TERMINAL VALUE PROBLEMS OF IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES¹

DAJUN GUO

*Shandong University
Department of Mathematics
Jinan 250100, PR of China*

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This paper uses cone theory and the monotone iterative technique to investigate the existence of minimal nonnegative solutions of terminal value problems for first order nonlinear impulsive integro-differential equations of mixed type in a Banach space.

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1. Introduction

The theory of impulsive differential equations has become an important area of investigation. Initial value problems of such equations have been discussed in detail in recent years (see [3]). In this paper, we shall use cone theory and the monotone iterative technique to investigate the existence of a minimal nonnegative solution of the terminal value problem (TVP) for a first order nonlinear impulsive integro-differential equation of mixed type in a Banach space.

2. Preliminaries

Let E be a real Banach space and P be a cone in E which defines a partial order in E : $x \leq y$ if and only if $y - x \in P$. P is said to be *normal* if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N \|y\|$, where θ denotes the zero element of E . P is said to be *regular* (or *fully regular*) if $x_1 \leq x_2 \leq \dots \leq x_n \leq \dots \leq y$ (or $x_1 \leq x_2 \leq \dots \leq x_n \leq \dots$ with $\sup_n \|x_n\| < \infty$) implies $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$ for some $x \in E$. The full regularity of P implies the regularity of P , and the regularity of P implies the normality of P (see [2], Theorem 1.2.1). Moreover, if E is weakly complete (in particular, reflexive), then the normality of P implies the regularity of

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P (see [1], Theorem 2.2).

Consider the TVP in E :

$$\begin{cases} x' = f(t, x, Tx, Sx), & t \in J, t \neq t_m, \\ \Delta x|_{t=t_m} = I_m(x(t_m)), & (m = 1, 2, 3, \dots), \\ x(\infty) = x^*, \end{cases} \quad (1)$$

where $J = [0, \infty)$, $f \in C(J \times P \times P \times P, -P)$, $0 < t_1 < \dots < t_m < \dots$, $t_m \rightarrow \infty$ and $m \rightarrow \infty$, $I_m \in C(P, -P)$ ($m = 1, 2, 3, \dots$), $x^* \in P$, $x(\infty) = \lim_{t \rightarrow \infty} x(t)$, and

$$(Tx)(t) = \int_0^t k(t, s)x(s)ds, \quad (Sx)(t) = \int_0^\infty h(t, s)x(s)ds, \quad (2)$$

$k \in C(D, R_+)$, $D = \{(t, s) \in J \times J : t \geq s\}$, $h \in C(J \times J, R_+)$. $\Delta x|_{t=t_m} = x(t_m^+) - x(t_m^-)$ which denotes the jump of $x(t)$ at $t = t_m$. Here $x(t_m^+)$ and $x(t_m^-)$ represent the right- and left-sided limits of $x(t)$ at $t = t_m$, respectively.

Let $PC(J, E) = \{x : x \text{ is a map from } J \text{ into } E \text{ such that } x(t) \text{ is continuous at } t \neq t_m \text{ and left continuous at } t = t_m \text{ and } x(t_m^+) \text{ exists for } m = 1, 2, 3, \dots\}$, $BPC(J, E) = \{x \in PC(J, E) : \sup_{t \in J} \|x(t)\| < \infty\}$ and $TPC(J, E) = \{x \in PC(J, E) : x(\infty) = \lim_{t \rightarrow \infty} x(t) \text{ exists}\}$. Evidently, $TPC(J, E) \subset BPC(J, E)$, and $BPC(J, E)$ is a Banach space with norm $\|x\|_B = \sup_{t \in J} \|x(t)\|$. Let $BPC(J, P) = \{x \in BPC(J, E) : x(t) \geq \theta \text{ for } t \in J\}$, $TPC(J, P) = \{x \in TPC(J, E) : x(t) \geq \theta \text{ for } t \in J\}$ and $J' = J \setminus \{t_1, \dots, t_m, \dots\}$. A map $x \in TPC(J, P) \cap C^1(J', E)$ is called a *non-negative solution* of TVP(1) if it satisfies (1).

