{n}{4}} = m \sin\left(\frac{n+2}{2n}\pi\right)$ for all $l \in \{0, 1, \dots, \frac{n-2}{2}\}$. Thus, using lemma 2.2, if $m \leq \left[2 \sin\left(\frac{n+2}{2n}\pi\right)\right]^{-1}$ the operator T_l is inverse positive on $F_{0,2\pi}^2$ for all $l \in \{0, 1, \dots, \frac{n-2}{2}\}$. Therefore lemma 2.3 implies that L_n is inverse positive on $F_{0,2\pi}^n$.

If $n = 2 + 4k$ for some $k \in \{1, 2, \dots\}$, then $\beta_l \leq \beta_{\frac{n-2}{4}} = m$ for all $l \in \{0, 1, \dots, \frac{n-2}{2}\}$ and as a consequence, T_l is inverse positive on $F_{0,2\pi}^2$ when $m \leq \frac{1}{2}$. By (2.3) and the two previous lemmas, we obtain that L_n is inverse positive on $F_{0,2\pi}^n$.

Now, we suppose that n is odd.

In this case, $p(\lambda) = 0$ if and only if $\lambda = -m$ or $\lambda = \lambda_l = \alpha_l \pm i\beta_l$,

$l = 0, \dots, \frac{n-3}{2}$. Thus

$$\lambda^n + m^n = (\lambda + m) \prod_{l=0}^{\frac{n-3}{2}} (\lambda^2 - 2\alpha_l \lambda + m^2),$$

and

$$L_n \equiv T_0 \circ T_1 \circ \dots \circ T_{\frac{n-3}{2}} \circ S_1.$$

Where $S_1 u = u' + mu$.

In this case $\beta_l \leq \beta_{\frac{n-1}{4}} = m \sin\left(\frac{n+1}{2n}\pi\right)$ for all $l \in \{0, 1, \dots, \frac{n-3}{2}\}$. Thus, if $m \geq [2 \sin\left(\frac{n+1}{2n}\pi\right)]^{-1}$ lemmas 2.2 and 2.3 imply that the operator L_n is inverse positive on $F_{0,2\pi}^n$. $\square$

Analogously we can prove the following result for $M < 0$.

Lemma 2.5: *Let $M < 0$. The following properties hold:*

1. *Let $n = 4k$, $k \in \{1, 2, \dots\}$.
If $M \geq -\left[\frac{\pi}{b-a}\right]^n$ then L_n is inverse negative on $F_{a,b}^n$.*
2. *Let $n = 2 + 4k$, $k \in \{0, 1, \dots\}$.
If $M \geq -\left[\frac{\pi}{(b-a) \sin\left(\frac{n+2}{2n}\pi\right)}\right]^n$ then L_n is inverse negative on $F_{a,b}^n$.*
3. *Let n be odd.
If $M \geq -\left[\frac{\pi}{(b-a) \sin\left(\frac{n+1}{2n}\pi\right)}\right]^n$ then L_n is inverse negative on $F_{a,b}^n$.*

Remark 2.1: Note that these estimates are not the best possible for all $n \in \mathbb{N}$.

In [2] it is proved that L_3 is inverse positive (inverse negative) on $F_{0,2\pi}^3$ if and only if $M \in (0, M_3^3](M \in [-M_3^3, 0))$. Where M_3 is the unique solution of the equation

$$\arctan\left(\frac{\sin\sqrt{3} m\pi}{\cos\sqrt{3} m\pi - e^{m\pi}}\right) + \pi = \frac{\sqrt{3}}{3} \log\left(\frac{e^{3m\pi} - e^{m\pi}}{\sqrt{1 + e^{2m\pi} - 2e^{m\pi} \cos\sqrt{3} m\pi}}\right).$$

Furthermore, L_4 is inverse negative on $F_{0,2\pi}^4$ if and only if $M \in [-M_4^4, 0)$, with M_4 given as the unique solution in $(\frac{1}{2}, 1)$ of the equation

$$-\tanh m\pi = \tan m\pi.$$

Note that the estimates obtained in lemmas 2.4 and 2.5 are the best possible for $n = 1$ and $n = 2$.

