ON CERTAIN RANDOM POLYGONS OF LARGE AREAS

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Consider the tessellation of a plane into convex random polygons determined by a unit intensity Poissonian line process. Let $M(A)$ be the ergodic intensity of random polygons with areas exceeding a value A . A two-sided asymptotic bound

$$\exp\{-2\sqrt{A/\pi} + c_0 A^{1/6}\} < M(A) < \exp\{-2\sqrt{A/\pi} + c_1 A^{1/6}\}$$

is established for large A , where $c_0 > 2.096$, $c_1 < 6.36$.

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1. The Problem Statement and Main Results

This paper is devoted to the discussion of a problem posed by D.G. Kendall in his foreword to the book [9]. The problem concerns the investigation of the tail of the distribution of the area of a random polygon.

Consider a unit intensity Poissonian line process in $\mathbb{R}^2$. Such a process can be determined by a planar Poissonian process of points (p_i, φ_i) , with a planar intensity $1/\pi$ in the band $\mathbb{R}_+ \times (0, 2\pi)$, in such a way that each of the points generates a random line with polar coordinates (p_i, φ_i) for the foot of the perpendicular from the origin O to the line. Processes of this kind have been investigated thoroughly by Miles [6-8] and others. (Miles uses an equivalent definition: $p_i \in \mathbb{R}$, $\varphi_i \in (0, \pi)$.) The line processes determines the tessellation of the plane into convex random polygons.

Consider the ergodic intensity $M(A)$ of random polygons with areas exceeding a value A .

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It is necessary to specify a precise definition of ergodic intensities for random polygons. The simplest way to do this is a reduction to the ergodic intensity of an appropriate point process. Evidently, all the sides of a random polygon have different lengths (almost surely). Thus every random polygon K can be associated with a well-defined point P_K , namely, its vertex such that, when running along the contour of K in the positive direction, the greatest of its sides has P_K as its endpoint.

We define the function $M(A)$ as the ergodic intensity of the planar point process of random points P_K considering only random polygons K with areas $A(K) > A$. In turn, the ergodic intensity of a planar point process is defined as the mean number of random points in a unit area. From the Korolyuk theorem [3], for a simple stationary point process, the probability that a random point falls into an area element is equivalent to the ergodic intensity of the point process multiplied by the area.

Two asymptotic bounds are established for $M(A)$, as follows.

Theorem 1: *The bound*

$$M(A) > \exp\left\{-2\sqrt{A/\pi} + c_0 A^{1/6}(1 + o(1))\right\} \quad (1)$$

holds for a constant $c_0 > 2.096$ as $A \rightarrow \infty$.

Theorem 2: *The bound*

$$M(A) < \exp\left\{-2\sqrt{A/\pi} + c_1 A^{1/6}(1 + o(1))\right\} \quad (2)$$

holds for a constant $c_1 < 6.36$ as $A \rightarrow \infty$.

2. Proof of Theorem 1

Consider an event Ω_r : {no random line crosses the circle C_r of the radius r , with the center in the origin}. Evidently, $P\{\Omega_r\} = e^{-2r}$. If the event Ω_r occurs then the circle C_r is surrounded by a random polygon K_ϖ (a so-called Crofton cell).

Let $\{(r + X(t), t), 0 \leq t \leq 2\pi\}$ be the graph of K_ϖ in polar coordinates. Then

$$A(K_\varpi) = \frac{1}{2} \int_0^{2\pi} (r + X(t))^2 dt. \quad (3)$$

It is convenient to consider the positive square root $a(K)$ of the area $A(K)$ of a polygon K . By the Bounjakowsky (Cauchy) inequality,

$$2\pi \int_0^{2\pi} X^2(t) dt \geq \left(\int_0^{2\pi} X(t) dt \right)^2.$$

Thus equation (3) implies the bound

$$a(K_\varpi) \geq r\sqrt{\pi} \left(1 + \frac{1}{2\pi r} \int_0^{2\pi} X(t) dt \right), \quad (4)$$

provided Ω_r occurred.