3. Main Results

Let us list some conditions.

$$(H_1) \quad k^* = \sup_{t \in J} \int_0^t k(t, s)ds < \infty, \quad h^* = \sup_{t \in J} \int_0^\infty h(t, s)ds < \infty \text{ and}$$

$$\lim_{t' \rightarrow t} \int_0^\infty |h(t', s) - h(t, s)| ds = 0, \quad t \in J.$$

$$(H_2) \quad \|f(t, x, y, z)\| \leq p(t) + q(t)(a\|x\| + b\|y\| + c\|z\|), \quad t \in J, x, y, z \in P, \text{ and}$$

$$\|I_m(x)\| \leq a_m + b_m\|x\|, \quad x \in P (m = 1, 2, 3, \dots),$$

where $p, q \in C(J, R_+)$ and $a \geq 0$, $b \geq 0$, $a_m \geq 0$, $b_m \geq 0$ ($m = 1, 2, 3, \dots$) satisfying

$$p^* = \int_0^\infty p(t)dt < \infty, \quad q^* = \int_0^\infty q(t)dt < \infty, \quad a^* = \sum_{m=1}^\infty a_m < \infty, \quad b^* = \sum_{m=1}^\infty b_m < \infty.$$

$$(H_3) \quad f(t, x, y, z) \text{ is nonincreasing in } x, y, z \in P \text{ and } I_m(x) \text{ is nonincreasing in } x \in P (m = 1, 2, 3, \dots), \text{ i.e.}$$

$$f(t, x, y, z) \leq f(t, \bar{x}, \bar{y}, \bar{z}), \quad t \in J, \quad x \geq \bar{x} \geq \theta, \quad y \geq \bar{y} \geq \theta, \quad z \geq \bar{z} \geq \theta$$

and

$$I_m(x) \leq I_m(\bar{x}), \quad x \geq \bar{x} \geq \theta \quad (m = 1, 2, 3, \dots).$$

It is easy to see that when (H_1) is satisfied, T and S , defined by (2), are bounded linear operators from $BPC(J, E)$ into $BPC(J, E)$.

Lemma 1: *If conditions (H_1) and (H_2) are satisfied, then for any $x \in BPC(J, P)$, the integral*

$$\int_0^\infty f(t, x(t), (Tx)(t), (Sx)(t)) dt \quad (3)$$

and the series

$$\sum_{m=1}^{\infty} I_m(x(t_m)) \quad (4)$$

are convergent.

Proof: Let $x \in BPC(J, P)$. By virtue of (H_1) and (H_2) , it is easy to see that

$$\begin{aligned} & \int_0^\infty \|f(s, x(s), (Tx)(s), (Sx)(s))\| ds \\ & \leq \int_0^\infty p(s) ds + (a + bk^* + ch^*) \|x\|_B \int_0^\infty q(s) ds < \infty \end{aligned}$$

and

$$\sum_{m=1}^{\infty} \|I_m(x(t_m))\| \leq \sum_{m=1}^{\infty} a_m + \|x\|_B \sum_{m=1}^{\infty} b_m < \infty,$$

so, integral (3) and series (4) are convergent.

Lemma 2: *Let conditions (H_1) and (H_2) be satisfied. Then $x \in TPC(J, P) \cap C^1(J', E)$ is a solution of TVP(1) if and only if $x \in BPC(J, P)$ is a solution to the following impulsive integral equation*

$$x(t) = x^* - \int_t^\infty f(s, x(s), (Tx)(s), (Sx)(s)) ds - \sum_{t \leq t_m \leq \infty} I_m(x(t_m)), \quad t \in J. \quad (5)$$

Proof: Let $x \in TPC(J, P) \cap C^1(J', E)$ be a solution of TVP(1). We first establish the following formula:

$$x(t) = x(0) + \int_0^t x'(s) ds + \sum_{0 < t_m < t} [x(t_m^+) - x(t_m)], \quad t \in J. \quad (6)$$