3. THE MONOTONE METHOD

In this section we study the existence of solutions of the problem (1.1)-(1.2) in the sector $[\alpha, \beta]$ or $[\beta, \alpha]$, where $[v, w] = \{u \in L^1(I) : v \leq u \leq w \text{ on } I\}$. We improve the following result given in [2], which generalizes theorem 5 in [15].

Theorem 3.1: *The following properties hold.*

1. *If there exists $\alpha \leq \beta$ ($\alpha \geq \beta$) lower and upper solutions respectively of the problem (1.1)-(1.2), and f satisfies the condition (H_1) ((H_2)) for some $M \in (0, A(n)]$ ($M \in [-A(n), 0)$) then there exists a solution of the problem (1.1)-(1.2) in $[\alpha, \beta]$ ($[\beta, \alpha]$). Furthermore, there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$ which converge uniformly to the extremal solutions in $[\alpha, \beta]$ ($[\beta, \alpha]$) of the problem (1.1)-(1.2).*
2. *The previous property is true when n is even and f satisfies the condition (H_2) for some $M \in (-[A(\frac{n}{2})]^2, 0]$.*

Using lemma 2.4 we prove the following result.

Theorem 3.2: *If there exists $\alpha \geq \beta$ lower and upper solutions respectively of the problem (1.1)-(1.2) and if any of the following properties are true:*

1. *Let $n = 4k$, $k \in \{1, 2, \dots\}$. Suppose that f satisfies the property (H_2) for some $M \in [-\left[\frac{\pi}{(b-a)\sin\left(\frac{n+2}{2n}\pi\right)}\right]^n, 0)$.*
2. *Let $n = 2 + 4k$, $k \in \{1, 2, \dots\}$. Suppose that f satisfies the property (H_2) for some $M \in [-\left[\frac{\pi}{b-a}\right]^n, 0)$.*
3. *Let n be odd. Suppose that f satisfies the property (H_2) for some $M \in [-\left[\frac{\pi}{(b-a)\sin\left(\frac{n+1}{2n}\pi\right)}\right]^n, 0)$.*

Then there exists u a solution of the problem (1.1)-(1.2) in $[\beta, \alpha]$.

Furthermore, there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly to the extremal solutions in $[\beta, \alpha]$ of

the problem (1.1)-(1.2).

Proof: We consider the problem:

$$u^{(n)}(t) - Mu(t) = f(t, \eta(t)) - M\eta(t) \text{ for a.e. } t \in I \quad (3.1)$$

$$u^{(i)}(a) - u^{(i)}(b) = \lambda_i, \quad i = 0, 1, \dots, n-1 \quad (3.2)$$

with $\eta \in L^1(I)$, $\beta(t) \leq \eta(t) \leq \alpha(t)$.

We have:

$$\begin{aligned} (\alpha - u)^{(n)}(t) - M(\alpha - u)(t) &\geq -f(t, \eta(t)) \\ &\quad + M\eta(t) + f(t, \alpha(t)) - M\alpha(t) \geq 0 \\ (\alpha - u)^{(i)}(a) - (\alpha - u)^{(i)}(b) &= 0; \quad i = 0, \dots, n-2 \\ (\alpha - u)^{(n-1)}(a) - (\alpha - u)^{(n-1)}(b) &\geq 0. \end{aligned}$$

Lemma 2.4 implies that $u \leq \alpha$.

Analogously we can prove that $u \geq \beta$.

Let $u_i = Q\eta_i$ the unique solution of the problem (3.1)-(3.2) for $\eta = \eta_i \in L^1(I)$. Since for $\beta \leq \eta_1 \leq \eta_2 \leq \alpha$,

$$\begin{aligned} (u_2 - u_1)^{(n)}(t) - M(u_2 - u_1)(t) &= f(t, \eta_2(t)) \\ &\quad - M\eta_2(t) - f(t, \eta_1(t)) + M\eta_1(t) \geq 0 \\ (u_2 - u_1)^{(i)}(a) - (u_2 - u_1)^{(i)}(b) &= 0; \quad i = 0, \dots, n-1, \end{aligned}$$

the following property holds:

$$\text{If } \beta \leq \eta_1 \leq \eta_2 \leq \alpha \text{ then } u_1 = Q\eta_1 \leq Q\eta_2 = u_2.$$

The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are obtained by recurrence: $\alpha_0 = \alpha$, $\beta_0 = \beta$, $\alpha_n = Q\alpha_{n-1}$ and $\beta_n = Q\beta_{n-1}$; $n \geq 1$.

