QUANTITATIVE RESULTS FOR PERTURBED STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT

The paper is devoted to Itô type stochastic differential equations (SDE's) with "small" perturbations. Our goal is to present strong results showing how "close" are the $2m$ -order moments of the solutions of the perturbed SDE's and the unperturbed SDE.

Key words: Stochastic Differential Equation, "Small" Perturbations, Moments of Order $2m$, Approximations of the Solutions.

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1. Introduction. Statement of the Problem

The object of this study are stochastic differential equations (SDE's) of the following type

$$X_t = X_{t_0} + \int_{t_0}^t a(s, X_s) ds + \int_{t_0}^t b(s, X_s) dw_s, \quad t \geq t_0 \geq 0. \quad (1)$$

Here $w = (w_t, t \geq 0)$ is a standard Wiener process defined on a given probability space $(\Omega, \mathcal{F}, P)$, $a(t, x)$ and $b(t, x)$, $t \geq t_0$, $x \in \mathbb{R}^1$, are measurable real-valued functions, and X_{t_0} is a random variable (r.v.) independent of w with $E\{X_{t_0}^2\} < \infty$. Finally, $\int_{t_0}^t b(\cdot) dw_s$ is the well-known stochastic integral in Itô sense.

Under general conditions, the SDE (1) has a unique (strong) solution $X = (X_t, t \geq t_0)$, which is a diffusion Markov process with a drift coefficient a and a diffusion coefficient b^2 . Let us adopt the following classical conditions: For some constants $K_1 > 0$ and $K_2 > 0$ and all $t \geq t_0$, $x, y \in \mathbb{R}^1$ we have

$$\begin{cases} |a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq K_1 |x - y| \\ a^2(t, x) + b^2(t, x) \leq K_2(1 + x^2). \end{cases} \quad (2)$$

Notice that standard references in the area of SDE's are the books by Gihman and Skorohod [4], Arnold [1], Liptser and Shiryaev [5] and Gard [3]. We shall systematically use basic facts from these sources without mentioning.

Now, along with (1), we consider another SDE:

$$X_t^\epsilon = X_{t_0}^{\epsilon_0} + \int_{t_0}^t \tilde{a}(s, X_s^\epsilon, \epsilon_1) ds + \int_0^t \tilde{b}(s, X_s^\epsilon, \epsilon_2) dw_s, \quad t \geq t_0. \quad (3)$$

Here $\epsilon_0, \epsilon_1, \epsilon_2$ are “small” positive parameters, e.g., each is in the interval $(0, 1]$ and ϵ stands for $(\epsilon_0, \epsilon_1, \epsilon_2)$; $X_{t_0}^{\epsilon_0}$, $\tilde{a}$, and $\tilde{b}$ are as X_{t_0} , a , and b above; w is the same Wiener process. Thus the SDE (3) has also a unique (strong) solution $X^\epsilon = (X_t^\epsilon, t \geq t_0)$.

Our goal is to compare the solutions X^ϵ and X of (3) and (1) in the case when their coefficients are related as follows:

$$\begin{cases} \tilde{a}(t, x, \epsilon_1) = a(t, x) + \alpha(t, x, \epsilon_1) \\ \tilde{b}(t, x, \epsilon_2) = b(t, x) + \beta(t, x, \epsilon_2). \end{cases} \quad (4)$$

The terms $\alpha(\cdot)$ and $\beta(\cdot)$ are called perturbations of the coefficients $a(\cdot)$ and $b(\cdot)$ which explains why (3) is called a “perturbed SDE” while the name “unperturbed SDE” is kept for (1).

Let us suppose that for some fixed natural number m we have $E\{(X_{t_0}^{\epsilon_0})^{2m}\} < \infty$, $E\{(X_{t_0})^{2m}\} < \infty$ and let for all $t \geq t_0$,

$$\begin{cases} E\{|X_{t_0}^{\epsilon_0} - X_{t_0}|^{2m}\} \leq \delta_0(\epsilon_0) \\ \sup_x |\alpha(t, x, \epsilon_1)| \leq \delta_1(t, \epsilon_1) \\ \sup_x |\beta(t, x, \epsilon_2)| \leq \delta_2(t, \epsilon_2). \end{cases} \quad (5)$$

