

VOLTERRA AND URYSOHN INTEGRAL EQUATIONS IN BANACH SPACES

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We use topological methods to present existence principles and theory for integral equations in Banach spaces.

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1. Introduction

In this paper we are concerned with Volterra and Urysohn equations in Banach spaces. The paper will be divided into two main sections. In Section 2 general existence principles are established for these equations. The technique relies on a nonlinear alternative of Leray-Schauder type [2]. Our results improve and extend results in [15]; in addition some of the results are new even in the finite dimensional setting. In Section 3 some applications are given. First an existence principle of Brezis-Browder type [1] is established for Hammerstein equations in Banach spaces. Also in Section 3 we give a notion of "solution tube" for singular second order differential equations in Hilbert spaces.

Throughout E will be a Banach space with norm $\|\cdot\|$. We denote by $C([0, T], E)$ the space of continuous functions $u: [0, T] \rightarrow E$. Let $u: [0, T] \rightarrow E$ be a measurable function. By $\int_0^T u(s) ds$ we mean the Bochner integral of u , assuming it exists (see [16] for properties of the Bochner integral). The semi inner products ([7, 9]) on E are defined by

$$\langle x, y \rangle_+ = \|x\| \lim_{t \rightarrow 0^+} \frac{\|x + ty\| - \|x\|}{t};$$

$$\langle x, y \rangle_- = \|x\| \lim_{t \rightarrow 0^-} \frac{\|x + ty\| - \|x\|}{t}.$$

Let Ω_E be the bounded subsets of E . Let $X \in \Omega_E$. The diameter of X is defined by

$$\text{diam}(X) = \sup\{\|x - y\| : x, y \in X\}.$$

The *Kuratowski measure of noncompactness* is the map $\alpha: \Omega_E \rightarrow [0, \infty]$ defined by

$$\alpha(X) = \inf\{\epsilon > 0: X \subseteq \bigcup_{i=1}^n X_i \text{ and } \text{diam}(X_i) \leq \epsilon\}.$$

Theorem 1.1: *Let $T > 0$ and E be a Banach space.*

(i) *Let $A \subseteq C([0, T], E)$ be bounded. Then*

$$\sup_{t \in [0, T]} \alpha(A(t)) \leq \alpha(A[0, T]) \leq 2\alpha(A)$$

where $A(t) = \{\phi(t): \phi \in A\}$ and $A[0, T] = \bigcup_{t \in [0, T]} \{\phi(t): \phi \in A\}$.

(ii) *Let $A \subseteq C([0, T], E)$ be bounded and equicontinuous. Then*

$$\alpha(A) = \sup_{t \in [0, T]} \alpha(A(t)) = \alpha(A[0, T]).$$

Proof: (i) For each $t \in [0, T]$ we have $A(t) \subseteq A[0, T]$ and so $\alpha(A(t)) \leq \alpha(A[0, T])$ which gives

$$\sup_{t \in [0, T]} \alpha(A(t)) \leq \alpha(A[0, T]).$$

The other inequality follows from the ideas in [7, page 24].

(ii) The result follows from [7, 9]. □

Let E_1 and E_2 be two Banach spaces and let $F: Y \subseteq E_1 \rightarrow E_2$ be continuous and map bounded sets into bounded sets. We call such an F a α -Lipschitzian map if there is a constant $k \geq 0$ with $\alpha(F(X)) \leq k\alpha(X)$ for all bounded sets $X \subseteq Y$. We also say F is a *Darbo* map if F is α -Lipschitzian with $k < 1$. Next we state a fixed point result due to Sadovskii [2].

Theorem 1.2: *Let K be a closed, convex subset of a Banach space B and let $N: K \rightarrow K$ be a bounded Darbo map. Then N has a fixed point in K .*

We also have the following nonlinear alternative of Leray-Schauder type for Darbo maps [2].

Theorem 1.3: *Let K be a closed, convex subset of a Banach space B . Assume U is a relatively open subset of K with $0 \in U$, $N(\bar{U})$ bounded and $N: \bar{U} \rightarrow K$ a Darbo map. Then either*

(A1) *N has a fixed point in \bar{U} ; or*

(A2) *there is a $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda Nu$.*

2. Existence Principles

In this section we establish existence principle for the Volterra integral equation

$$y(t) = h(t) + \int_0^t K(t, s, y(s))ds, \quad t \in [0, T] \tag{2.1}$$

and the Urysohn integral equation

$$y(t) = h(t) + \int_0^T K(t, s, y(s))ds, \quad t \in [0, T]. \tag{2.2}$$

We will look for solutions in $C([0, T], E)$; here E is a Banach space with norm $\| \cdot \|$. The ideas involved in establishing existence principles for both (2.1) and (2.2) are essentially the same; as a result we will examine (2.1) in detail and just state the results for (2.2).