From Miles' theory, the probability of a random line not crossing a convex figure

equals $e^{-S/\pi}$, where S is the perimeter of the figure. The event $\Omega_r \cap \{X(t) > x\}$ means no crossing by a random line of the convex hull of C_r completed by the point $(r+x, t)$. The perimeter of the hull equals $2r(\pi + \tan \alpha - \alpha)$, where $\alpha = \arccos(r/(r+x))$. Hence,

$$P\{X(t) > x \mid \Omega_r\} = \exp\{-2r(\tan \alpha - \alpha)/\pi\}. \quad (5)$$

It is convenient to introduce a monotonic transformation $\alpha(t) = \arccos(r/(r+X(t)))$. Since $\{X(t) > x\} = \{\alpha(t) > \alpha\}$, the lefthand side of equation (5) is just $P(\alpha(t) > \alpha \mid \Omega_r)$. Moreover, consider truncated random variables

$$\bar{\alpha}(t) = \begin{cases} \alpha(t) & \text{if } \alpha(t) < \alpha_0, \\ 0 & \text{if } \alpha(t) > \alpha_0 \end{cases}$$

and $\bar{X}(t) = r((1/\cos \bar{\alpha}(t)) - 1)$. Thanks to equation (5), the conditional p.d.f. $f(\alpha)$ of the random variable $\bar{\alpha}(t)$ has the form

$$f(\alpha) = (2r/\pi) \tan^2 \alpha \exp\{-2r(\tan \alpha - \alpha)/\pi\}$$

inside the interval $(0, \alpha_0)$, provided Ω_r occurred. Choose α_0 as $\alpha_0 = (\ln r/r)^{1/3} \ln \ln r$. It can be easily seen that, $\tan \alpha - \alpha \cong \alpha^3/3$, $\tan \alpha \cong \alpha$ uniformly in the interval $\{0 < \alpha < \alpha_0\}$; hence

$$E\bar{\alpha}^m(t \mid \Omega_r) \cong (2r/\pi) \int_0^{\alpha_0} \alpha^{m+2} \exp\{-2r\alpha^3/(3\pi)\} d\alpha; \text{ as } r \rightarrow \infty$$

for any positive m . Moreover, the corresponding integral over the interval $\{\alpha > \alpha_0\}$ can be proved to decrease more rapidly than r^{-N} for a given N ; therefore,

$$E\bar{\alpha}^m \cong (2r/\pi) \int_0^{\infty} \alpha^{m+2} \exp\{-2r\alpha^3/(3\pi)\} d\alpha, \text{ as } r \rightarrow \infty.$$

In particular,

$$E\{\alpha^2(t) \mid \Omega_r\} \cong (3\pi/(2r))^{2/3} \Gamma(5/3) \text{ as } r \rightarrow \infty; \quad (6)$$

$$E\{\alpha^4(t) \mid \Omega_r\} \cong O(r^{-4/3}) \text{ as } r \rightarrow \infty. \quad (7)$$

As $(1/\cos z) > 1 + z^2/2$ for any $z: 0 < z < \pi/2$, we have the inequality

$$\bar{X}(t) > r\bar{\alpha}^2(t)/2. \quad (8)$$

From Equations (8), (6) and (4),

$$E\{a(K_{\varpi}) \mid \Omega_r\} \geq r\sqrt{\pi}(1 + \frac{1}{2} \left(\frac{3\pi}{2r}\right)^{2/3} \Gamma(5/3)(1 + o(1))) \text{ as } r \rightarrow \infty.$$

Note that $\text{cov}(\bar{\alpha}(t), \bar{\alpha}(\tau)) = 0$ as soon as the angle distance between t and τ exceeds $2\alpha_0$. Applying also equation (7), one obtains the bound

$$\text{Var} \left\{ \int_0^{2\pi} \bar{\alpha}^2(t) dt \mid \Omega_r \right\} = O(r^{-5/3} (\ln r)^{1/3} \ln \ln r). \quad (9)$$

Equations (9), (8), (6) and (4) imply the bound

$$P\left\{a(K_{\varpi}) \geq r\sqrt{\pi}\left(1 + \frac{1}{2}\left(\frac{3\pi}{2r}\right)^{2/3}\Gamma(5/3)(1-\delta)\right) \mid \Omega_r\right\} \rightarrow 1 \text{ as } r \rightarrow \infty \quad (10)$$

for any $\delta > 0$.