By standard arguments we prove that $\{\alpha_n\}$ and $\{\beta_n\}$ converge to the extremal solutions on $[\beta, \alpha]$ of the problem (1.1)-(1.2). $\square$

Analogously, using lemma 2.5 we can prove the following theorem.

Theorem 3.3: *If there exists $\alpha \leq \beta$ lower and upper solutions respectively of the problem (1.1)-(1.2) and any of the following properties are*

verified:

1. Let $n = 4k$, $k \in \{1, 2, \dots\}$. Suppose that f satisfies the property (H_1) for some $M \in (0, [\frac{\pi}{b-a}]^n]$.
2. Let $n = 2 + 4k$, $k \in \{1, 2, \dots\}$. Suppose that f satisfies the property (H_1) for some $M \in (0, [\frac{\pi}{(b-a)\sin(\frac{n+2}{2n}\pi)}]^n]$.
3. Let n be odd. Suppose that f satisfies the property (H_1) for some $M \in (0, [\frac{\pi}{(b-a)\sin(\frac{n+1}{2n}\pi)}]^n]$.

Then there exists a solution of the problem (1.1)-(1.2) in $[\alpha, \beta]$.

Furthermore there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$ which converge uniformly to the extremal solutions in $[\alpha, \beta]$ of the problem (1.1)-(1.2).

Remark 3.1: Similarly to the remark 2.1, note that the estimates obtained for the function f in theorems 3.2 and 3.3 are not the best possible for all $n \in \mathbb{N}$.

4. THE METHOD OF MIXED MONOTONY

In this section we study the method of mixed monotony, studied by Khavanin and Lakshmikantham in [5], in which they consider the initial and periodic first order problems. In this case, under stronger conditions on the function f it is possible to guarantee the unicity of the solution when we have an n th-order system.

In [5] the following results are obtained.

Theorem 4.1: Consider the following system

$$u'(t) = f(t, u(t)); t \in [0, T]$$

with $f \in C([0, T] \times \mathbb{R}^N, \mathbb{R}^N)$.

If there exists $F \in C([0, T] \times \mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$, $\alpha, \beta \in C^1([0, T], \mathbb{R}^N)$ which satisfy the following conditions:

- (i) $\alpha'(t) \geq F(t, \alpha(t), \beta(t))$, $\beta'(t) \leq F(t, \beta(t), \alpha(t))$. With $\beta \leq \alpha$ on $[0, T]$.
- (ii) $F(t, u, v)$ is nondecreasing on u and nonincreasing on v .
- (iii) $F(t, u, u) = f(t, u)$ and

$$-B(z_1 - z_2) \leq F(t, y_1, z_1) - F(t, y_2, z_2) \leq B(y_1 - y_2),$$

with $\beta(t) \leq y_2 \leq y_1 \leq \alpha(t)$, $\beta(t) \leq z_2 \leq z_1 \leq \alpha(t)$ and B an $N \times N$ matrix with nonnegative elements.

Then:

If $\beta(0) \leq u_0 \leq \alpha(0)$, then there exist two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ nonincreasing and nondecreasing respectively which converge uniformly to the unique solution of the problem

$$u'(t) = f(t, u(t)); u(0) = u_0.$$

Furthermore, if $(T = 2\pi)$ $\beta(0) \leq \beta(2\pi)$ and $\alpha(0) \geq \alpha(2\pi)$ with $I \neq e^{2B\pi}$ the same result is valid for the problem

$$u'(t) = f(t, u(t)); u(0) = u(2\pi).$$

Theorem 4.2: If there exists $\alpha, \beta \in C^1([0, T], \mathbb{R}^N)$, with $\beta \leq \alpha$ on $[0, T]$ verifying:

$$\alpha'(t) \geq f(t, \alpha(t)) + B(\alpha(t) - \beta(t)) \text{ and } \beta'(t) \leq f(t, \beta(t)) - B(\alpha(t) - \beta(t)),$$

and f satisfies

$$-B(x - y) \leq f(t, x) - f(t, y) \leq B(x - y),$$

with $\beta(t) \leq y \leq x \leq \alpha(t)$, where B is an $N \times N$ matrix with nonnegative elements, then the conclusions of theorem 4.1 are valid.