Theorem 2.1: *Let $K: [0, T] \times [0, t] \times E \rightarrow E$. Suppose*

$$\left\{ \begin{array}{l} \text{there exists a constant } \gamma \geq 0 \text{ with } \alpha(K([0, T] \times [0, t] \times \Omega)) \leq \gamma \alpha(\Omega) \\ \text{for each bounded set } \Omega \subseteq E \end{array} \right. \quad (2.3)$$

$$2\gamma T < 1 \quad (2.4)$$

$$\left\{ \begin{array}{l} K: [0, T] \times [0, t] \times E \rightarrow E \text{ is } L^1\text{-Carathéodory uniformly in } t; \text{ by this we mean} \\ \text{(i) for each } t \in [0, T], \text{ the map } u \rightarrow K_t(s, u) \text{ is continuous for almost all } s \in [0, t] \\ \text{(note for each } t \in [0, T], K_t: [0, t] \times E \rightarrow E \text{ is defined by } K_t(s, u) = K(t, s, u)) \\ \text{(ii) for each } t \in [0, T], \text{ the map } s \rightarrow K_t(s, u) \text{ is measurable for all } u \in E \\ \text{(iii) for each } t \in [0, T] \text{ and for each } r > 0 \text{ there exists } h_{t,r} \in L^1([0, T], \mathbf{R}) \\ \text{such that } \|u\| \leq r \text{ implies } \|K_t(s, u)\| \leq h_{t,r}(s) \text{ for almost all } s \in [0, t] \\ \text{(iv) for each } r > 0 \text{ there exists } h_r \in L^1([0, T], \mathbf{R}) \text{ and } \alpha > 0 \\ \text{such that for } x < z \text{ in } [0, T], \int_x^z h_{z,r}(s) ds \leq (\int_x^z h_r(s) ds)^\alpha \end{array} \right. \quad (2.5)$$

$$\left\{ \begin{array}{l} \text{for any } r > 0 \text{ and any } z, t \in [0, T] \text{ we have that} \\ \int_0^{t_1} \sup \|u\| \leq r \|K(z, s, u) - K(t, s, u)\| ds \rightarrow 0 \text{ as } z \rightarrow t \\ \text{where } t_1 = \min\{t, z\} \end{array} \right. \quad (2.6)$$

and

$$h \in C([0, T], E) \quad (2.7)$$

hold. In addition assume that there is a constant M_0 , independent of λ , with

$$\|y\|_0 = \sup_{[0, T]} \|y(t)\| \neq M_0$$

for any solution $y \in C([0, T], E)$ to

$$y(t) = \lambda \left(h(t) + \int_0^t K(t, s, y(s)) ds \right), \quad t \in [0, T] \quad (2.8)_\lambda$$

for each $\lambda \in [0, 1]$. Then (2.1) has a solution in $C([0, T], E)$.

Remark: Theorem 2.1 improves a result in [15, Section 3].

Proof: Define the operator $N:C([0, T], E) \rightarrow C([0, T], E)$ by

$$Ny(t) = h(t) + \int_0^t K(t, s, y(s))ds.$$

Now $(2.8)_\lambda$ is equivalent to the fixed point problem $y = \lambda Ny$. We would like to apply Theorem 1.3. First we show $N:C([0, T], E) \rightarrow C([0, T], E)$ is continuous. To see this, let $u_n \rightarrow u$ in $C([0, T], E)$. Then there exists $r > 0$ with $\|u_n(s)\| \leq r$ and $\|u(s)\| \leq r$ for all $s \in [0, T]$. Also for each $t \in [0, T]$ there exists $h_{t,r} \in L^1([0, T], \mathbf{R})$ with

$$\|K(t, s, v)\| \leq h_{t,r}(s) \text{ for a.e. } s \in [0, T] \text{ and all } \|v\| \leq r.$$

Now from (2.5) we have for each $t \in [0, T]$ that

$$K(t, s, u_n(s)) \rightarrow K(t, s, u(s)) \text{ for almost all } s \in [0, T].$$

This together with the Lebesgue dominated convergence theorem yields $Nu_n(t) \rightarrow Nu(t)$ pointwise on $[0, T]$. Now (2.6) guarantees that the convergence is uniform (i.e., the argument below will show that for any $\epsilon > 0$ there exists $\delta > 0$ such that for $t, t' \in [0, T]$ with $|t - t'| < \delta$ we have $\|Nu_n(t') - Nu_n(t)\| < \epsilon$ for all n and $\|Nu(t') - Nu(t)\| < \epsilon$). Hence $Nu \in C([0, T], E)$ and N is continuous.

Next, let Ω be a bounded subset of $C([0, T], E)$. We first claim that $N\Omega$ is bounded and equicontinuous on $[0, T]$. Then there exists $r > 0$ with $\|u(s)\| \leq r$ for all $s \in [0, T]$ and $u \in \Omega$. Also there exists $h_{t,r}$ and h_r as in (2.5). Now $N\Omega$ is bounded since for $t \in [0, T]$ and $u \in \Omega$ we have

$$\begin{aligned} \|Nu(t)\| &\leq \sup_{[0, T]} \|h(t)\| + \int_0^t \sup_{\|v\| \leq r} \|K(t, s, v)\| ds \\ &\leq \sup_{[0, T]} \|h(t)\| + \left(\int_0^T h_r(w)dw \right)^\alpha. \end{aligned}$$

Also, for $t, t' \in [0, T]$ with $t' > t$ and $u \in \Omega$, we have

$$\begin{aligned} \|Nu(t') - Nu(t)\| &\leq \|h(t') - h(t)\| + \int_0^t \sup_{\|v\| \leq r} \|K(t', s, v) - K(t, s, v)\| ds \\ &\quad + \int_t^{t'} \sup_{\|v\| \leq r} \|K(t', s, v)\| ds \\ &\leq \|h(t') - h(t)\| + \int_0^t \sup_{\|v\| \leq r} \|K(t', s, v) - K(t, s, v)\| ds \\ &\quad + \left(\int_t^{t'} h_r(s)ds \right)^\alpha. \end{aligned}$$