Choose r as follows:

$$r = \frac{a}{\sqrt{\pi}}\left(1 - Ca^{-2/3}(1-2\delta)\right) \quad (11)$$

where

$$C = \pi(3/2)^{2/3}\Gamma(5/3)/2. \quad (12)$$

Then equation (10) implies the bound

$$P\{a(K_{\varpi}) > a \mid \Omega_r\} \rightarrow 1 \text{ as } a \rightarrow \infty \quad (13)$$

and hence

$$P\{\Omega_r \cap \{a(K_{\varpi}) > a\}\} \cong P\{\Omega_r\} = e^{-2r}. \quad (14)$$

Inserting r , defined by equation (11), into equation (14), and changing a^2 to A leads to the following bound:

$$P\{C_r \subset K_{\varpi}; A(K_{\varpi}) > A\} > \exp\{-2\sqrt{A/\pi} + c_0 A^{1/6}(1 + o(1))\}$$

with $c_0 = (3/2)^{2/3}\sqrt{\pi}\Gamma(5/3) > 2.096$.

To make a passage from probabilities to ergodic intensities, we will prove that a typical Crofton cell lies inside a certain circle of the radius $3r$, provided the event Ω_r occurred. Consider three concentric circles C_r, C_{2r}, C_{3r} of the radii $r, 2r, 3r$, respectively. The event $\Omega_r \cap \{X(t) > 2r\}$ implies the event $\Omega_r \cap \{X(\tau) > r, \tau \in \Delta_t\}$ where Δ_t is an interval of a positive length. If N is large enough then

$$\Omega_r \cap \{X(\tau) > r, \tau \in \Delta_t\} \subset \bigcup_{k=0}^{N-1} \Omega_r \cap \{X(2\pi k/N) > r\}$$

for every $t, 0 \leq t \leq 2\pi$. Therefore, due to equation (5) with $\alpha = \pi/3$,

$$P\left\{\max_t X(t) > 2r \mid \Omega_r\right\} \leq N \exp\{-2r(\sqrt{3} - \pi/3)/\pi\} = o(1) \text{ as } r \rightarrow \infty.$$

We have the relation

$$\Omega_r \cap \left\{\max_t X(t) \leq 2r\right\} \subset \Omega_r \cap \left\{P_{K_{\varpi}} \in C_{3r}\right\};$$

since

$$A(C_{3r}) = 9\pi r^2,$$

we have

$$9\pi r^2 M(A) \geq P\left\{\Omega_r \cap \left\{\max_t X(t) \leq 2r\right\}\right\} \cong 9\pi r^2 P\{\Omega_r\}. \quad (15)$$

Equations (11-15) imply the relation in equation (1).

3. A Bound for Ergodic Intensities $M_n(A)$

Evidently,

$$M(A) = \sum_{n=3}^{\infty} M_n(A),$$

where $M_n(A)$ is the contribution of n -gons to $M(A)$. The following lemma presents an upper bound for $M_n(A)$; we set $A = a^2$.

Lemma 1: *A bound*

$$M_n(a^2) \leq \frac{cn}{(2n-3)!a} (2\pi)^{2n} L_{n-1} \quad (16)$$

holds true with

$$L_{n-1} = \int \cdots \int_{y_i > 0, y_1 + \dots + y_n > 2a(1 + \Delta_n)/\sqrt{\pi}} \exp\{-(y_1 + \dots + y_n)\} dy_1 \dots dy_{n-1}, \quad (17)$$

where y_n is a function of the remaining y_i 's;

$$y_n > (y_1 + \dots + y_{n-1})/(n-1), \quad (18)$$

$$\Delta_n = \left(\frac{n}{\pi} \tan \frac{\pi}{n}\right)^{1/2} - 1. \quad (19)$$

