INVARIANT PROBABILITIES FOR FELLER-MARKOV CHAINS¹

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ABSTRACT

We give necessary and sufficient conditions for the existence of invariant probability measures for Markov chains that satisfy the Feller property.

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1. Introduction

The existence of invariant probabilities for Markov chains is an important issue for studying the long-term behavior of the chains and also for analyzing Markov control processes under the long-run expected average cost criterion. Inspired by the latter control problems, we present in this paper, two **necessary and sufficient** conditions for the existence of invariant probabilities for Markov chains that satisfy the Feller property. Our study extends previous results using stronger assumptions, such as the **strong** Feller property in Beneš [1], nondegeneracy assumptions (see condition (2) in Beneš [2]), and a uniform countable-additivity hypothesis in Liu and Susko [8]. As can be seen in the references, it is also worth noting that there are many reported results providing (only) **sufficient** conditions for the existence of invariant measures; in contrast however, our conditions are also necessary.

The setting for this paper is specified in Section 2, and our main result is presented in Section 3.

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2. Notation and Definitions

Let X be a σ -compact metric space, and let $\{x_t, t = 0, 1, \dots\}$ be an X -valued Markov chain with time-homogeneous kernel P , i.e.,

$$P(B | x) = \text{Prob}(x_{t+1} \in B | x_t = x) \forall t = 0, 1, \dots, x \in X, B \in \mathfrak{B}(X),$$

where $\mathfrak{B}(X)$ denotes the Borel σ -algebra of X . A probability measure (p.m.) μ on $\mathfrak{B}(X)$ is said to be *invariant* for P if

$$\mu(B) = \int_X P(B | x) \mu(dx) \quad \forall B \in \mathfrak{B}(X).$$

Here, we give necessary and sufficient conditions for the existence of invariant p.m.'s when P satisfies the *Feller property*:

$$x \mapsto \int u(y) P(dy | x) \text{ is in } C(X) \text{ whenever } u \in C(X), \quad (1)$$

where $C(X)$ denotes the space of all bounded and continuous functions on X . Our conditions use a *moment* function, defined as follows.

Definition: A nonnegative Borel-measurable function v on X is said to be a *moment* if, as $n \rightarrow \infty$, $\inf \{v(x) | x \notin K_n\} \uparrow \infty$ for some sequence of compact sets $K_n \uparrow X$.

Moment functions have been used by several authors to study the existence of invariant measures for Markov processes (e.g., see Beneš [1, 2], Hernández-Lerma [6], Liu and Susko [8], and Meyn and Tweedie [9]). The key feature used in these studies is the following (easily proved) fact.

Lemma: Let M be a family of p.m.'s on X . If there exists a moment v on X such that $\sup_{\mu \in M} \int v d\mu < \infty$, then M is *tight*, i.e., for every positive ϵ there exists a compact set K such that $\mu(K) > 1 - \epsilon$ for all $\mu \in M$.

Therefore by Prohorov's Theorem [3, 9], the family M in the lemma is relatively compact, i.e., every sequence in M contains a weakly convergent subsequence.

Our theorem below (see Section 3) extends a result by Beneš [2] where our conditions (a) and (b) are new and, most importantly, we do *not* require Beneš' "nondegeneracy" condition, according to which

$$\lim_{x \rightarrow \infty} P^t(K | x) = 0 \text{ for } t = 1, 2, \dots, K \text{ compact,}$$

with $P^t(\cdot | x)$ being the t -step transition probability given the initial state $x_0 = x$. This condition excludes important classes of ergodic Markov chains, such as those that have a "minorant"; see e.g., Dynkin and Yushkevich [5], or condition R1 in Hernández-Lerma *et al.* [7]. See also Remarks 2 and 3 (Section 3) for additional comments on related results.

3. The Theorem

If ν is a p.m. on X , $E_\nu(\cdot)$ stands for the expectation given the "initial distribution" ν .

Theorem: If P satisfies the Feller property, then the following conditions (a), (b), and (c) are equivalent:

(a) There exists a p.m. ν and a moment v such that

$$\limsup_{n \rightarrow \infty} J_n(\nu) < \infty,$$

$$\text{where } J_n(\nu) := n^{-1} E_\nu \left[\sum_{t=0}^{n-1} v(x_t) \right];$$

(b) There exists a p.m. ν and a moment v such that

$$\limsup_{\alpha \uparrow 1} V_\alpha(\nu) < \infty,$$

$$\text{where } V_\alpha(\nu) := (1 - \alpha) E_\nu \left[\sum_{t=0}^{\infty} \alpha^t v(x_t) \right];$$

(c) There exists an invariant probability for P .

Proof: We will show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

(a) *implies* (b): This follows from a well-known Abelian theorem (e.g., see Sznajder and Filar [11], Theorem 2.2), which states that

$$\limsup_{\alpha \uparrow 1} V_\alpha(\nu) \leq \limsup_{n \rightarrow \infty} J_n(\nu).$$

(Since a direct proof that (a) implies (c) is surprisingly simple, it will also be included here; see Remark 1 below.)