Consequently $N\Omega$ is equicontinuous on $[0, T]$. We now show

$$\alpha(N\Omega) \leq 2\gamma T \alpha(\Omega). \tag{2.9}$$

For $t \in [0, T]$, we have

$$\begin{aligned} \alpha(N\Omega(t)) &= \alpha\left(\left\{h(t) + \int_0^t K(t, s, u(s))ds : u \in \Omega\right\}\right) \\ &\leq \alpha(\overline{t\Omega}\{K(t, s, u(s)) : y \in \Omega, s \in [0, t]\}) \\ &= t\alpha(\{K(t, s, u(s)) : y \in \Omega, s \in [0, t]\}) \\ &\leq T\alpha(K([0, T] \times [0, t] \times \Omega[0, t])) \\ &\leq T\gamma\alpha(\Omega[0, t]) \end{aligned}$$

where $\Omega[0, t] = \bigcup_{s \in [0, t]} \{\phi(s) : \phi \in \Omega\}$. Theorem 1.1 (i) implies

$$\alpha(N\Omega(t)) \leq 2\gamma T \alpha(\Omega). \tag{2.10}$$

In addition since $N\Omega$ is bounded and equicontinuous on $[0, T]$ we have from Theorem 1.1 (ii) that

$$\alpha(N\Omega) = \sup_{t \in [0, T]} \alpha(N\Omega(t))$$

and this together with (2.10) implies that (2.9) is true. Let

$$U = \{u \in C([0, T], E) : \|u\|_0 < M_0\}, B = K = C([0, T], E).$$

Now Theorem 1.3 (notice (A2) cannot occur) implies that (2.1) has a solution in \bar{U} . \square
 A special case of (2.1) is

$$y(t) = h(t) + \int_0^t k(t, s)f(s, y(s))ds, t \in [0, T] \tag{2.11}$$

where k takes values in \mathbf{R} .

Theorem 2.2: Let $k: [0, T] \times [0, t] \rightarrow \mathbf{R}$ and $K(t, s, u) = k(t, s)f(s, u)$. Assume (2.3), (2.4) and (2.7) hold. Also suppose

$$\left\{ \begin{array}{l} f: [0, T] \times E \rightarrow E \text{ is a } L^q\text{-Carathéodory function (here } q > 1 \text{ is a} \\ \text{constant); by this we mean} \\ \text{(i) the map } t \mapsto f(t, z) \text{ is measurable for all } z \in E \\ \text{(ii) the map } z \mapsto f(t, z) \text{ is continuous for almost all } t \in [0, T] \\ \text{(iii) for each } r > 0 \text{ there exists } \mu_r \in L^q([0, T], \mathbf{R}) \text{ such that} \\ \quad \|z\| \leq r \text{ implies } \|f(t, z)\| \leq \mu_r(t) \text{ for almost all } t \in [0, T] \end{array} \right. \tag{2.12}$$

$$\left\{ \begin{array}{l} k_t(s) \in L^p([0, t], \mathbf{R}) \text{ for each } t \in [0, T]; \text{ here } \frac{1}{p} + \frac{1}{q} = 1 \text{ and for} \\ \text{each } t \in [0, T], k_t: [0, t] \rightarrow \mathbf{R} \text{ is defined by } k_t(s) = k(t, s) \end{array} \right. \quad (2.13)$$

and

$$\left\{ \begin{array}{l} \text{for any } t_1, t_2 \in [0, T] \text{ we have that} \\ \int_0^{t_3} |k_{t_1}(s) - k_{t_2}(s)|^p ds \rightarrow 0 \text{ as } t_1 \rightarrow t_2 \\ \text{where } t_3 = \min\{t_1, t_2\} \end{array} \right. \quad (2.14)$$

are satisfied. In addition, assume there is a constant M_0 , independent of λ , with $\|y\|_0 \neq M_0$ for any solution $y \in C([0, T], E)$ to

$$y(t) = \lambda \left(h(t) + \int_0^t k(t, s)f(s, y(s))ds \right), \quad t \in [0, T] \quad (2.15)_\lambda$$

for each $\lambda \in [0, 1]$. Then (2.11) has a solution in $C([0, T], E)$.

Remark: One could also discuss the case $q = 1$ in Theorem 2.2.

Proof: The results follows from Theorem 2.1 once we show (2.5) and (2.6) are true. Notice first for any $r > 0$ and any $u \in E$ with $\|u\| \leq r$ there exists $\mu_r \in L^q([0, T], \mathbf{R})$ with $\|f(s, u)\| \leq \mu_r(s)$ for almost all $s \in [0, T]$. Then for each $t \in [0, T]$ and $\|u\| \leq r$ we have

$$\|K(t, s, u)\| \leq |k(t, s)| \mu_r(s) \equiv h_{t,r}(s) \text{ for almost all } s \in [0, T].$$

By Hölder's inequality for $x, z \in [0, T]$ with $x < z$, we have

$$\begin{aligned} \int_x^z h_{z,r}(s)ds &\leq \left(\int_0^z |k_z(s)|^p ds \right)^{\frac{1}{p}} \left(\int_x^z \mu_r^q(s)ds \right)^{\frac{1}{q}} \\ &\leq \max_{z \in [0, T]} \left(\int_0^z |k_z(s)|^p ds \right)^{\frac{1}{p}} \left(\int_x^z \mu_r^q(s)ds \right)^{\frac{1}{q}} \end{aligned}$$

and so

$$\int_x^z h_{z,r}(s)ds \leq \left(\int_x^z h_r(s)ds \right)^\alpha$$

where

$$\alpha = \frac{1}{q} > 0 \text{ and } h_r = [c\mu_r]^q \text{ with } c = \max_{z \in [0, T]} \left(\int_0^z |k_z(s)|^p ds \right)^{\frac{1}{p}}.$$