Proof: Let K be a random n -gon such that its vertex P_K (see Section 1 for a definition) falls into the circle C_ρ of a small radius ρ , with center at the origin. Denote vertices of K by $P_1, \dots, P_n$ in the positive direction; for definiteness, set $P_n = P_K$ (i.e., the endpoint of the side of the greatest length). Also, denote the exterior angle of K corresponding to the vertex P_i by θ_i , and set $x_1 = |P_n P_1|$; $x_i = |P_{i-1} P_i|$, $2 \leq i \leq n$. For simplicity, we will also give the line $P_{i-1} P_i$ the name X_i . An n -gon K can be coded as $(\theta, \mathbf{x})$, where $\theta = (\theta_1, \dots, \theta_{n-2})$ and $\mathbf{x} = (x_1, \dots, x_{n-1})$. Given a position of the line X_1 , these parameters determine x_n and θ_{n-1} uniquely as $\rho \rightarrow 0$. For example, $x_n = |OP_{n-1}|$ in the limit. By definition, see Section 1, $x_n = \max_{1 \leq i \leq n} \{x_i\}$. It is well known that the perimeter of an n -gon K does not exceed the perimeter of a regular n -gon, given a value of the area. Thus, $x_1 + \dots + x_n > 2a\sqrt{\pi}(1 + \Delta_n)$ as soon as $A(K) \geq a^2$. These notes explain the bounds for y_i in equations (17-19), where $y_i = x_i/\pi$. Consider the probability of the occurrence of a random polygon in elementary volumes $d\theta$, $d\mathbf{x}$, and $\pi\rho^2$ for parameters, $\theta, \mathbf{x}$, and P_K , respectively. This differential probability can be considered as the product of the following expressions:

- (i) 2ρ = the probability of a random line X_1 crossing C_ρ ;
- (ii) $(2\pi)^{-n+2} d\theta_1 \dots d\theta_{n-2}$ = the probability of the choice of the directions of random lines $X_2, \dots, X_{n-1}$;
- (iii) $2^{n-2} \sin \theta_1 dx_1 \dots \sin \theta_{n-2} dx_{n-2}$ = the probability of the crossing of X_i by X_{i+1} in the intervals dx_i , given θ_i ;
- (iv) $2\rho \sin \theta_{n-1} dx_{n-1} / (\pi x_n)$ = the probability of the occurrence of a random line crossing both an interval dx_{n-1} of the line X_{n-1} and the circle C_ρ ; and
- (v) $\exp\{-(x_1 + \dots + x_n)/\pi\}$ = the probability of no random line crossing the polygon K . [The above formulation is close to that of Miles [6-8]].

Applying a usual principle

$$\int f(\theta, \mathbf{x}) d\theta d\mathbf{x} \leq \int d\theta \int \sup_{\theta} f(\theta, \mathbf{x}) d\mathbf{x}$$

we may integrate in θ separately:

$$\begin{aligned} & \int \cdots \int \sin \theta_1 \cdots \sin \theta_{n-1} (d\theta_1 \cdots d\theta_{n-2}) \\ & < \int \cdots \int \theta_1 \cdots \theta_{n-2} (2\pi - \theta_1 - \cdots - \theta_{n-2}) d\theta_1 \cdots d\theta_{n-2} \\ & = (2\pi)^{2n-3} / (2n-3)!. \end{aligned} \quad (20)$$

Having collected expressions (i) to (v), the bound in equation (20), applying the inequality $x_n > (x_1 + \cdots + x_n)/n > 2a\sqrt{\pi}/n$, and also having divided by $\pi\rho^2$, one obtains the bound in equations (16) and (17) directly.

4. A Proof of Theorem 2

Let L_{n-1}^0 denote the contribution of the domain $\{y_1 + \cdots + y_{n-1} < 2a(1 + \Delta_n)/\sqrt{\pi}\}$ to the integral L_{n-1} [see equation (17)]; $L_{n-1}^1 = L_{n-1} - L_{n-1}^0$. Due to equation (16), we have

$$M_n(a^2) \leq M_n^0(a^2) + M_n^1(a^2)$$

where $M_n^j(a^2)$ is defined as the righthand side of equation (16) with L_{n-1} changed by L_{n-1}^j ($j = 0, 1$). We have

$$\begin{aligned} L_{n-1}^0 & < \exp\{-2a(1 + \Delta_n)/\sqrt{\pi}\} \int \cdots \int_{y_i > 0, y_1 + \cdots + y_{n-1} < 2a(1 + \Delta_n)/\sqrt{\pi}} dy_1 \cdots dy_{n-1} \\ & = (2a(1 + \Delta_n)/\sqrt{\pi})^{n-1} \exp\{-2a(1 + \Delta_n)/\sqrt{\pi}\} / (n-1)!. \end{aligned} \quad (21)$$