Using lemma 2.5 we prove the following result.

Theorem 4.3: Let

$$u^{(n)}(t) = f(t, u(t)) \text{ for a.e. } t \in [a, b] \quad (4.1)$$

$$u_j^{(i)}(a) - u_j^{(i)}(b) = \lambda_{i,j} \in \mathbb{R}, \quad i = 0, \dots, n-1; \quad j = 1, \dots, N, \quad (4.2)$$

with $f: I \times \mathbb{R}^N \rightarrow \mathbb{R}$ a Carathéodory function and $n \geq 2$.

If there exists a Carathéodory function $F: I \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ and $\alpha, \beta \in W^{n,1}(I, \mathbb{R}^N)$, $\alpha \leq \beta$ on I , verifying the following properties:

(i)
$$\alpha^{(n)}(t) \geq F(t, \alpha(t), \beta(t)) \text{ for a.e. } t \in I$$

$$\begin{aligned}\alpha_j^{(i)}(a) - \alpha_j^{(i)}(b) &= \lambda_{i,j}; i = 0, \dots, n-2; j = 1, \dots, N \\ \alpha_j^{(n-1)}(a) - \alpha_j^{(n-1)}(b) &\geq \lambda_{n-1,j}; j = 1, \dots, N.\end{aligned}$$

(ii)

$$\begin{aligned}\beta^{(n)}(t) &\leq F(t, \beta(t), \alpha(t)) \text{ for a.e. } t \in I \\ \beta_j^{(i)}(a) - \beta_j^{(i)}(b) &= \lambda_{i,j}; i = 0, \dots, n-2; j = 1, \dots, N \\ \beta_j^{(n-1)}(a) &= \beta_j^{(n-1)}(b) \leq \lambda_{n-1,j}; j = 1, \dots, N.\end{aligned}$$

(iii) $F(t, u, v)$ is nonincreasing on u and nondecreasing on v .

(iv) $F(t, u, u) = f(t, u)$ and

$$F(t, y, z) - F(t, z, y) = -B(y - z),$$

B being an $N \times N$ matrix with nonnegative elements such that $\exp(C(b-a)) \neq I$. Where C is given by the expression

$$C \equiv \left(\begin{array}{c|c} 0 & I_{(n-1)N} \\ \hline -B & 0 \end{array} \right).$$

Here $I_{(n-1)N}$ is the $(n-1)N \times (n-1)N$ identity matrix.

Then there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$, with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly to the unique solution of the problem (4.1)-(4.2).

Proof: Let $M_1 = -\left[\frac{\pi}{b-a}\right]^n$ and $\eta, \nu \in L^1(I, \mathbb{R}^N)$, $\eta, \nu \in [\alpha, \beta]$.

Consider the following linear problem for each $j = 1, \dots, N$:

$$u_j^{(n)}(t) + M_1 u_j(t) = F_j(t, \eta(t), \nu(t)) + M_1 \eta_j(t) \text{ for a.e. } t \in [a, b] \quad (4.3)$$

$$u_j^{(i)}(a) - u_j^{(i)}(b) = \lambda_{i,j} \in \mathbb{R}, i = 0, \dots, n-1; j = 1, \dots, N. \quad (4.4)$$

Let $u = A[\eta, \nu]$ be the unique solution of the problem (4.3)-(4.4) for each η, ν .

First, we prove that $\alpha \leq A[\alpha, \beta] = \alpha_1$,

$$\begin{aligned}(\alpha_j^{(n)} - \alpha_{1,j}^{(n)})(t) + M_1(\alpha_j - \alpha_{1,j})(t) &\geq 0 \\ (\alpha_j^{(i)} - \alpha_{1,j}^{(i)})(a) - (\alpha_j^{(i)} - \alpha_{1,j}^{(i)})(b) &= 0; i = 0, \dots, n-2 \\ (\alpha_j^{(n-1)} - \alpha_{1,j}^{(n-1)})(a) - (\alpha_j^{(n-1)} - \alpha_{1,j}^{(n-1)})(b) &\geq 0.\end{aligned}$$

Thus, lemma 2.5 implies that $\alpha \leq \alpha_1$ on I .

Similarly, we obtain that $\beta \geq \beta_1 = A[\beta, \alpha]$.