Thus we can expect that if the quantities $\delta_0(\epsilon_0)$, $\delta_1(t, \epsilon_1)$, $\delta_2(t, \epsilon_2)$ are small for small $\epsilon_0, \epsilon_1, \epsilon_2$, then the process X^ϵ is close to X . Recall that X^ϵ and X are $2m$ -integrable in the sense that for each $t \geq t_0$ the r.v.'s $|X_t^\epsilon|^{2m}$ and $|X_t|^{2m}$ are P -integrable. Thus, the following quantity

$$\Delta_t^\epsilon = E\{|X_t^\epsilon - X_t|^{2m}\}$$

is well-defined and we are interested in conditions guaranteeing that $\Delta_t^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$. This means that for a fixed t , the $2m$ -order moments of X_t^ϵ and X_t are close. Furthermore, we describe a few cases when $\Delta_t^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ on intervals whose length tend to infinity.

2. Preliminary Result

Let us prove first a result which is of independent interest. This result plays a key role for the statements in the next section.

Theorem A: Suppose conditions (2) and (5) are satisfied for the SDE's (1) and (3). Then for any $t \geq t_0$,

$$\begin{aligned} \Delta_t^\epsilon \leq & \left\{ \delta_0^{1/m}(\epsilon_0) \exp \left[M(t - t_0) + 2 \int_{t_0}^t \delta_1(s, \epsilon_1) ds \right] \right. \\ & \left. + \int_{t_0}^t [2\delta_1(s, \epsilon_1) + (2m - 1)\delta_2^2(s, \epsilon_2)] \exp \left[M(t - s) + 2 \int_s^t \delta_1(\tau, \epsilon_1) d\tau \right] ds \right\}^m, \end{aligned} \quad (6)$$

where $M = 2K + 2(2m - 1)K^2$ ($K = \max[K_1, K_2]$, and K_1 and K_2 are the constants from (2)).

Proof: If we write explicitly the difference $Z_t^\epsilon = X_t^\epsilon - X_t$, $t \geq t_0$, apply the Itô formula to $(Z_t^\epsilon)^{2m}$ and take expectations, we find that

$$\Delta_t^\epsilon = E\{(Z_t^\epsilon)^{2m}\} = E\{(Z_{t_0}^\epsilon)^{2m}\} + 2mE\{I_1(t)\} + m(2m - 1)E\{I_2(t)\} + 2mE\{I_3(t)\}, \quad (7)$$

where

$$I_1(t) = \int_{t_0}^t [\tilde{a}(s, X_s^\epsilon, \epsilon_1) - a(s, X_s)](Z_s^\epsilon)^{2m-1} ds,$$

$$I_2(t) = \int_{t_0}^t [\tilde{b}(s, X_s^\epsilon, \epsilon_2) - b(s, X_s)]^2 (Z_s^\epsilon)^{2m-2} ds,$$

$$I_3(t) = \int_{t_0}^t [\tilde{b}(s, X_s^\epsilon, \epsilon_2) - b(s, X_s)](Z_s^\epsilon)^{2m-1} dw_s.$$

The existence of the $2m$ -order moments of X_t^ϵ and X_t , $t \geq t_0$ and the conditions on $\tilde{b}(\cdot)$ and $b(\cdot)$ allow us to use one of the properties of the stochastic integrals, thus concluding that

$$E\{I_3(t)\} = 0.$$

Let us estimate $I_1(t)$ and $I_2(t)$. In view of (2), we see that

$$I_1(t) \leq K \int_{t_0}^t |X_s^\epsilon - X_s| |Z_s^\epsilon|^{2m-1} ds + \int_{t_0}^t \delta_1(s, \epsilon_1) |Z_s^\epsilon|^{2m-1} ds.$$

Now we take the expectations of both sides of the last inequality and applying Hölder's inequality to $E\{|Z_s^\epsilon|^{2m-1}\}$ (see Shiryaev [7]) we find that

$$E\{I_1(t)\} \leq \int_{t_0}^t \Delta_s^\epsilon ds + \int_{t_0}^t \delta_1(s, \epsilon_1) (\Delta_s^\epsilon)^{(2m-1)/(2m)} ds. \quad (8)$$

Similar arguments imply that

$$E\{I_2(t)\} \leq 2K^2 \int_{t_0}^t \Delta_s^\epsilon ds + \int_{t_0}^t \delta_2^2(s, \epsilon_2) (\Delta_s^\epsilon)^{(m-1)/m} ds. \quad (9)$$