In fact, let $t_m \leq t \leq t_{m+1}$. Then

$$x(t_1) - x(0) = \int_0^{t_1} x'(s) ds, \quad x(t_2) - x(t_1^+) = \int_{t_1}^{t_2} x'(s) ds,$$

$$x(t_m) - x(t_{m-1}^+) = \int_{t_{m-1}}^{t_m} x'(s) ds, \quad x(t) - x(t_m^+) = \int_{t_m}^t x'(s) ds.$$

Summing up these equations, we get

$$x(t) - x(0) - \sum_{i=1}^m [x(t_i^+) - x(t_i)] = \int_0^t x'(s) ds$$

(i.e., (6) holds). Substituting (1) into (6), we obtain

$$x(t) = x(0) + \int_0^t f(s, x(s), (Tx)(s), (Sx)(s)) ds + \sum_{0 < t_m < t} I_m(x(t_m)), \quad t \in J. \quad (7)$$

By Lemma 1, integral (3) and series (4) are convergent, hence, from (1) and (7) we get

$$x^* = x(0) + \int_0^\infty f(s, x(s), (Tx)(s), (Sx)(s)) ds + \sum_{m=1}^\infty I_m(x(t_m)). \quad (8)$$

Solving $x(0)$ from (8) and substituting it into (7), we find that $x(t)$ satisfies equation (5).

Conversely, if $x \in BPC(J, P)$ is a solution of equation (5), direct differentiation of (5) implies that $x \in C^1(J', E)$ and $x(t)$ satisfies TVP(1). $\square$

Consider operator A defined by

$$(Ax)(t) = x^* - \int_t^\infty f(s, x(s), (Tx)(s), (Sx)(s)) ds - \sum_{t \leq t_m < \infty} I_m(x(t_m)). \quad (9)$$

Lemma 3: *If conditions (H_1) and (H_2) are satisfied, then A defined by (9) is an operator from $BPC(J, P)$ into $BPC(J, P)$.*

Proof: Let $x \in BPC(J, P)$. Since $f \in C(J \times P \times P \times P, -P)$, $I_m \in C(P, -P)$ and $x^* \in P$, we see that $(Ax)(t) \geq \theta$ for $t \in J$, and clearly $Ax \in PC(J, P)$. By (H_1) and (H_2) , we have

$$\begin{aligned} \|(Ax)(t)\| &\leq \|x^*\| + \int_t^\infty p(s) ds + (a + bk^* + ch^*) \|x\|_B \int_t^\infty q(s) ds \\ &\quad + \sum_{t \leq t_m \leq \infty} a_m + \|x\|_B \sum_{t \leq t_m < \infty} b_m \\ &\leq \|x^*\| + p^* + a^* + [b^* + (a + bk^* + ch^*)q^*] \|x\|_B, \quad t \in J, \end{aligned}$$

and therefore

$$\|Ax\|_B \leq \|x^*\| + p^* + a^* + [b^* + (a + bk^* + ch^*)q^*] \|x\|_B. \quad (10)$$

Hence $Ax \in BPC(J, P)$. $\square$

In the following, let $J_0 = [0, t_1]$, $J_m = (t_m, t_{m+1}]$ ($m = 1, 2, 3, \dots$).

Theorem 1: *Let cone P be fully regular and conditions (H_1) , (H_2) , (H_3) be satisfied. Assume that*

$$r = b^* + (a + bk^* + ch^*)q^* < 1, \quad (11)$$

where constants $k^*, h^*, a, b, c, q^*, b^*$ are defined by (H_1) and (H_2) . There exists a nondecreasing sequence $\{x_n\} \subset TPC(J, P) \cap C^1(J', E)$ which converges on J (uniformly in each J_m , $m = 0, 1, 2, \dots$) to the minimal solution $\bar{x} \in TPC(J, P) \cap C^1(J', E)$ of TVP(1) in $TPC(J, P) \cap C^1(J', E)$, i.e., for any solution $x \in$

