ON THE APPROXIMATION OF AN INTEGRAL BY A SUM OF RANDOM VARIABLES

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We approximate the integral of a smooth function on $[0, 1]$, where values are only known at n random points (i.e., a random sample from the uniform- $(0, 1)$ distribution), and at 0 and 1. Our approximations are based on the trapezoidal rule and Simpson's rule (generalized to the non-equidistant case), respectively. In the first case, we obtain an n^2 -rate of convergence with a degenerate limiting distribution; in the second case, the rate of convergence is as fast as $n^{3^{1/2}}$, whereas the limiting distribution is Gaussian then.

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1. Introduction and Main Results

Suppose we (can) only observe the values of a smooth function $f: [0, 1] \rightarrow \mathbb{R}$ at the points $U_0, U_1, \dots, U_n, U_{n+1}$, where $U_1, U_2, \dots, U_n$ are the order statistics ($U_1 \leq U_2 \leq \dots \leq U_n$) of n independent uniformly- $(0, 1)$ distributed random variables and $U_0 = 0$, $U_{n+1} = 1$. It is our aim to estimate the integral

$$I = \int_0^1 f(x) dx \tag{1}$$

from these observations, i.e., by only using $(U_i, f(U_i))$, $i = 0, 1, \dots, n+1$. The first

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estimator we will employ is constructed by using the 'trapezoidal rule' on each sub-interval $[U_{i-1}, U_i]$, $i = 1, \dots, n+1$. This rule approximates an integral $\int_a^b g(x)dx$ simply by $\frac{1}{2}(b-a)(g(a) + g(b))$ and it can easily be shown (see, e.g., Isaacson and Keller [2], p. 304) that

$$\frac{1}{2}(b-a)(g(a) + g(b)) - \int_a^b g(x)dx = \frac{1}{12}(b-a)^3 g''(\eta), \quad (2)$$

where $\eta \in (a, b)$. Writing $D_i = U_i - U_{i-1}$, $i = 1, \dots, n+1$, for the spacings of the U_i 's, our estimator of I becomes

$$I_n = \sum_{i=1}^{n+1} \frac{1}{2} D_i (f(U_{i-1}) + f(U_i)). \quad (3)$$

Using (2), we will prove the following limiting result for the standardized difference of I_n and I :

Theorem 1: *If $|f'''|$ is bounded, then*

$$n^2(I_n - I) \xrightarrow{P} \frac{1}{2}(f'(1) - f'(0)), \quad \text{as } n \rightarrow \infty. \quad (4)$$

A much better and probabilistically more interesting estimator is obtained by applying a 3-points formula, i.e., for a given $c \in (a, b)$, we approximate $\int_a^b g(x)dx$ by $w_1 g(a) + w_2 g(c) + w_3 g(b)$ in such a way that the approximation error is zero in the case g is a polynomial of second degree. If the 3 points are equidistant, this approximation is known as Simpson's rule. It is not hard to show that

$$w_1 = \frac{1}{6}(b-a)\left(2 - \frac{b-c}{c-a}\right), \quad w_2 = \frac{1}{6} \frac{(b-a)^3}{(c-a)(b-c)}, \quad w_3 = \frac{1}{6}(b-a)\left(2 - \frac{c-a}{b-c}\right), \quad (5)$$

and it follows (see again Isaacson and Keller [2], p. 304) that

$$w_1 g(a) + w_2 g(c) + w_3 g(b) - \int_a^b g(x)dx = -\frac{1}{6} \int_a^b (x-a)(x-c)(x-b) g^{(3)}(\eta) dx, \quad (6)$$

where $\eta = \eta(x) \in (a, b)$. Hence, our estimator of I in (1), again denoted by I_n , becomes

$$\begin{aligned} I_n = & \sum_{i=1}^{\frac{n+1}{2}} \frac{1}{6} (D_{2i-1} + D_{2i}) \left\{ \left(2 - \frac{D_{2i}}{D_{2i-1}}\right) f(U_{2i-2}) \right. \\ & \left. + \frac{(D_{2i-1} + D_{2i})^2}{D_{2i-1} D_{2i}} f(U_{2i-1}) + \left(2 - \frac{D_{2i-1}}{D_{2i}}\right) f(U_{2i}) \right\}, \end{aligned} \quad (7)$$

where, for convenience, n is taken to be odd. Formula (6) will be used to prove our main result:

Theorem 2: *Let n be odd. If $|f^{(5)}|$ is bounded, then*

$$n^{\frac{3}{2}}(I_n - I) \xrightarrow{d} \sqrt{\frac{35}{3} \int_0^1 (f^{(3)}(x))^2 dx} Z, \quad \text{as } n \rightarrow \infty, \quad (8)$$

where Z is a standard normal random variable.