Equation (21) implies the bound $\sum_{n=3}^N M_n^0(a^2) = O(-2a(1 + \varepsilon)/\sqrt{\pi})$ as $a \rightarrow \infty$ for a given N ; thus, only the case of large n should be investigated. Applying Stirling's formula to the factorials and also applying the relation

$$\Delta_n \cong \pi^2/(6n^2) = o(1/n),$$

one obtains the bound

$$M_n^0(a^2) < c_1(n^4/a^2)(2\pi^{3/2}e^3a/n^3)^n \exp\{-2a(1 + \pi^2/(6n^2))/\sqrt{\pi n^2}\}.$$

To apply a usual asymptotic analysis, see Dingle [2], introduce a variable

$$x = n/(2\pi^{3/2}e^3a)^{1/3}$$

and search the maximum of the expression

$$\begin{aligned} & \ln(\exp\{-2a\Delta_n/\sqrt{\pi}\}/x^{3n}) \\ & \approx \{-18.211x \ln x - 0.0504x^{-2}\}a^{1/3}. \end{aligned}$$

As the result of computations, the value 6.36 is obtained as an upper bound. By a standard argument in Dingle [2] it can be shown that

$$M^0(a^2) < \exp\{-2a/\sqrt{\pi} + 6.36a^{1/3}\} \quad (22)$$

for large a .

As for $M_n^1(a^2)$, consider two cases:

- (i) $n \geq a/\ln a$.
- (ii) $n < a/\ln a$.

In the case (i) it is sufficient to note that $L_{n-1}^1 < 1$ whereas

$$\sum_n \frac{n}{(2n-3)!} (2\pi)^{2n} = O(e^{-2(1-\delta)a}) \text{ as } a \rightarrow \infty$$

for a given $\delta > 0$, and thus

$$\sum_n M_n^1(a^2) < \exp\{-2a/\sqrt{\pi}\} \quad (23)$$

for large a . In case (ii), we omit the factor $(1 + \Delta_n)$ in equation (17) and note that

$$\int \dots \int_{y_i > 0, y_1 + \dots + y_n > x} e^{-(y_1 + \dots + y_n)} dy_1 \dots dy_n < (ex/n)^n e^{-x}$$

as soon as $n < x$. Hence,

$$\begin{aligned} M_{n+1}^1(a^2) & < \frac{c(n+1)}{a(2n-1)!} \left(\frac{2ae}{\sqrt{\pi n}} \right)^n \\ & \times (2\pi)^{2n+2} \exp\{-2a(1+1/n)/\pi^{1/2}\} = Q_n(a), \text{ say.} \end{aligned} \quad (24)$$

From equation (24), we have a relation

$$Q_{n+1}(a)/Q_n(a) \cong 2a\pi^{3/2}n^{-3} \exp\{2a(1+o(1))/(\pi^2 n^{1/2})\} \quad (25)$$

for large n . The righthand side of equation (25) is large as $n < a^{(1-\varepsilon)/2}$ and small as $n > a^{1/2}$. Hence $\arg \max Q_n(a) = a^\theta$ where $(1-\varepsilon)/2 < \theta < 1/2$. For such n , equation (24) implies the relation

$$M_{n+1}(a^2) < \exp\{-2(a+a^{1/2})/\pi^{1/2}\};$$

hence,

$$\Sigma_{n < a/\ln a} M_{n+1}^1(a^2) < \exp\{-2a/\pi^{1/2}\}. \quad (26)$$

Combining the bounds in equations (22), (23) and (26) leads to the desired equation (2).

5. Remarks

The problem considered here is closely related to a “long-standing conjecture of D.G. Kendall” concerning shapes of random polygons. In a version suggested by Miles [8], this conjecture is as follows: Let $\mu(A)dA$ be the ergodic intensity of random polygons of the type considered as above, and $\mu_\varepsilon(A)dA$ be the ergodic intensity of those contours which, moreover, are surrounded by concentric circles of radii $\sqrt{A/\pi}(1 \pm \varepsilon)$. Then:

$$\mu_\varepsilon(A)/\mu(A) \rightarrow 1 \text{ as } A \rightarrow \infty \quad (27)$$

for a given $\varepsilon > 0$. For two different proofs of equation (27), both based on an inequality of Bonnesen [1], see Kovalenko [4, 5].

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