(b) *implies* (c): Suppose that (b) holds and for each $\alpha \in (0, 1)$, let μ_α be the probability measure on X defined as

$$\mu_\alpha(B) := (1 - \alpha) \sum_{t=0}^{\infty} \alpha^t \int_X P^t(B | z) \nu(dz), \quad B \in \mathfrak{B}(X).$$

Then we may write $V_\alpha(\nu)$ as $V_\alpha(\nu) = \int v d\mu_\alpha$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\alpha_n \uparrow 1$ and, by (b),

$$\limsup_{\alpha \uparrow 1} V_\alpha(\nu) = \lim_{n \rightarrow \infty} V_{\alpha_n}(\nu) = \lim_{n \rightarrow \infty} \int v d\mu_{\alpha_n} < \infty.$$

By the lemma in Section 2, $\{\mu_{\alpha_n}\}$ is tight and therefore, by Prohorov's Theorem, $\{\mu_{\alpha_n}\}$ contains a weakly convergent subsequence, which we denote by $\{\mu_{\alpha_n}\}$ again; that is, there exists a probability measure μ on X such that

$$\lim_n \int u d\mu_{\alpha_n} = \int u d\mu \quad \forall u \in C(X).$$

We claim that μ is invariant for P .

To see this, first note that by the Markov property, we may write μ_α as

$$\mu_\alpha(B) = (1 - \alpha) \nu(B) + \alpha \int P(B | x) \mu_\alpha(dx) \quad \forall \alpha \in (0, 1), \quad B \in \mathfrak{B}(X).$$

Hence, for any $u \in C(X)$,

$$\int u d\mu_\alpha = (1 - \alpha) \int u(x) \nu(dx) + \alpha \int \int u(y) P(dy | x) \mu_\alpha(dx),$$

and furthermore, note that by the Feller property (1), $\int u(y)P(dy | \cdot)$ is in $C(X)$. Thus, replacing α by α_n and letting $n \rightarrow \infty$, we obtain

$$\int u d\mu = \int \int u(y)P(dy | x)\mu(dx). \quad (2)$$

Finally, since $u \in C(X)$ was arbitrary, we conclude from (2) that μ is invariant for P .

(c) *implies* (a): Let ν be an invariant probability for P , and let $\{K_n\}$ be an increasing sequence of compact sets such that $K_n \uparrow X$ and $\nu(K_{n+1} - K_n) < 1/n^3$, $n = 1, 2, \dots$ (Here we have used the fact that every p.m. on a σ -compact metric space is tight; see [3], p. 9.) Define a function $v(\cdot) := 0$ on K_1 and $v(x) := n$ for $x \in K_{n+1} - K_n$, $n \geq 1$. Then v is a moment and

$$\limsup_{n \rightarrow \infty} J_n(\nu) = \int v(x)\nu(dx) \leq \sum_{n=1}^{\infty} n^{-2} < \infty. \quad \square$$

Remark 1: We will prove directly that (a) implies (c). Suppose that (a) holds and for every $n = 1, 2, \dots$, let μ_n be the probability measure on X defined as

$$\mu_n(B) := n^{-1} \sum_{t=0}^{n-1} \int P^t(B | z)\nu(dz), \quad B \in \mathfrak{B}(X),$$

so that we may rewrite the condition in (a) as

$$\limsup_{n \rightarrow \infty} \int v d\mu_n < \infty.$$

Hence, by the lemma in Section 2, $\{\mu_n\}$ has a subsequence $\{\mu_{n_i}\}$ which converges weakly to some probability measure μ . We will show that (cf. (2))

$$\int Lu(x)\mu(dx) = 0 \quad \forall u \in C(X), \quad (3)$$

where $Lu(x) := \int u(y)P(dy | x) - u(x)$, thus showing that μ is invariant for P .

Indeed, for any bounded measurable function u on X , the sequence

$$M_n(u) := u(x_n) - \sum_{t=0}^{n-1} Lu(x_t), \quad n = 1, 2, \dots,$$

with $M_0(u) := u(x_0)$, is a martingale, which implies

$$E_\nu[M_n(u)] = E_\nu[M_0(u)] \forall n,$$

i.e.,

$$E_\nu[u(x_n)] - n \int Lu(x)\mu_n(dx) = \int u(x)\nu(dx).$$

Finally, let u be in $C(X)$; replace n by n_i ; multiply by $1/n_i$; and then let $i \rightarrow \infty$ to get (3).

Remark 2: In [8], it is shown that

$$\sup_{t \geq 1} \int \int g(y)P^t(dy | x)\nu(dx) < \infty \quad (4)$$

for some moment g and initial p.m. ν , is also a necessary and sufficient condition for existence of invariant probabilities *provided that the Markov chain satisfies the uniform countable-additivity property*

$$\lim_{A \downarrow \emptyset} \sup_{x \in K} P(A | x) = 0 \quad (5)$$

for every compact set K in x .

Note that (4) is stronger than our condition (a) and that (5) implies: *For every compact set $K \subset X$, the family of p.m.'s $\{P(\cdot | x)\}_{x \in K}$ is tight.*

Remark 3: It is worth noting that the theorem still holds if we replace “limsup” by “lim inf” in both conditions (a) and (b). Now, (b) $\Rightarrow$ (a) by a well-known Abelian theorem [11]. With similar arguments as in Remark 1, (a) $\Rightarrow$ (c). We finally prove (c) $\Rightarrow$ (b) by exhibiting the same moment function v and show that

$$\liminf_{\alpha \uparrow 1} V_{\alpha}(\nu) = \liminf_{\alpha \uparrow 1} (1 - \alpha) E_{\nu} \sum_{t=0}^{\infty} \alpha^t v(x_t) = \int v(x) \nu(dx) \leq \sum_{t=0}^{\infty} n^{-2} < \infty.$$

In conclusion, we mention that the theorem can be extended in the obvious way to continuous-time Markov processes, as in [2]. Conditions for *uniqueness* and *ergodicity* of invariant measures can be found, for instance, in [4, 7, 10] and references therein.

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