Consequently (2.5) is true. Finally (2.6) follows since if $t > z$, we have

$$\int_0^z \sup_{\|u\| \leq r} \|K(z, s, u) - K(t, s, u)\| ds = \int_0^z |k(z, s) - k(t, s)| \sup_{\|u\| \leq r} \|f(s, u)\| ds$$

$$\leq \left(\int_0^z |k_z(s) - k_t(s)|^p ds \right)^{\frac{1}{p}} \left(\int_0^T \mu_r^q(s) ds \right)^{\frac{1}{q}} \rightarrow 0 \text{ as } z \rightarrow t.$$

□

Essentially the same reasoning as in Theorem 2.1 establishes the following existence principle for the Urysohn integral equation (2.2).

Theorem 2.3: *Let $K: [0, T] \times [0, T] \times E \rightarrow E$. Suppose*

$$\left\{ \begin{array}{l} \text{there exists a constant } \gamma \geq 0 \text{ with } \alpha(K([0, T] \times [0, T] \times \Omega)) \leq \gamma \alpha(\Omega) \\ \text{for each bounded set } \Omega \subseteq E \end{array} \right. \quad (2.16)$$

$$2\gamma T < 1 \quad (2.17)$$

$$\left\{ \begin{array}{l} K: [0, T] \times [0, T] \times E \rightarrow E \text{ is } L^1\text{-Carathéodory in } t; \\ \text{by this we mean for each } t \in [0, T]; \\ \text{(i) the map } u \rightarrow K_t(s, u) \text{ is continuous for almost all } s \in [0, T] \\ \text{(ii) the map } s \rightarrow K_t(s, u) \text{ is measurable for all } u \in E \\ \text{(iii) for each } r > 0 \text{ there exists } h_{t,r} \in L^1([0, T], \mathbf{R}) \text{ such that} \\ \quad \|u\| \leq r \text{ implies } \|K_t(s, u)\| \leq h_{t,r}(s) \text{ for almost all } s \in [0, T] \end{array} \right. \quad (2.18)$$

$$\left\{ \begin{array}{l} K \text{ is integrably bounded in } t \text{ i.e. } \sup_{t \in [0, T]} \int_0^T h_{t,r}(s) ds < \infty \\ \text{where } h_{t,r}(s) \text{ is as in (2.18) (iii)} \end{array} \right. \quad (2.19)$$

$$\lim_{z \rightarrow t} \int_0^T \sup_{\|u\| \leq r} \|K(z, s, u) - K(t, s, u)\| ds = 0 \quad (2.20)$$

and

$$h \in C([0, T], E) \quad (2.21)$$

hold. In addition assume there is a constant M_0 , independent of λ , with $\|y\|_0 \neq M_0$ for any solution $y \in C([0, T], E)$ to

$$y(t) = \lambda \left(h(t) + \int_0^T K(t, s, y(s)) ds \right), \quad t \in [0, T] \quad (2.22)_\lambda$$

for each $\lambda \in [0, 1]$. Then (2.2) has a solution in $C([0, T], E)$.

Theorem 2.3 immediately yields the following result for the Hammerstein integral equation

$$y(t) = h(t) + \int_0^T k(t,s)f(s,y(s))ds, t \in [0, T]. \tag{2.23}$$

Theorem 2.4: *Let $k[0, T] \times [0, T] \rightarrow \mathbf{R}$ and $K(t, s, u) = k(t, s)f(s, u)$. Assume (2.16), (2.17) and (2.21) hold. Also suppose*

$$f: [0, T] \times E \rightarrow E \text{ is a } L^q\text{-Carathéodory function (here } q > 1 \text{ is a constant)} \tag{2.24}$$

$$k_t(s) \in L^p([0, T], \mathbf{R}) \text{ for each } t \in [0, T]; \text{ here } \frac{1}{p} + \frac{1}{q} = 1 \tag{2.25}$$

and

$$\text{the map } t \rightarrow k_t \text{ is continuous from } [0, T] \text{ to } L^p([0, T], \mathbf{R}) \tag{2.26}$$

are satisfied. In addition assume there is a constant M_0 , independent of λ , with $\|y\|_0 \neq M_0$ for any solution $y \in C([0, T], E)$ to

$$y(t) = \lambda \left(h(t) + \int_0^T k(t,s)f(s,y(s))ds \right), t \in [0, T] \tag{2.27}_\lambda$$

for each $\lambda \in [0, 1]$. Then (2.23) has a solution in $C([0, T], E)$.

3. Applications

In this section we use the existence principles of Section 2 to establish existence theory for various integral equations. We begin by discussing the Hammerstein equation

$$y(t) = h(t) + \int_0^T k(t,s)f(s,y(s))ds, t \in [0, T]. \tag{3.1}$$

Remark: An existence theory of “superlinear” type could easily be developed for (3.1) (or indeed the Urysohn integral equation (2.2)) using the ideas in [13]; however since the reasoning involved is essentially the same, we as a result will not include results of this type here.

We first establish a result of Brezis-Browder type [1] for (3.1).