Therefore, from (7), (8) and (9), we get

$$\begin{aligned} \Delta_t^\epsilon &\leq \delta_0(\epsilon_0) + mM \int_{t_0}^t \Delta_s^\epsilon ds + 2m \int_{t_0}^t \delta_1(s, \epsilon_1) (\Delta_s^\epsilon)^{(2m-1)/(2m)} ds \\ &\quad + m(2m-1) \int_{t_0}^t \delta_2^2(s, \epsilon_2) (\Delta_s^\epsilon)^{(m-1)/m} ds. \end{aligned} \quad (10)$$

Now we use the following elementary inequality

$$v^{r_2} \leq v^{r_1} + v$$

which is valid for any nonnegative number v and $0 < r_1 \leq r_2 < 1$. Setting

$$r_1 = (m-1)/m, \quad r_2 = (2m-1)/(2m), \text{ and } v = \Delta_s^\epsilon$$

we find that

$$(\Delta_s^\epsilon)^{(2m-1)/(2m)} \leq (\Delta_s^\epsilon)^{(m-1)/m} + \Delta_s^\epsilon$$

and hence (10) takes the form

$$\begin{aligned} \Delta_t^\epsilon &\leq \delta_0(\epsilon_0) + m \int_{t_0}^t [M + 2\delta_1(s, \epsilon_1)] \Delta_s^\epsilon ds \\ &+ \int_{t_0}^t [2m\delta_1(s, \epsilon_1) + m(2m-1)\delta_2^2(s, \epsilon_2)] (\Delta_s^\epsilon)^{(m-1)/m} ds. \end{aligned} \quad (11)$$

The last tool we need is the following generalized Gronwall-Bellman inequality (see Filatov and Sharova [2]):

If a nonnegative function $u(t)$, $t \geq t_0$, satisfies the integral inequality

$$u(t) \leq C + \int_{t_0}^t A(s)u(s)ds + \int_{t_0}^t B(s)[u(s)]^\gamma ds,$$

where $C \geq 0$, $0 \leq \gamma < 1$, and functions $A(t)$ and $B(t)$, $t \geq t_0$, are nonnegative and continuous, then

$$u(t) \leq \left\{ C^{1-\gamma} \exp \left[(1-\gamma) \int_{t_0}^t A(s)ds \right] + (1-\gamma) \int_{t_0}^t B(s) \exp \left[(1-\gamma) \int_s^t A(\tau)d\tau \right] ds \right\}^{1/(1-\gamma)}.$$

Obviously, it remains to apply this inequality to (11) by letting

$$u(t) = \Delta_t^\epsilon, \quad C = \delta_0(\epsilon_0), \quad \gamma = (m-1)/m,$$

$$A(s) = m[M + 2\delta_1(s, \epsilon_1)],$$

$$B(s) = 2m\delta_1(s, \epsilon_1) + m(2m-1)\delta_2^2(s, \epsilon_2).$$

Thus we arrive at the desired relation (6). Theorem A is proved. $\square$

3. Basic Results. Proofs

Since the magnitude of the perturbations of SDE (1) is determined by the quantities $\delta_0(\epsilon_0)$, $\delta_1(t, \epsilon_1)$ and $\delta_2(t, \epsilon_2)$ (see (4) and (5)) it is natural to impose some conditions on these quantities and see how $\Delta_t^\epsilon = E\{|X_t^\epsilon - X_t|^{2m}\} \rightarrow 0$ as $\epsilon \rightarrow 0$ and on which intervals this convergence holds.

Three specific cases will be considered.

In the statements below (Theorems 1, 2 and 3) we assume (with mentioning it again) that the general conditions of Theorem A are satisfied. We also use the constant $M = 2K + 2(2m - 1)K^2$.

Theorem 1: Suppose that for all $t \geq t_0$,

$$\delta_0(\epsilon_0) = \epsilon_0, \delta_1(t, \epsilon_1) = \epsilon_1 \text{ and } \delta_2(t, \epsilon_2) = \epsilon_2. \quad (12)$$

Define the numbers ϵ and T_1 as follows:

$$\epsilon = \max[\epsilon_0^{1/m}, \epsilon_1, \epsilon_2^2] \text{ and } T_1 = (1 - \rho)/(M + \rho),$$

where $\rho \in (0, 1)$ is arbitrarily chosen.

