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OSCILLATION OF THE SOLUTIONS OF A CLASS OF IMPULSIVE DIFFERENTIAL EQUATIONS WITH A DEVIATING ARGUMENT

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Sufficient conditions are found for oscillation of all solutions of a class of impulsive differential equations with deviating argument.

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1. Introduction

Impulsive differential equations with deviating argument are an adequate mathematical apparatus for simulation of processes which depend on their prehistory and are subject to short-time disturbances. Such processes occur in the theory of optimal control, theoretical physics, population dynamics, pharmacokinetics, biotechnologies, industrial robotics, economics, etc. In spite of numerous possibilities for their applications, the theory of these equations is developing rather slowly due to difficulties of theoretical and technical nature.

In the recent twenty years, a large number of studies devoted to the oscillation were published. To the best of our knowledge, no other publications on this subject have every been published.

In the present paper, we establish sufficient conditions for oscillation of all solutions of a class of impulsive differential equations with fixed moments of impulse effect and a deviating argument.

Let us note that in contrast to [1], the present paper deals with the oscillatory properties of more general non-homogeneous impulsive differential-difference equation.

2. Preliminary Notes

Let $h \in \mathbb{R}_+ = (0, \infty)$; $p, q \in C(\bar{\mathbb{R}}_+, \mathbb{R}_+)$, $\bar{\mathbb{R}}_+ = [0, \infty)$, $\{\tau_k\}_{k=1}^{\infty}$ be a monotone increasing unbounded sequence of nonnegative numbers, and $\{b_k\}_{k=1}^{\infty}$ be a sequence of real numbers.

Consider the following impulsive differential equations with a deviating argument

$$\left.\begin{aligned} x'(t) + q(t)x(t) + p(t)x(t-h) = 0, \quad t \neq \tau_k \\ \Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k) = b_k x(\tau_k) \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} x'(t) + q(t)x(t) + p(t)x(t-h) = b(t), \quad t \neq \tau_k \\ \Delta x(\tau_k) = b_k x(\tau_k), \end{aligned}\right\} \tag{2}$$

with the initial condition

$$x(t) = \varphi(t), \quad -h \leq t \leq 0, \tag{3}$$

where $\varphi \in C([-h, 0], \mathbb{R})$.

We construct the sequence

$$\{t_i\}_{i=1}^{\infty} = \{\tau_i\}_{i=1}^{\infty} \cup \{\tau_{ih}\}_{i=1}^{\infty}$$

where $\tau_{ih} = \tau_i + h$ and $t_i < t_{i+1}$ for $i \in \mathbb{N}$.

Definition 1: By a *solution* of the equation (1) with initial function (3), we mean any function $x: [-h, \infty) \rightarrow \mathbb{R}$, for which the following conditions are valid:

1. If $-h \leq t \leq 0$, then $x(t) = \varphi(t)$.
2. If $0 \leq t \leq t_1 = \tau_1$, then x coincides with the solution of the equation
$$x'(t) + q(t)x(t) + p(t)x(t-h) = 0.$$
3. If $t_i < t \leq t_{i+1}$, $t_i \in \{\tau_i\}_{i=1}^{\infty} \backslash \{\tau_{ih}\}_{i=1}^{\infty}$, then x coincides with the solution of the problem
$$x'(t) + q(t)x(t) + p(t)x(t-h) = 0$$
$$x(t_i + 0) = (1 + b_{k_i})x(t_i),$$
where the number k_i is determined from the equality $t_i = \tau_{k_i}$.
4. If $t_i < t \leq t_{i+1}$, $t_i \in \{\tau_{ih}\}_{i=1}^{\infty} \backslash \{\tau_i\}_{i=1}^{\infty}$, then x coincides with the solution of the problem
$$x'(t) + q(t)x(t+0) + p(t)x(t-h+0) = 0$$
$$x(t_i + 0) = x(t_i).$$
5. If $t_i < t \leq t_{i+1}$, $t_i \in \{\tau_i\}_{i=1}^{\infty} \cap \{\tau_{ih}\}_{i=1}^{\infty}$, then x coincides with the solution of the problem

$$x'(t)+q(t)x(t+0)+p(t)x(t-h+0)=0$$

$$x(t_i+0)=(1+b_{k_i})x(t_i),$$

where k_i is determined from the equality $t_i=\tau_{k_i}$.

The definition of a solution of the problem (2), (3) is analogous to Definition 1, where $b:[0,\infty)\to\mathbb{R}$.

Definition 2: A nonzero solution x of the equation (1) is said to be *nonoscillating* if there exists a point $t_0\geq 0$ such that $x(t)$ has a constant sign for $t\geq t_0$. Otherwise, the solution x is said to *oscillate*.