Theorem 3.1: *Let $k: [0, T] \times [0, T] \rightarrow \mathbf{R}$ and $K(t, s, u) = k(t, s)f(s, u)$ and assume (2.16), (2.17), (2.21), (2.24), (2.25) and (2.26) hold. In addition suppose*

$$\left\{ \begin{array}{l} \text{there exists } R > 0 \text{ and a constant } a_0 > 0 \text{ with} \\ \langle f(t, y), y \rangle_+ \geq a_0 \|y\| \|f(t, y)\| \text{ for } \|y\| \geq R \text{ and a.e. } t \in [0, T] \end{array} \right. \tag{3.2}$$

$$\left\{ \begin{array}{l} \text{there exists constants } \eta > 0, \gamma \text{ with } \gamma \geq q - 1 \text{ and a function } \phi \in L^p([0, T], \mathbf{R}) \\ \text{with } \|y\| \geq \eta \|f(t, y)\|^\gamma + \phi(t) \text{ for } \|y\| \geq R \text{ and a.e. } t \in [0, T] \end{array} \right. \tag{3.3}$$

and
$$\left\{ \begin{array}{l} \text{there exists a constant } A_0 \geq 0 \text{ with for any } u \in C([0, T], E), \\ \int_0^T \langle f(t, u(t)), \int_0^T k(t, s) f(s, u(s)) ds \rangle_+ dt \leq A_0 \end{array} \right. \quad (3.4)$$

are satisfied. Then (3.1) has a solution in $C([0, T], E)$.

Proof: Let y be a solution of (2.27) $_{\lambda}$. We have (recall $\langle x, y + z \rangle_+ \leq \langle x, y \rangle_+ + \langle x, z \rangle_+$ where $x, y, z \in B$, a Banach space),

$$\begin{aligned} \int_0^T \langle f(t, y(t)), y(t) \rangle_+ dt &\leq \int_0^T \langle f(t, y(t)), h(t) \rangle_+ dt \\ &\quad + \lambda \int_0^T \langle f(t, y(t)), \int_0^T k(t, s) f(s, y(s)) ds \rangle_+ dt \end{aligned}$$

and so
$$\int_I \langle f(t, y(t)), y(t) \rangle_+ dt \leq \int_0^T \|h(t)\| \|f(t, y(t))\| dt + |A_0|. \quad (3.5)$$

Let

$$I = \{t \in [0, T]: \|y(t)\| \geq R\} \text{ and } J = \{t \in [0, T]: \|y(t)\| < R\}.$$

Notice

$$\begin{aligned} \int_0^T \langle f(t, y(t)), y(t) \rangle_+ dt &\geq a_0 \int_I \|y(t)\| \|f(t, y(t))\| dt \\ &\geq \eta a_0 \int_I \|f(t, y(t))\|^{\gamma+1} dt + a_0 \int_I \phi(t) \|f(t, y(t))\| dt. \end{aligned}$$

Put this into (3.5) to obtain

$$\begin{aligned} \eta a_0 \int_I \|f(t, y(t))\|^{\gamma+1} dt &\leq a_0 \int_I |\phi(t)| \|f(t, y(t))\| dt + \int_J \|y(t)\| \|f(t, y(t))\| dt \\ &\quad + \int_0^T \|h(t)\| \|f(t, y(t))\| dt + |A_0|. \end{aligned}$$

Since f is L^q -Carathéodory there exists $\mu_R \in L^q([0, T], \mathbf{R})$ such that $\|u\| \leq R$ implies $\|f(t, u)\| \leq \mu_R(t)$ for a.e. $t \in [0, T]$. Thus

$$\begin{aligned} \int_J \|y(t)\| \|f(t, y(t))\| dt + \int_J \|h(t)\| \|f(t, y(t))\| dt + |A_0| &\leq R \int_0^T \mu_R(t) dt + \int_0^T \|h(t)\| \mu_R(t) dt + |A_0| \equiv A_1 \end{aligned}$$

and so

$$\begin{aligned} \eta\alpha_0 \int_I \|f(t, y(t))\|^{\gamma+1} dt \\ \leq a_0 \int_I |\phi(t)| \|f(t, y(t))\| dt + \int_I \|h(t)\| \|f(t, y(t))\| dt + A_1. \end{aligned}$$

Apply Hölder's inequality to obtain

$$\begin{aligned} \eta\alpha_0 \int_I \|f(t, y(t))\|^{\gamma+1} dt \\ \leq a_0 T^{\frac{p\gamma - (\gamma+1)}{p(\gamma+1)}} \left(\int_0^T |\phi(s)|^p ds \right)^{\frac{1}{p}} \left(\int_I \|f(t, y(t))\|^{\gamma+1} dt \right)^{\frac{1}{\gamma+1}} \\ + \left(\int_0^T \|h(t)\|^{\frac{\gamma+1}{\gamma}} dt \right)^{\frac{\gamma}{\gamma+1}} \left(\int_I \|f(t, y(t))\|^{\gamma+1} dt \right)^{\frac{1}{\gamma+1}}. \end{aligned}$$