$TPC(J, P) \cap C^1(J', E)$ of $TVP(1)$, we have

$$x(t) \geq \bar{x}(t), \quad t \in J. \quad (12)$$

Moreover,

$$\bar{x}(t) \geq \bar{x}(t') \geq x^*, \quad 0 \leq t < t' < \infty, \quad (13)$$

and

$$\| \bar{x} \|_B \leq (1-r)^{-1} (\| x^* \| + p^* + a^*), \quad (14)$$

where r is given by (11) and p^*, a^* are defined by (H_2) .

Proof: Let $x_0(t) = \theta$, $x_n(t) = (Ax_{n-1})(t)$ ($n = 1, 2, 3, \dots$), i.e.,

$$\begin{aligned} x_n(t) = & x^* - \int_t^\infty f(s, x_{n-1}(s), (Tx_{n-1})(s), (Sx_{n-1})(s)) ds \\ & - \sum_{t \leq t_m < \infty} I_m(x_{n-1}(t_m)), \quad t \in J (n = 1, 2, 3, \dots). \end{aligned} \quad (15)$$

By Lemma 3, $x_n \in BPC(J, P)$ ($n = 0, 1, 2, \dots$) and $x_1(t) \geq \theta = x_0(t)$ for $t \in J$, so, (15) and (H_3) imply that

$$\theta = x_0(t) \leq x_1(t) \leq x_2(t) \leq \dots \leq x_n(t) \leq \dots, \quad t \in J. \quad (16)$$

On the other hand, from (10) we know

$$\| x_n \|_B = \| Ax_{n-1} \|_B \leq d + r \| x_{n-1} \|_B, \quad (n = 1, 2, 3, \dots),$$

where $d = \| x^* \| + p^* + a^*$ and r is given by (11), thus

$$\begin{aligned} \| x_n \|_B & \leq d + r(d + r \| x_{n-2} \|_B) \leq d + rd + r^2(d + r \| x_{n-3} \|_B) \\ & \leq d + rd + \dots + r^{n-1}d + r^n \| x_0 \|_B = d + rd + \dots + r^{n-1}d = d(1-r^n)(1-r)^{-1} \\ & \leq d(1-r)^{-1}, \quad (n = 1, 2, 3, \dots). \end{aligned} \quad (17)$$

It follows from (16), (17), and the full regularity of P that the following limit exists:

$$\lim_{n \rightarrow \infty} x_n(t) = \bar{x}(t), \quad t \in J. \quad (18)$$

Now we have, by (17),

$$\begin{aligned} & \| f(s, x_{n-1}(s), (Tx_{n-1})(s), (Sx_{n-1})(s)) \| \\ & \leq p(s) + (a + bk^* + ch^*) \| x_{n-1} \|_{Bq(s)} \\ & \leq p(s) + (a + bk^* + ch^*)d(1-r)^{-1}q(s), \quad s \in J \quad (n = 1, 2, 3, \dots), \end{aligned} \quad (19)$$

so, from (15) we know that functions $\{x_{mn}(t)\}$ ($n = 0, 1, 2, \dots$) are equicontinuous in $\bar{J}_m$ ($m = 0, 1, 2, \dots$), where $\bar{J}_m = [t_m, t_{m+1}]$ and

$$x_{mn}(t) = \begin{cases} x_n(t), & t_m < t \leq t_{m+1}; \\ x_n(t_m^+), & t = t_m. \end{cases}$$

Hence, observing (18) and using the Ascoli-Arzelà theorem, we see that $\{x_{mn}(t)\}$ ($n = 0, 1, 2, \dots$) is compact in $C(\bar{J}_m, E)$ ($m = 0, 1, 2, \dots$). and therefore, by diagonal method, $\{x_n(t)\}$ has a subsequence which converges to $\bar{x}(t)$ uniformly in each J_m ($m = 0, 1, 2, \dots$). Since P is also normal and $\{x_n(t)\}$ is nondecreasing, on account of (16), we conclude that the entire sequence $\{x_n(t)\}$ converges to $\bar{x}(t)$ uniformly in

each J_m ($m = 0, 1, 2, \dots$), hence, $\bar{x} \in PC(J, P)$. Moreover, from (17) we know that $\bar{x} \in BPC(J, P)$ and $\|\bar{x}\|_B \leq d(1-r)^{-1}$, i.e., (14) holds.