In work [1], the oscillatory properties of the impulsive differential equation with a deviating argument,

$$\left.\begin{aligned} x'(t)+p(t)x(t-h)=0, &\quad t\neq\tau_k \\ \Delta x(\tau_k)=b_kx(\tau_k), & \end{aligned}\right\} \tag{4}$$

are studied. To make this presentation self-contained, we formulate the most significant results of [1].

Theorem 1: [1] *Let the following conditions hold:*

1. $p\in C(\bar{\mathbb{R}}_+,\mathbb{R}_+)$.
2. *There exists a constant* $T>0$ *such that for any* $k\in\mathbb{N}$ *we have* $\tau_{k+1}-\tau_k\geq T$.
3. $\limsup_{k\to\infty}\frac{1}{1+b_k}\int_{\tau_k}^{\tau_k+\Delta}p(s)ds>1$, *where* $\Delta=\min(h,T)$.

Then all solutions of the equation (4) *oscillate.*

Theorem 2: [1] *Let the following conditions hold:*

1. $p\in C(\bar{\mathbb{R}}_+,\mathbb{R}_+)$.
2. *There exists a constant* $T>h>0$ *such that for any* $k\in\mathbb{N}$ *we have* $\tau_{k+1}-\tau_k\geq T$.
3. *There exists a constant* $M>0$ *such that for any* $k\in\mathbb{N}$ *we have* $0\leq b_k\leq M$.
4. $\liminf_{t\to\infty}\int_{t-h}^{t}p(s)ds>\frac{1+M}{e}$.

Then all solutions of the equation (4) *oscillate.*

Introduce the following conditions:

H1. There exists a positive constant T such that for any $k\in\mathbb{N}$ we have $\tau_{k+1}-\tau_k\geq T>h$.

H2. $b_k\neq-1$ for all $k\in\mathbb{N}$.

H3. $b_k>-1$ for all $k\in\mathbb{N}$.

3. Main Results

Theorem 3: *Let the following conditions hold:*

1. *Conditions* H1 *and* H2 *are satisfied.*
2. $q(t)+\frac{1}{1+b_k}p(t)>0,\ t\in\bar{\mathbb{R}}_+,\ k\in\mathbb{N}$.

Then all solutions of the problem (1), (3) (*i.e., the problem described by equation*

(1) *with the initial condition* (3)) *oscillate.*

Proof: Let a nonoscillating solution x of problem (1), (3) exist. Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_0 \geq 0$. Then $x(t-h) > 0$ for $t \geq t_0 + h$ too.

From (1) it follows that $x(t)$ is a decreasing function on the set $(t_0 + h, \tau_s) \cup \left[\bigcup_{i=s}^{\infty} (\tau_i, \tau_{i+1})\right]$, $\tau_{s-1} < t_0 + h < \tau_s$.

We integrate (1) from τ_k to $\tau_k + h$ $(k \geq s)$ and obtain that

$$x(\tau_k + h) - x(\tau_k + 0) + \int_{\tau_k}^{\tau_k + h} q(s)x(s)ds + \int_{\tau_k}^{\tau_k + h} p(s)x(s-h)ds = 0,$$

$$x(\tau_k + h) - x(\tau_k + 0) + \inf_{s \in [\tau_k, \tau_k + h]} x(s) \int_{\tau_k}^{\tau_k + h} q(s)ds$$

$$+ \inf_{s \in [\tau_k, \tau_k + h]} x(s-h) \int_{\tau_k}^{\tau_k + h} p(s)ds \leq 0,$$

and

$$x(\tau_k + h) - x(\tau_k + 0) + x(\tau_k + h) \int_{\tau_k}^{\tau_k + h} q(s)ds + x(\tau_k) \int_{\tau_k}^{\tau_k + h} p(s)ds \leq 0. \quad (5)$$

Since $x(\tau_k) = \frac{x(\tau_k + 0)}{1 + b_k}$ and $x(\tau_k + h) < x(\tau_k + 0)$, then from (5) it follows that

$$\left.\begin{aligned} &x(\tau_k + h)\left[1 + \int_{\tau_k}^{\tau_k + h} q(s)ds\right] + x(\tau_k + 0)\left[\frac{1}{1 + b_k}\int_{\tau_k}^{\tau_k + h} p(s)ds - 1\right] \leq 0 \\ &x(\tau_k + h)\int_{\tau_k}^{\tau_k + h}\left[q(s) + \frac{1}{1 + b_k}p(s)\right]ds \leq 0. \end{aligned}\right\} \quad (6)$$

The last inequality contradicts condition 2 of Theorem 3. □

Corollary 1: *Let the following conditions hold:*

1. *Conditions* H1 *and* H2 *are satisfied.*
2. $\limsup_{k \to \infty} \frac{1}{1 + b_k} \int_{\tau_k}^{\tau_k + h} p(s)ds > 1.$