Then the following relation holds:

$$\sup_t \Delta_t^\epsilon \rightarrow 0 \text{ as } \epsilon \rightarrow 0 \text{ for } t \in [t_0, t_0 + T_1 \ln(1/\epsilon)).$$

Proof: In Theorem A we have found the upper bound (6) for Δ_t^ϵ and now by using (12) we can go further. After a substitution we arrive at

$$(\Delta_t^\epsilon)^{1/m} \leq \epsilon_0^{1/m} \exp[M(t - t_0) + 2\epsilon_1(t - t_0)] + \int_{t_0}^t [2\epsilon_1 + (2m - 1)\epsilon_2^2 \exp[M(t - s) + 2\epsilon_1(t - s)]] ds.$$

Since $\epsilon \rightarrow 0$ implies that $\epsilon_1 \rightarrow 0$, we can assume that $2\epsilon_1 < \rho$. Taking into account that $\epsilon_0^{1/m} \leq \epsilon$, $\epsilon_1 \leq \epsilon$ and $\epsilon_2^2 \leq \epsilon$ we find that

$$(\Delta_t^\epsilon)^{1/m} \leq C_1 \epsilon e^{(M + \rho)t} + (4\epsilon)/(M + \rho), \quad (13)$$

where the constant C_1 depends on t_0 and m but not on t .

Obviously, (13) implies that $\Delta_t^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ for each t on any finite interval $[t_0, t_1]$ with fixed $t_1 > t_0$. However, we can use (13) and make one step further by extending the time-interval on which $\Delta_t^\epsilon \rightarrow 0$. Indeed, if we take $T_1 = (1 - \rho)/(M + \rho)$ we find from (13) that

$$\Delta_t^\epsilon \leq [C_1 \epsilon^\rho + (4\epsilon)/(M + \rho)]^m \text{ for any } t \in [t_0, t_0 + T_1 \ln(1/\epsilon))$$

and hence $\sup_t \Delta_t^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ on the interval $[t_0, t_0 + T_1 \ln(1/\epsilon))$. Note that the length of this interval tends to infinity as $\epsilon \rightarrow 0$. Theorem 1 is proved. $\square$

Theorem 2: Suppose that $t_0 > 1$ and let for all $t \geq t_0$,

$$\delta_0(\epsilon_0) = t_0^{-1/\epsilon_0}, \delta_1(t, \epsilon_1) = t^{-1/\epsilon_1} \text{ and } \delta_2(t, \epsilon_2) = t^{-1/\epsilon_2}. \quad (14)$$

Define ϵ and T_2 as follows:

$$\epsilon = \max[m\epsilon_0, \epsilon_1, \epsilon_2/2] \text{ and } T_2 = (1/M) \ln(t_0 - \rho),$$

where ρ is an arbitrary number in the interval $(0, t_0)$.

Then,

$$\sup_t \Delta_t^\epsilon \rightarrow 0 \text{ as } \epsilon \rightarrow 0 \text{ for } t \in [t_0, t_0 + T_2/\epsilon).$$

Proof: From (6) and (14), we find that

$$(\Delta_t^\epsilon)^{1/m} \leq t_0^{-1/(m\epsilon_0)} \exp[M(t-t_0) + 2\epsilon_1(t_0^{1-1/\epsilon_1} - t^{1-1/\epsilon_1})/(1-\epsilon_1)] \\ + \int_{t_0}^t [2s^{-1/\epsilon_1} + (2m-1)s^{-2/\epsilon_2}] \exp[M(t-s) + 2\epsilon_1(s^{1-1/\epsilon_1} - t^{1-1/\epsilon_1})/(1-\epsilon_1)] ds.$$

Obviously, we can take $\epsilon_1 < 1/2$, and, since $s^{-1/(m\epsilon_0)} \leq s^{-1/\epsilon}$, $s^{-1/\epsilon_1} \leq s^{-1/\epsilon}$ and $s^{-2/\epsilon_2} \leq s^{-1/\epsilon}$ for any $s > 1$, we derive that

$$(\Delta_t^\epsilon)^{1/m} \leq t_0^{-1/\epsilon} e^{M(t-t_0)+2} + 4m \int_{t_0}^t s^{-1/\epsilon} e^{M(t-s)+2} ds.$$