Let $\eta_1, \eta_2, \nu \in [\alpha, \beta]$, with $\eta_1 \leq \eta_2$. Let $u_1 = A[\eta_1, \nu]$ and $u_2 = A[\eta_2, \nu]$. We have that

$$\begin{aligned} (u_{1,j} - u_{2,j})^{(n)}(t) + M_1(u_{1,j} - u_{2,j})(t) &= F_j(t, \eta_1, \nu) + M_1\eta_{1,j} \\ &\quad - F_j(t, \eta_2, \nu) - M_1\eta_{2,j} \geq 0 \end{aligned}$$

which implies that $u_1 \leq u_2$.

Analogously, one can prove that $A[\eta, \nu_1] \leq A[\eta, \nu_2]$ if $\nu_1 \geq \nu_2$.

It is now easy to define the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$, $\alpha_{n+1} = A[\alpha_n, \beta_n]$ and $\beta_{n+1} = A[\beta_n, \alpha_n]$.

Clearly, $\alpha \leq \alpha_1 \leq \dots \leq \alpha_n \leq \beta_n \leq \dots \leq \beta_1 \leq \beta$ on I .

By standard arguments we can show that $\lim_{n \rightarrow \infty} \alpha_n = \phi$ and $\lim_{n \rightarrow \infty} \beta_n = \psi$ exist uniformly on I and ϕ and ψ satisfy

$$\begin{aligned} \phi^{(n)}(t) &= F(t, \phi, \psi), \quad \psi^{(n)}(t) = F(t, \psi, \phi) \\ \phi_j^{(i)}(a) - \phi_j^{(i)}(b) &= \psi_j^{(i)}(a) - \psi_j^{(i)}(b) = \lambda_{i,j}; \end{aligned}$$

$i = 0, \dots, n-1$; $j = 1, \dots, N$.

That is

$$(\phi - \psi)^{(n)}(t) = F(t, \phi, \psi) - F(t, \psi, \phi) = -B(\phi - \psi) \quad (4.5)$$

$$(\phi - \psi)^{(i)}(a) = (\phi - \psi)^{(i)}(b); \quad i = 0, \dots, n-1. \quad (4.6)$$

Now, we define $p(t) = ((\phi - \psi)(t), (\phi - \psi)'(t), \dots, (\phi - \psi)^{(n-1)}(t)) \in \mathbb{R}^{nN}$. Therefore $p' = Cp$, $p(a) = p(b)$. Since $p(b) = \exp(C(b-a))p(a)$, we obtain that $p \equiv 0$ and, in consequence, $\phi = \psi$. That is, $\phi^{(n)}(t) = F(t, \phi, \phi) = f(t, \phi)$, which concludes the proof. $\square$

Similarly, using lemma 2.4 we prove the following result.

Theorem 4.4: *The conclusions obtained in theorem 4.3 are valid if $\alpha \geq \beta$ and the properties (iii) and (iv) are changed by*

(iii)' $F(t, u, v)$ is nondecreasing on u and nonincreasing on v .

(iv)' $F(t, u, u) = f(t, u)$ and

$$F(t, y, z) - F(t, z, y) = B(y - z),$$

B being an $N \times N$ matrix with nonnegative elements as such that $\exp(D(b - a)) \neq I$, where D is defined as follows:

$$D \equiv \left(\begin{array}{c|c} 0 & I_{(n-1)N} \\ \hline -B & 0 \end{array} \right).$$

Here $I_{(n-1)N}$ is the $(n-1)N \times (n-1)N$ identity matrix.

As consequence of the two previous lemmas we prove the following result.

Theorem 4.5: Let $n \geq 2$. Suppose that there exist α and $\beta \in W^{n,1}(I, \mathbb{R}^N)$, $\alpha \leq \beta$ ($\alpha \geq \beta$) and f a Carathéodory function, satisfying

$$-B(x - y) \leq f(t, x) - f(t, y) \leq B(x - y),$$

with $y \leq x$ between $\alpha(t)$ and $\beta(t)$, where B is an $N \times N$ matrix with nonnegative elements.