Then all solutions of the problem (1), (3) *oscillate.*

Proof: Analogously to the proof of Theorem 3, we obtain (6) which implies

$$x(\tau_k + 0)\left[\frac{1}{1 + b_k}\int_{\tau_k}^{\tau_k + h} p(s)ds - 1\right] \leq 0.$$

The last inequality contradicts condition 2 of Corollary 1. □

Corollary 2: *Let conditions* H1 *and* H3 *hold.*

Then all solutions of the problem (1),(3) *oscillate.*

Proof: From H3 it follows that $\frac{1}{1+b_k} > 0$. From the condition that $p, q \in C(\bar{\mathbb{R}}_+, \mathbb{R}_+)$ we conclude that

$$q(t) + \frac{1}{1+b_k} p(t) > 0,\ t \in \bar{\mathbb{R}}_+,\ k \in \mathbb{N}.$$

Then from Theorem 3, it follows that all solutions of the problem (1), (3) oscillate.

We shall carry out equivalent transformations of impulsive equation (1). Set

$$x(t) = e^{-\int_{t_0}^{t} q(s)ds} z(t),\quad t \geq t_0 \geq 0,$$

and substitute it into (1) to obtain

$$z'(t) + p_1(t) z(t-h) = 0,\quad t \neq \tau_k, \tag{7}$$

where

$$p_1(t) = p(t) \exp\left[\int_{t-h}^{t} q(s)ds\right].$$

From the second equation in (1) it follows that

$$\Delta z(\tau_k) = b_k z(\tau_k). \tag{8}$$

Hence the equation (1) is equivalent to (7), (8). Then, Theorem 1 and Theorem 2 can be applied to impulsive differential equation (7), (8), and to (1), respectively. □

Theorem 4: *Let the following conditions hold:*

1. *There exists a positive constant T such that for $k \in \mathbb{N}$ we have $\tau_{k+1} - \tau_k \geq T$.*
2. $b_k \neq -1,\ k \in \mathbb{N}$.
3. $\limsup_{k\to\infty} \frac{1}{1+b_k} \int_{\tau_k}^{\tau_k+\Delta} p(s) \exp\left[\int_{s-h}^{s} q(s_1)ds_1\right] ds > 1$, *where* $\Delta = \min(h, T)$.

Then all solutions of the problem (1), (3) *oscillate.*

Theorem 5: *Let the following conditions hold:*

1. *Condition* H1 *is valid.*
2. *There exists a constant $M > 0$ such that for any $k \in \mathbb{N}$, the inequalities $0 \leq b_k \leq M$ are valid.*
3. $\liminf_{t\to\infty} \int_{t-h}^{t} p(s) \exp\left[\int_{s-h}^{s} q(s_1)ds_1\right] ds > \frac{1+M}{e}$.

Then all solutions of the problem (1), (3) *oscillate.*

Consider the nonhomogeneous impulsive differential equation

$$\left. \begin{aligned} x'(t) + p(t)x(t-h) &= b(t), \quad t \neq \tau_k \\ \Delta x(\tau_k) &= b_k x(\tau_k). \end{aligned} \right\} \tag{9}$$

Lemma 1: *Let the following conditions hold:*

1. *Condition* H1 *and condition* 2 *of Theorem* 5 *are valid.*
2. *There exists a function* $w \in C^1(\bar{\mathbb{R}}_+, \bar{\mathbb{R}}_+)$ *such that* $w'(t) = b(t)$.
3. *There exist two sequences* $\{t'_k\}_{k=1}^{\infty}$, $\{t''_k\}_{k=1}^{\infty} \subset \mathbb{R}_+$ *and two constants* $q_1, q_2 \in \mathbb{R}_+$ *such that*
   (*i*) $\lim_{k\to\infty} t'_k = \lim_{k\to\infty} t''_k = \infty$,
   (*ii*) $w(t'_k) = q_1 \leq w(t) \leq q_2 = w(t''_k)$, $k \in \mathbb{N}$, $t \in \mathbb{R}_+$.
4. $\liminf\limits_{k\to\infty} \int_{\tau_k}^{\tau_k + h} p(s)ds > 1 + M + \dfrac{M(q_2 - q_1)}{S + q_1 - q_2}$, *where* $S = \text{const}$, $S > q_2 - q_1 > 0$.