From (18) and (19), we see that

$$f(s, x_{n-1}(s), (Tx_{n-1})(s), (Sx_{n-1})(s)) \rightarrow f(s, \bar{x}(s), (T\bar{x})(s), (S\bar{x})(s))$$

as $n \rightarrow \infty$, $s \in J$, (20)

and

$$\begin{aligned} & \|f(s, x_{n-1}(s), (Tx_{n-1})(s), (Sx_{n-1})(s)) - f(s, \bar{x}(s), (T\bar{x})(s), (S\bar{x})(s))\| \\ & \leq 2p(s) + 2(a + bk^* + ch^*)d(1-r)^{-1}q(s), \quad s \in J \quad (n = 1, 2, 3, \dots). \end{aligned} \quad (21)$$

In addition, (17), (18) and (H_2) imply that

$$I_m(x_{n-1}(t_m)) \rightarrow I_m(\bar{x}(t_m)) \quad \text{as } n \rightarrow \infty \quad (m = 1, 2, 3, \dots) \quad (22)$$

and

$$\sum_{m=j}^{\infty} \|I_m(x_{n-1}(t_m))\| \leq \sum_{m=j}^{\infty} a_m + d(1-r)^{-1} \sum_{m=j}^{\infty} b_m \quad (n = 1, 2, 3, \dots), \quad (23)$$

$$\sum_{m=j}^{\infty} \|I_m(\bar{x}(t_m))\| \leq \sum_{m=j}^{\infty} a_m + d(1-r)^{-1} \sum_{m=j}^{\infty} b_m. \quad (24)$$

Observing (20)-(24) and taking limits in (15) as $n \rightarrow \infty$, we obtain by virtue of the dominated convergence theorem that

$$\bar{x}(t) = x^* - \int_t^{\infty} f(s, \bar{x}(s), (T\bar{x})(s), (S\bar{x})(s)) ds - \sum_{t \leq t_m < \infty} I_m(\bar{x}(t_m)), \quad t \in J, \quad (25)$$

which by Lemma 2 implies that $\bar{x} \in TPC(J, P) \cap C^1(J', E)$ and $\bar{x}(t)$ is a solution of TVP(1). From (25) we see clearly that (13) holds.

Finally, we prove the minimal property of $\bar{x}(t)$. Let $x \in TPC(J, P) \cap C^1(J', E)$ by any solution of TVP(1). By Lemma 2, $x(t)$ satisfies equation (5). We have $x(t) \geq \theta = x_0(t)$ for $t \in J$. Assume that $x(t) \geq x_{n-1}(t)$ for $t \in J$. Then (15), (5) and (H_3) imply that $x(t) \geq x_n(t)$ for $t \in J$. Hence, by induction, $x(t) \geq x_n(t)$ for $t \in J$ ($n = 0, 1, 2, \dots$), and by taking the limit, we get $x(t) \geq \bar{x}(t)$ for $t \in J$, i.e., (12) holds. The proof is complete. $\square$