Hence,

$$(\Delta_t^\epsilon)^{1/m} \leq C_2 t_0^{-1/\epsilon} e^{Mt}, \quad (15)$$

where C_2 is a constant depending on t_0 and m but not on t . It follows from (15) that $\Delta_t^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ for each t on any finite interval $[t_0, t_2]$ with fixed $t_2 > t_0$. Moreover, with $T_2 = (1/M) \ln(t_0 - \rho)$ we easily find that

$$\Delta_t^\epsilon \leq C_2^m (1 - \rho/t_0)^{m/\epsilon} \text{ for all } t \in [t_0, t_0 + T_2/\epsilon].$$

Since $0 < 1 - \rho/t_0 < 1$, the conclusion is that $\sup_t \Delta_t^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ on the interval $[t_0, t_0 + T_2/\epsilon]$ whose length tends to infinity. Theorem 2 is proved. $\square$

Theorem 3: Suppose that $t_0 > 0$ and let for all $t \geq t_0$

$$\delta_0(\epsilon_0) = e^{-t_0/\epsilon_0}, \delta_1(t, \epsilon_1) = e^{-t/\epsilon_1} \text{ and } \delta_2(t, \epsilon_2) = e^{-t/\epsilon_2}. \quad (16)$$

For an arbitrary $\rho \in (0, t_0)$, define

$$\epsilon = \max[m\epsilon_0, \epsilon_2/2] \text{ and } T_3 = (t_0 - \rho)/M.$$

Then,

$$\sup_t \Delta_t^\epsilon \rightarrow 0 \text{ as } \epsilon \rightarrow 0 \text{ for } t \in [t_0, t_0 + T_3/\epsilon].$$

Proof: Taking $\epsilon_1 < 1$ (we can do this since $\epsilon_1 \rightarrow 0$) we easily see that $\int_{t_0}^t e^{-s/\epsilon_1} ds < 1$ and $\int_s^t e^{-\tau/\epsilon_1} d\tau < 1$. Then, substituting (16) into (6) and suitably transforming the right-hand side of (6) we finally arrive at the relation:

$$(\Delta_t^\epsilon)^{1/m} \leq C_3 e^{Mt - t_0/\epsilon},$$

where C_3 depends on t_0 , m and M but not on t .

Therefore, $\Delta_t^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ for each t on any finite interval $[t_0, t_3]$ with fixed $t_3 > t_0$. Even more, if $T_3 = (t_0 - \rho)/M$, then

$$\sup_t \Delta_t^\epsilon \leq C_3^m e^{-\rho/\epsilon} \rightarrow 0 \text{ as } \epsilon \rightarrow 0 \text{ for } t \in [t_0, t_0 + T_3/\epsilon].$$

Again the convergence to zero holds on intervals whose lengths tends to infinity. Theorem 3 is proved. $\square$

4. Additional Remarks

(a) In Theorems 1, 2 and 3 not only did we prove that $\sup_t \Delta_t^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ but, in addition, we can specify the rate of convergence. It is a power rate in Theorem 1 and exponential rate in Theorems 2 and 3. Moreover, we can specify the rate of getting to infinity of the lengths of the corresponding intervals. Obviously, both rates depend on the magnitude of perturbations.

(b) Instead of $\Delta_t^\epsilon = E\{|X_t^\epsilon - X_t|^{2m}\}$, we can consider the quantity

$$\tilde{\Delta}_t^\epsilon = E\left\{\sup_{t \in [t_0, T]} |X_t^\epsilon - X_t|^{2m}\right\}$$

as a measure of closeness between the processes X^ϵ and X . If we establish a sup-version of Theorem A and use some additional arguments, we can provide conditions under which $\tilde{\Delta}_T^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ on finite fixed intervals or on intervals whose lengths tend to infinity.

(c) The results of the present paper can be used when studying stability properties of SDE's under perturbations. Another possibility is to look for the so-called expansions of the solution X^ϵ of the perturbed SDE (3) assuming some smoothness of the coefficients $\tilde{a}(\cdot)$ and $\tilde{b}(\cdot)$.

(d) Similar questions can be raised for more general SDE's driven by arbitrary semimartingales not just by the standard Wiener process (see Protter [6]).

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