Then all solutions of the problem (9), (3) *satisfying the inequalities* $|x(\tau_k)| > S$, $k \in \mathbb{N}$, *oscillate.*

Proof: Let $x(t)$ be a nonoscillating solution of the problem (9), (3). Suppose that $x(t) > 0$ for $t \geq t_0 \geq 0$. It is clear that $x(t-h) > 0$ for $t \geq t_0 + h$. Set

$$z(t) = x(t) - w(t) + q_1 \tag{10}$$

and substitute it in (9). We obtain

$$\left. \begin{aligned} z'(t) + p(t)z(t-h) &\leq 0, \quad t \neq \tau_k \\ \Delta z(\tau_k) &= b_k z(\tau_k) + b_k(w(\tau_k) - q_1). \end{aligned} \right\} \tag{11}$$

From condition 3 of Lemma 1, it follows that

$$z(\tau_k) = x(\tau_k) - w(\tau_k) + q_1 \geq S + q_1 - q_2, \quad t \in \bar{\mathbb{R}}_+.$$

We consider the following two cases:

Case 1: Let $z(t) > 0$ be a solution of (11) for $t \geq t_1 \geq t_0$. It is clear that $z(t-h) > 0$ for $t \geq t_1 + h$. From $p \in C(\bar{\mathbb{R}}_+, \mathbb{R}_+)$ and (11) it follows that z is a decreasing function on the set $(t_1 + h, \tau_s) \cup \left[\bigcup_{i=s}^{\infty} (\tau_i, \tau_{i+1})\right]$ where the number s is chosen so that $\tau_{s-1} \leq t_1 + h < \tau_s$.

We integrate (11) to obtain

$$z(\tau_k + h) - z(\tau_k + 0) + \int_{\tau_k}^{\tau_k + h} p(s)z(s-h)ds \leq 0,$$

whence, taking into account that $z(\tau_k + h) > 0$ we obtain the inequality

$$\inf_{s \in [\tau_k, \tau_k + h]} z(s-h) \int_{\tau_k}^{\tau_k + h} p(s)ds < z(\tau_k + 0). \tag{12}$$

From (11) it follows that

$$z(\tau_k+0)=(1+b_k)z(\tau_k)+b_k(w(\tau_k)-q_1). \tag{13}$$

From (12) and (13) we obtain that

$$z(\tau_k)\int_{\tau_k}^{\tau_k+h} p(s)ds \le (1+b_k)z(\tau_k)+b_k[w(\tau_k)-q_1]$$

and

$$\int_{\tau_k}^{\tau_k+h} p(s)ds \le 1+b_k+\frac{b_k[w(\tau_k)-q_1]}{z(\tau_k)} \le 1+M+\frac{M(q_2-q_1)}{S+q_1-q_2}.$$

The last inequality contradicts condition 4 of Lemma 1.

Case 2: Let $z(t)<0$ be a solution of (9) for $t \ge t_1 \ge t_0$. From the substitution (10), it follows that

$$z(t_k')=x(t_k')-w(t_k')+q_1=x(t_k'). \tag{14}$$

But $x(t)>0$ for $t\ge t_0$, hence $x(t_k')>0$ for $t_k'>t_1$. Then from (14) we deduce that the inequalities $z(t_k')=x(t_k')>0$, $t_k'>t_1$, $k\in\mathbb{N}$, hold true. The established inequalities contradict the assumption made. □

Theorem 6: *Let the following conditions hold:*

1. *Conditions* 1,2 *and* 3 *of Lemma* 1 *are valid.*
2. *There exists a constant* $L>1$ *such that*

$$\int_0^{\tau_k} q(s)ds \ge \ln L, \quad k\in\mathbb{N}.$$

3. $\liminf\limits_{k\to\infty}\int_{\tau_k}^{\tau_k+h} p_1(s)ds > 1+M+\frac{M[q_2-q_1]}{SL+q_1-q_2}$, *where* $S=\text{const}>0$, $p_1(t)=p(t)\exp\left[\int_{t-h}^{t} q(s)ds\right]$, $t\in\bar{\mathbb{R}}_+$.

Then all solutions of the problem (2), (3) *satisfying the inequalities* $|x(\tau_k)|>S$, $k\in\mathbb{N}$, *oscillate.*

Proof: Set

$$x(t)=z(t)e^{-\int_0^t q(s)ds},$$

and substitute it into (2) to obtain

$$\left.\begin{aligned} z'(t)+p_1(t)z(t-h)&=b(t),\quad t\neq\tau_k\\ \Delta z(\tau_k)&=b_kz(\tau_k). \end{aligned}\right\} \tag{15}$$

Taking into account condition 2 of Theorem 6, we derive the estimate

$$|z(\tau_k)| = |x(\tau_k)|e^{\int_0^{\tau_k} q(s)ds} \ge Se^{\ln L}=SL,\quad k\in\mathbb{N}.$$

From Lemma 1 it follows that all solutions of the problem (15), (3) for which $|z(\tau_k)| > SL$, $k \in \mathbb{N}$, oscillate. The latter immediately yields that all solutions of the problem (2), (3) for which $|x(\tau_k)| \geq S$, oscillate. □

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