There exists a constant A_2 with

$$\int_I \|f(t, y(t))\|^{\gamma+1} dt \leq A_2.$$

Returning to (2.27) $_\lambda$ we have for $t \in [0, T]$ that

$$\begin{aligned} \|y(t)\| &\leq \sup_{[0, T]} \|h(t)\| + \int_J \|k(t, s)f(s, y(s))\| ds + \int_I \|k(t, s)f(s, y(s))\| ds \\ &\leq \sup_{[0, T]} \|h(t)\| + \sup_{t \in [0, T]} \left(\int_0^T |k(t, s)|^p ds \right)^{\frac{1}{p}} \left(\int_0^T \mu_R(s) ds \right)^{\frac{1}{q}} \\ &\quad + T^{\frac{p\gamma - (\gamma+1)}{p(\gamma+1)}} \sup_{t \in [0, T]} \left(\int_0^T |k(t, s)|^p ds \right)^{\frac{1}{p}} A_2^{\frac{1}{\gamma+1}} \equiv M_0 \end{aligned}$$

and so

$$\sup_{[0, T]} \|y(t)\| \leq M_0$$

for any solution y to (2.27) $_\lambda$. The result follows from Theorem 2.4. □

Essentially the same reasoning as in Theorem 3.1 establishes the following existence result for the Volterra equation

$$y(t) = h(t) + \int_0^t k(t, s)f(s, y(s))ds, t \in [0, T]. \tag{3.6}$$

Theorem 3.2: Let $k: [0, T] \times [0, t] \rightarrow \mathbf{R}$ and $K(t, s, u) = k(t, s)f(s, u)$ and assume (2.3), (2.4), (2.7), (2.12), (2.13), (2.14), (3.2) and (3.3) hold. In addition, suppose

$$\left\{ \begin{array}{l} \text{there exists a constant } A_0 \geq 0 \text{ with for any } u \in C([0, T], E), \\ \int_0^T \langle f(t, u(t)), \int_0^t k(t, s)f(s, u(s))ds \rangle_+ dt \leq A_0 \end{array} \right. \quad (3.6)$$

is satisfied. Then (3.6) has a solution in $C([0, T], E)$.

Remark: Existence theory of “growth type” could also easily be developed for (3.6) (or indeed (2.1)) using the ideas in [12].

For our next application we will examine the abstract Dirichlet boundary value problem

$$\left\{ \begin{array}{l} y'' + f(t, y) = 0 \text{ a.e. on } [0, 1] \\ y(0) = y(1) = 0; \end{array} \right. \quad (3.8)$$

here $y: [0, 1] \rightarrow H$ where H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We give a notion of “solution tube” for such problems in the Hilbert space setting. Our theory was motivated by ideas in [4, 5]. We will assume that $f: [0, 1] \times H \rightarrow H$ is a L^1_{loc} -Carathéodory function. By this we mean

- (i) the map $t \mapsto f(t, z)$ is measurable for all $z \in E$;
- (ii) the map $z \mapsto f(t, z)$ is continuous for almost all $t \in [0, 1]$;
- (iii) for any $r > 0$ there exists $h_r \in L^1_{loc}(0, 1)$ with $\|f(t, z)\| \leq h_r(t)$ for almost all $t \in [0, 1]$ and all $\|z\| \leq r$; also $\int_0^1 x(1-x)h_r(x)dx < \infty$ with $\lim_{t \rightarrow 0^+} t^2(1-t)h_r(t) = 0$ if $\int_0^1 (1-x)h_r(x)dx = \infty$ and $\lim_{t \rightarrow 1^-} t(1-t)^2h_r(t) = 0$ if $\int_0^1 xh_r(x)dx = \infty$.

Remark: It is worth remarking that other boundary data (homogeneous and non-homogeneous) could also be considered here. However in our opinion (3.8) is the “most difficult” to examine (i.e., the “most singular”) and as a result we will concentrate our study on (3.8).

By a solution to (3.8) we mean a function $y \in AC([0, 1], H) \cap C^1([0, 1], H)$ with $y' \in AC_{loc}((0, 1), H)$ which satisfies the differential equation in (3.8) almost everywhere, and the stated boundary data. One can check (see [10, 14]) that solving (3.8) is equivalent to finding a function $y \in C([0, 1], H)$ to

$$y(t) = \int_0^1 k(t, s)f(s, y(s))ds, \quad t \in [0, 1] \quad (3.9)$$

where

$$k(t, s) = \begin{cases} (1-t)s, & 0 \leq s \leq t \\ (1-s)t, & t \leq s \leq 1. \end{cases}$$

Remark: Notice h_r in (iii) above is not necessarily in $L^1[0, 1]$.

Theorem 3.3: Let $K(t, s, u) = k(t, s)f(s, u)$ and suppose the following conditions are satisfied:

$$\left\{ \begin{array}{l} \text{there exists a constant } 0 \leq 2\gamma < 1 \text{ with } \alpha(K([0, 1] \times [0, 1] \times \Omega)) \leq \gamma\alpha(\Omega) \\ \text{for each bounded set } \Omega \subseteq H \end{array} \right. \tag{3.10}$$

$$f: [0, 1] \times H \rightarrow H \text{ is a } L^1_{loc}\text{-Carathéodory function} \tag{3.11}$$

$$\left\{ \begin{array}{l} \text{for any } t_0 \in (0, 1), t_1 \in (0, 1) \text{ with } t_0 < t_1, \text{ there exists} \\ v \in AC([0, 1], H) \cap C^1((0, 1), H) \text{ with } v' \in AC_{loc}((0, 1), H), \\ \text{and } M \in AC([0, 1], [0, \infty)) \cap C^1((0, 1), \mathbf{R}) \text{ with } M' \in AC_{loc}((0, 1), \mathbf{R}) \\ (v \text{ and } M \text{ are independent of } t_0, t_1) \text{ with} \\ \langle y - v(t), -f(t, y) - v''(t) \rangle \geq M''(t)M(t) \\ \text{for a.e. } t \in [t_0, t_1] \text{ and all } y \in H \text{ with } \|y - v(t)\| = M(t) \text{ and } M(t) \neq 0 \end{array} \right. \tag{3.12}$$