If α and β satisfies

$$\alpha^{(n)}(t) \geq f(t, \alpha(t)) + B |\beta(t) - \alpha(t)| \text{ for a.e. } t \in I$$

$$\alpha_j^{(i)}(a) - \alpha_j^{(i)}(b) = \lambda_{i,j}; i = 0, 1, \dots, n-2, j = 1, \dots, N$$

$$\alpha_j^{(n-1)}(a) - \alpha_j^{(n-1)}(b) \geq \lambda_{n-1,j}; j = 1, \dots, N$$

and

$$\beta^{(n)}(t) \leq f(t, \beta(t)) - B |\beta(t) - \alpha(t)| \text{ for a.e. } t \in I$$

$$\beta_j^{(i)}(a) - \beta_j^{(i)}(b) = \lambda_{i,j}; i = 0, 1, \dots, n-2, j = 1, \dots, N$$

$$\beta_j^{(n-1)}(a) - \beta_j^{(n-1)}(b) \leq \lambda_{n-1,j}; j = 1, \dots, N.$$

And $\exp(C(b - a)) \neq I$ ($\exp(D(b - a)) \neq I$) (C and D given in theorems 4.3 and 4.4).

Then there exists a unique solution u between α and β of the problem (4.1)-(4.2). Furthermore, there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$, with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly to the solution u .

Proof: If $\alpha \leq \beta$ we define F as follows:

$$F(t, u, v) = \frac{1}{2}[f(t, u) + f(t, v) - B(u - v)].$$

It is easy to prove that the function F satisfies the conditions of theorem 4.3. If $\alpha \geq \beta$ the function F is defined as follows:

$$F(t, u, v) = \frac{1}{2}[f(t, u) + f(t, v) + B(u - v)].$$

Clearly, the function F satisfies the conditions of theorem 4.4. $\square$

REFERENCES

- [1] Bernfeld, S.R. and Lakshmikantham, V., *An Introduction to Nonlinear Boundary Value Problems*, Academic Press, New York 1974.
- [2] Cabada, A., The method of lower and upper solutions for second, third, fourth and higher order boundary value problems, *J. Math. Anal. Appl.* (to appear).
- [3] Cabada, A. and Nieto, J.J., A generalization of the monotone iterative technique for nonlinear second order periodic boundary value problems, *J. Math. Anal. Appl.* **151** (1990), 181-189.
- [4] Cabada, A. and Nieto, J.J., Extremal solutions of second order nonlinear periodic boundary value problems, *Appl. Math. Comput.* **40** (1990), 135-145.
- [5] Khavanin, M. and Lakshmikantham, V., The method of mixed monotony and first order differential systems, *Nonlinear Anal.* **10:9** (1986), 873-877.
- [6] Ladde, G.S, Lakshmikantham, V. and Vatsala, A.S., *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, Boston 1985.
- [7] Nieto, J.J., Nonlinear second order periodic boundary value problems, *J. Math. Anal. Appl.* **130** (1988), 22-29.
- [8] Nieto, J.J., Nonlinear second order periodic boundary value problems with Carathéodory functions, *Appl. Anal.* **34** (1989), 111-128.
- [9] Nieto, J.J., Periodic solutions for third order ordinary differential equations, *Commentat. Math. Univ. Carol.* **32** (1991), 495-499.
- [10] Nieto, J.J. and Cabada, A., A generalized upper and lower solution method for nonlinear second order ordinary differential equations, *J. Appl. Math. Stoch. Anal.* **5** (1992), 157-166.
- [11] Omari, P., A monotone method for constructing extremal solutions of second order scalar boundary value problems, *Appl. Math. Comp.* **18** (1986), 257-275.
- [12] Omari, P., Nonordered lower and upper solutions and solvability of the periodic problem for the Liénard and the Rayleigh equations, *Rend. Ist. Mat. Univ. Trieste* **20** (1991), 54-64.

- [13] Omari, P. and Trombetta, M., Remarks on the lower and upper solutions methods for second and third order periodic boundary value problems, *Appl. Math. Comput.* **50** (1992), 1-21.
- [14] Rudolf, B. and Kubacek, Z., Remarks on J.J. Nieto's paper: Nonlinear second order periodic boundary value problems, *J. Math. Anal. Appl.* **146** (1990), 203-206.
- [15] Šeda, V., Nieto, J.J. and Gera, M., Periodic boundary value problems for nonlinear higher order ordinary differential equations, *Appl. Math. Comput.* **48** (1992), 71-82.