Example 1: Consider the TVP of infinite system for scalar nonlinear impulsive integro-differential equations

$$\left\{ \begin{aligned} x'_n &= -\frac{e^{-t}}{2^{n+3}}(1 + x_n + \sqrt{x_{n+1} + 2x_{2n+1}}) - \frac{e^{-2t}}{3^n} \left(\int_0^t e^{-(t+1)s} x_n(s) ds \right)^{1/3} \\ &\quad - \frac{e^{-t}}{4^n} \left(\int_0^{\infty} \frac{x_{2n}(s) ds}{1+t+s^2} \right)^{1/5}, \quad 0 \leq t < \infty, \quad t \neq m, \\ \Delta x_n |_{t=m} &= -\frac{1}{2^{n+m+2}} [x_n(m) + x_{n+2}(m)], \quad (m = 1, 2, 3, \dots), \\ x_n(\infty) &= \frac{1}{n^2}, \quad (n = 1, 2, 3, \dots). \end{aligned} \right. \quad (26)$$

Corollary: TVP(26) has a minimal, nonnegative and continuously differentiable on $[0, \infty) \setminus \{1, 2, 3, \dots\}$ solution $\{x_n(t)\}$ ($n = 1, 2, 3, \dots$) satisfying

$$\sup_{0 \leq t < \infty} \sum_{n=1}^{\infty} x_n(t) < \infty.$$

Proof: Let $E = \ell^1 = \{x = (x_1, \dots, x_n, \dots) : \sum_{n=1}^{\infty} |x_n| < \infty\}$ with norm $\|x\| = \sum_{n=1}^{\infty} |x_n|$ and $P = \{x = (x_1, \dots, x_n, \dots) \in \ell^1 : x_n \geq 0, n = 1, 2, 3, \dots\}$. Thus, P is a normal cone in E . Since ℓ^1 is weakly complete, we conclude that P is regular. We now prove that P is fully regular. Let $x_k = (x_{k1}, \dots, x_{kn}, \dots) \in \ell^1$ ($k = 1, 2, 3, \dots$) satisfy $x_1 \leq x_2 \leq \dots \leq x_k \leq \dots$ and $M = \sup_k \|x_k\| < \infty$. Then, $x_{1n} \leq x_{2n} \leq \dots \leq x_{kn} \leq \dots \leq M$ ($n = 1, 2, 3, \dots$), so, $\lim_{k \rightarrow \infty} x_{kn} = y_n$ ($n = 1, 2, 3, \dots$) exist. For any positive integer i , we have $\sum_{n=1}^i |x_{kn}| \leq M$ ($k = 1, 2, 3, \dots$), so, by letting $k \rightarrow \infty$, we find $\sum_{n=1}^i |y_n| \leq M$. Since i is arbitrary, it follows that $\sum_{n=1}^{\infty} |y_n| \leq M < \infty$, and therefore $y = (y_1, \dots, y_n, \dots) \in \ell^1$. It is clear that $x_1 \leq x_2 \leq \dots \leq x_k \leq \dots \leq y$, consequently, the regularity of P implies that $\|x_k - x\| \rightarrow 0$ as $k \rightarrow \infty$ for some $x \in \ell^1$. Hence the full regularity of P is proven.

Now, system (26) can be regarded as a TVP of the form (1), where $k(t, s) = e^{-(t+1)s}$, $h(t, s) = (1 + t + s^2)^{-1}$, $x = (x_1, \dots, x_n, \dots)$, $y = (y_1, \dots, y_n, \dots)$, $z = (z_1, \dots, z_n, \dots)$, $f = (f_1, \dots, f_n, \dots)$, in which

$$f_n(t, x, y, z) = -\frac{e^{-t}}{2^{n+3}}(1 + x_n + \sqrt{x_{n+1} + 2x_{2n+1}}) - \frac{e^{-2t}}{3^n} y_n^{1/3} - \frac{e^{-t}}{4^n} z_{2n}^{1/5},$$

and $t_m = m$, $I_m = (I_{m1}, \dots, I_{mn}, \dots)$ with

$$I_{mn}(x) = -\frac{1}{2^{n+m+2}}(x_n + x_{n+2}), \quad (m, n = 1, 2, 3, \dots),$$

and $x^* = (1, \dots, \frac{1}{n^2}, \dots) \in P$. Evidently, $f \in C(J \times P \times P \times P, -P)$ and $I_m \in C(P, -P)$ ($m = 1, 2, 3, \dots$). (H_1) is obviously satisfied since