$$\left\{ \begin{array}{l} \text{for any } t_0 \in (0, 1), t_1 \in (0, 1) \text{ with } t_0 < t_1, \text{ there exists } v \text{ and} \\ M \text{ as in (3.12) with} \\ \frac{\langle y - v(t), -f(t, v(t)) - v''(t) \rangle}{\|y - v(t)\|} \geq M''(t) \\ \text{for a.e. } t \in [t_0, t_1] \text{ and all } y \in H \text{ with } \|y - v(t)\| > M(t) \text{ and } M(t) = 0 \end{array} \right. \tag{3.13}$$

and

$$\|v(0)\| \leq M(0) \text{ and } \|v(1)\| \leq M(1). \tag{3.14}$$

Then (3.8) has a solution y with $\|y(t) - v(t)\| \leq M(t)$ for all $t \in [0, 1]$.

Proof: Consider the problem

$$y(t) = \int_0^1 k(t, s)f(s, p(s, y(s)))ds, t \in [0, 1] \tag{3.15}$$

where

$$p(t, y) = \min \left\{ 1, \frac{M(t)}{\|y - v(t)\|} \right\} y + \left(1 - \min \left\{ 1, \frac{M(t)}{\|y - v(t)\|} \right\} \right) v(t),$$

i.e.

$$p(t, y) = \begin{cases} y, & \text{if } \|y - v(t)\| \leq M(t) \\ M(t) \frac{y - v(t)}{\|y - v(t)\|} + v(t), & \text{if } \|y - v(t)\| > M(t) \end{cases}$$

is the radial retraction of H onto $\{y: \|y - v(t)\| \leq M(t)\}$. We now show (3.15) has solution in $C([0, 1], H)$ by applying Theorem 1.2. Define the operator $N: C([0, 1], H) \rightarrow C([0, 1], H)$ by

$$Ny(t) = \int_0^1 k(t, s)f(s, p(s, y(s)))ds.$$

Let $u_n \rightarrow u$ in $C([0, 1], H)$. Then

$$\begin{aligned} \|Nu_n(t) - Nu(t)\| &\leq (1-t) \int_0^t s \|f(s, p(s, u_n(s))) - f(s, p(s, u(s)))\| ds \\ &\quad + t \int_t^1 (1-s) \|f(s, p(s, u_n(s))) - f(s, p(s, u(s)))\| ds \\ &\leq \int_0^t (1-s)s \|f(s, p(s, u_n(s))) - f(s, p(s, u(s)))\| ds \\ &\quad + \int_t^1 (1-s)s \|f(s, p(s, u_n(s))) - f(s, p(s, u(s)))\| ds \\ &= \int_0^1 (1-s)s \|f(s, p(s, u_n(s))) - f(s, p(s, u(s)))\| ds. \end{aligned}$$

So $N: C([0, 1], H) \rightarrow C([0, 1], H)$ is continuous. Now let $\Omega \subseteq C([0, 1], H)$ be bounded i.e., there exists $r > 0$ with $\|u(s)\| \leq r$ for all $s \in [0, 1]$ and $u \in \Omega$. There exists h_r as in the definition of L^1_{loc} -Carathéodory with

$$\|f(s, u)\| \leq h_r(s) \text{ for a.e. } s \in [0, 1] \text{ and all } \|u\| \leq r.$$

Now $N\Omega$ is bounded since for $t \in [0, 1]$ and $u \in \Omega$ we have

$$\|Nu(t)\| \leq (1-t) \int_0^t sh_r(s)ds + t \int_t^1 (1-s)h_r(s)ds \leq \int_0^1 s(1-s)h_r(s)ds.$$

Notice also for $u \in \Omega$ and $t \in [0, 1]$ that

$$(Nu)'(t) = \int_0^t sf(s, u(s))ds + \int_t^1 (1-s)f(s, u(s))ds$$

so we have

$$\|(Nu)'(t)\| \leq \int_0^t h_r(s)ds + \int_t^1 (1-s)h_r(s)ds \equiv \tau_r(t). \tag{3.16}$$

It is easy to check since $\int_0^1 s(1-s)h_r(s)ds < \infty$ that $\tau_r \in L^1[0, 1]$. Consequently $N\Omega$ is equicontinuous on $[0, 1]$. Next we show

$$\alpha(N\Omega) \leq 2\gamma\alpha(\Omega). \tag{3.17}$$

For $t \in [0, 1]$, we have

$$\alpha(N\Omega(t)) = \alpha\left(\left\{\int_0^1 K(t, s, p(s, u(s)))ds : u \in \Omega\right\}\right)$$

$$\begin{aligned} &\leq \alpha(\{K(t, s, p(s, u(s))): u \in \Omega, s \in [0, 1]\}) \\ &\leq \alpha(K([0, 1] \times [0, 1] \times \overline{c\bar{o}}(\Omega[0, 1] \cup v[0, 1]))) \end{aligned}$$

since if $u \in \Omega$ and $s \in [0, 1]$ we have

$$p(s, u(s)) = \lambda_s u(s) + (1 - \lambda_s)v(s) \in \overline{c\bar{o}}(\Omega[0, 1] \cup v[0, 1])$$

where

$$\lambda_s = \min \left\{ 1, \frac{M(s)}{\|u(s) - v(s)\|} \right\}.$$

Thus

$$\begin{aligned} \alpha(N \Omega(t)) &\leq \gamma\alpha(\overline{c\bar{o}}(\Omega[0, 1] \cup v[0, 1])) = \gamma\alpha(\Omega[0, 1] \cup v[0, 1]) \\ &= \gamma\alpha(\Omega[0, 1]) \leq 2\gamma\alpha(\Omega) \end{aligned}$$

and so

$$\alpha(N \Omega) = \sup_{t \in [0, 1]} \alpha(N \Omega(t)) \leq 2\gamma\alpha(\Omega).$$

Thus (3.17) is true. Theorem 1.2 implies that (3.15) has a solution $y \in C([0, 1], H)$. Next we *claim* $\|y(t) - v(t)\| \leq M(t)$ for $t \in [0, 1]$. If the claim is true then y is a solution of (3.9) and consequently y is a solution of (3.8).