$$k^* = \sup_{t \in J} \int_0^t e^{-(t+1)s} ds = \sup_{t \in J} \frac{1}{t+1} (1 - e^{-(t+1)t}) \leq 1,$$

$$h^* = \sup_{t \in J} \int_0^{\infty} \frac{ds}{1 + t + s^2} \leq \int_0^{\infty} \frac{ds}{1 + s^2} = \frac{\pi}{2},$$

and

$$\int_0^{\infty} \left| \frac{1}{1 + t' + s^2} - \frac{1}{1 + t + s^2} \right| ds = \int_0^{\infty} \frac{|t' - t|}{(1 + t' + s^2)(1 + t + s^2)} ds \leq \frac{\pi}{2} |t' - t| \rightarrow 0$$

as $t' \rightarrow t$. It is easy to verify the following scalar inequality:

$$u^\alpha \leq 1 - \alpha + \alpha u, \quad 0 \leq u < \infty, \quad 0 < \alpha < 1,$$

so, for $t \in J$, $x, y, z \in P$,

$$|f_n(t, x, y, z)|$$

$$\begin{aligned} &\leq \frac{e^{-t}}{2^{n+3}}(1 + x_n + \frac{1}{2}(x_{n+1} + 2x_{2n+1})) + \frac{e^{-2t}}{3^n}(\frac{2}{3} + \frac{1}{3}y_n) + \frac{e^{-t}}{4^n}(\frac{4}{5} + \frac{1}{5}z_{2n}) \\ &\leq \frac{e^{-t}}{2^{n+3}}(1 + \|x\|) + \frac{e^{-2t}}{3^n}(\frac{2}{3} + \frac{1}{3}\|y\|) + \frac{e^{-t}}{4^n}(\frac{4}{5} + \frac{1}{5}\|z\|), \end{aligned}$$

and therefore,

$$\begin{aligned} \|f(t, x, y, z)\| &= \sum_{n=1}^{\infty} |f_n(t, x, y, z)| \leq e^{-t} \left(\sum_{n=1}^{\infty} \frac{1}{2^{n+3}} + \frac{2}{3} \sum_{n=1}^{\infty} \frac{1}{3^n} + \frac{4}{5} \sum_{n=1}^{\infty} \frac{1}{4^n} \right) \\ &\quad + e^{-t} \left(\|x\| \sum_{n=1}^{\infty} \frac{1}{2^{n+3}} + \frac{1}{3} \|y\| \sum_{n=1}^{\infty} \frac{1}{3^n} + \frac{1}{5} \|z\| \sum_{n=1}^{\infty} \frac{1}{4^n} \right) \\ &= \frac{87}{120} e^{-t} + e^{-t} \left(\frac{1}{8} \|x\| + \frac{1}{6} \|y\| + \frac{1}{15} \|z\| \right). \end{aligned}$$

In addition, we have, for $x \in P$,

$$|I_{mn}(x)| \leq \frac{1}{2^{n+m+1}} \|x\|,$$

and so

$$\|I_m(x)\| = \sum_{n=1}^{\infty} |I_{mn}(x)| \leq \frac{1}{2^{m+1}} \|x\|.$$

Hence (H_2) is satisfied for $p(t) = (87/120)e^{-t}$, $q(t) = e^{-t}$, $a = 1/8$, $b = 1/6$, $c = 1/15$, $a_m = 0$ and $b_m = 1/2^{m+1}$ ($m = 1, 2, 3, \dots$), and therefore $p^* = 87/120$, $q^* = 1$, $a^* = 0$ and $b^* = 1/2$.

On the other hand, (H_3) is obviously satisfied, and

$$r = b^* + (a + bk^* + ch^*)q^* \leq \frac{1}{2} + \left(\frac{1}{8} + \frac{1}{6} + \frac{\pi}{30}\right) < 1,$$

i.e., (11) holds. Hence the assertion follows from Theorem 1. $\square$

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