It remains to prove the claim. If the claim is not true then

$$\|y(t) - v(t)\| - M(t)$$

has its positive (absolute) maximum at, say, $t_2 \in (0, 1)$. Choose $t_0 > 0$, $t_1 < 1$, $t_0 < t_2 < t_1$ with $\|y(t) - v(t)\| - M(t) > 0$ for $t \in (t_0, t_1)$ and

$$\|y(t_1) - v(t_1)\| - M(t_1) < \|y(t_2) - v(t_2)\| - M(t_2); \tag{3.18}$$

this is possible since $\|y(1) - v(1)\| - M(1) \leq 0$. Also we have

$$(\|y - v\| - M)'(t_2) = 0. \tag{3.19}$$

In addition for a.e. $t \in (t_0, t_1)$ we have

$$\begin{aligned} &\|y(t) - v(t)\|'' \\ &= \frac{\langle y(t) - v(t), y''(t) - v''(t) \rangle + \|y'(t) - v'(t)\|^2}{\|y(t) - v(t)\|} - \frac{[\langle y(t) - v(t), y'(t) - v'(t) \rangle]^2}{\|y(t) - v(t)\|^3} \\ &\geq \frac{\langle y(t) - v(t), y''(t) - v''(t) \rangle}{\|y(t) - v(t)\|} \\ &= \frac{\langle y(t) - v(t), -f(t, p(t, y(t))) - v''(t) \rangle}{\|y(t) - v(t)\|} \\ &\geq M''(t). \end{aligned}$$

To see the last inequality there are two cases to consider, namely $M(t) \neq 0$ and $M(t) = 0$. If $M(t) \neq 0$ we have

$$\frac{\langle y(t) - v(t), -f(t, p(t, y(t))) - v''(t) \rangle}{\|y(t) - v(t)\|} = \frac{\langle p(t, y(t)) - v(t), -f(t, p(t, y(t))) - v''(t) \rangle}{M(t)} \geq M''(t)$$

by (3.12), whereas if $M(t) = 0$ we have

$$\frac{\langle y(t) - v(t), -f(t, p(t, y(t))) - v''(t) \rangle}{\|y(t) - v(t)\|} = \frac{\langle y(t) - v(t), -f(t, v(t)) - v''(t) \rangle}{\|y(t) - v(t)\|} \geq M''(t)$$

by (3.13). Consequently,

$$(\|y - v\| - M)''(t) \geq 0 \text{ for a.e. } t \in (t_0, t_1). \tag{3.20}$$

Now (3.19) and (3.20) imply

$$(\|y - v\| - M)'(t) \geq 0 \text{ for } t \in (t_2, t_1)$$

and consequently

$$\|y(t_1) - v(t_1)\| - M(t_1) \geq \|y(t_2) - v(t_2)\| - M(t_2).$$

This contradicts (3.18). Thus our claim is true and we are finished. □

Remark: Let $H = \mathbf{R}$ and suppose $\alpha, \beta \in AC([0, 1], \mathbf{R}) \cap C^1([0, 1], \mathbf{R})$ with $\alpha', \beta' \in AC_{loc}((0, 1), \mathbf{R})$, are respectively lower and upper solutions of (3.8) (i.e., $\alpha'' + f(t, \alpha) \geq 0$ a.e. on $[0, 1]$, $\alpha(0) \leq 0$, $\alpha(1) \leq 0$ and $\beta'' + f(t, \beta) \leq 0$ a.e. on $[0, 1]$, $\beta(0) \geq 0$, $\beta(1) \geq 0$) with $\alpha(t) \leq \beta(t)$ for $t \in [0, 1]$. Its easy to check that

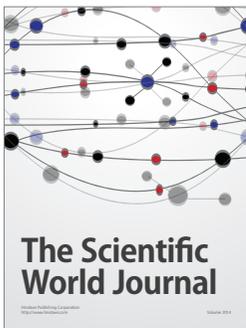
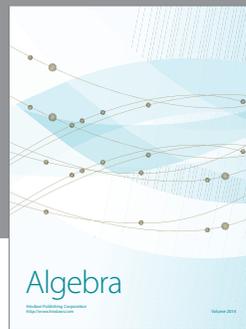
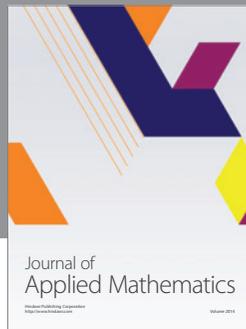
$$v = \frac{\alpha + \beta}{2} \text{ and } M = \frac{\beta - \alpha}{2}$$

satisfy (3.12), (3.13) and (3.14) in this case. Of course, (3.10) is satisfied with $\gamma = 0$. Consequently a special case of Theorem 3.3 is the result in [3, 10].

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