Remark 1: The present techniques can be easily adapted to cover the situation where the U_i 's are the order statistics of n independent random variables with common distribution function G (on $(0, 1)$) having a smooth density g . The adaptation is based on the quantile transform, transforming a uniform random variable V into a random variable $G^{-1}(V)$ with distribution function G . In this case, under regularity conditions on g , we obtain that the weak limit in Theorem 1 becomes $\frac{1}{2} \int_0^1 (f''(x)/g^2(x)) dx$ instead of $\frac{1}{2} \int_0^1 f''(x) dx = \frac{1}{2}(f'(1) - f'(0))$. In Theorem 2, the limiting random variable is again centered normal but now the standard deviation becomes

$$\sqrt{\frac{35}{3} \int_0^1 \frac{(f^{(3)}(x))^2}{g^7(x)} dx}.$$

On the other hand, the uniform distribution seems very relevant because of the following. Since $\int_0^1 f(x) dx$ can be considered as the mean 'output', given that the x -values are 'equally important', it seems desirable to estimate $\int_0^1 f(x)g(x)dx = \int_0^1 f(G^{-1}(y))dy$ in the case the random variables are distributed according to G . But if G is known, we can replace the pairs $(U_i, f(U_i))$ (just below (1)), with U_i 's being the order statistics from G , by $(G(U_i), f(U_i)) = (G(U_i), f(G^{-1}(G(U_i))))$. This brings us back to the 'uniform distribution setup' with f replaced by $f \circ G^{-1}$, but that is just the function whose integral we wanted to estimate as argued above!

This idea leads to possible ways of applying the results. Suppose U_i represents some uncontrollable physical random quantity, like temperature, humidity or light intensity with a known distribution function G having density g . Suppose also that we can measure f (the output or yield) only at the U_i and that we are interested in the mean output $I_g = \int_0^1 f(x)g(x)dx$. Then one can use our theorems to obtain rapidly converging estimators of I_g . In particular, when measuring the f -values is hard or expensive, one can get good estimators based on a few observations.

Also note that for the trapezoidal rule in Theorem 1 and f'' being constant, the uniform distribution is optimal, since $\int_0^1 g^{-2}(x)dx \geq \int_0^1 1dx = 1$. (This can be easily seen by using Jensen's inequality:

$$\begin{aligned} \int_0^1 \frac{1}{g^2(x)} dx &= \int_0^1 \frac{1}{g^3(x)} g(x) dx = \mathbb{E} \frac{1}{g^3(X)} \geq \left(\mathbb{E} \frac{1}{g(X)} \right)^3 \\ &= \left(\int_0^1 \frac{1}{g(x)} g(x) dx \right)^3 = 1, \end{aligned}$$

where X is a random variable with density g .) A similar remark applies to Theorem 2 with $f^{(3)}$ being constant.

Remark 2: There are various other ways to extend our results, which we will not pursue here, e.g., applying m -points formulas for $m > 3$ (Simpson's rule is 'by far the

most frequently used in obtaining approximate integrals', Davis and Rabinowitz [1], p. 45), combining trapezoidal rules to eliminate the bias $\frac{1}{2}(f'(1) - f'(0))$, proving a 'second order' limit result for $n^2(I_n - I) - \frac{1}{2}(f'(1) - f'(0))$ in Theorem 1, or treating the case n 'even' in Theorem 2. We are not pursuing these extensions because we believe they are not very interesting and/or they do not give good results.

Remark 3: We briefly compare our results with the deterministic, equidistant case, i.e., $U_i = \frac{i}{n+1}$, $i = 0, 1, \dots, n+1$. It is well-known that the limit in Theorem 1 is $\frac{1}{12}(f'(1) - f'(0))$ in that case, which means that we loose a factor of 6 by having random U_i 's. (Essentially, this 6 is coming from the third moment of a standard exponential random variable.) From Theorem 2, it is well-known that in the equidistant case (Simpson's rule), the rate is n^4 . So, there our loss is of order $n^{1/2}$. Nevertheless, from statistical point of view, $n^{3/2}$ is a remarkably fast rate of convergence.

2. Proofs

The following well-known lemma will be used frequently; it can be found in, e.g., Shorack and Wellner [3], p. 721.

Lemma 1: Let $E_1, \dots, E_{n+1}$ be independent exponential random variables with mean 1 and S_{n+1} be their sum. With D_i , $i = 1, \dots, n+1$, as before, we have

$$(D_1, \dots, D_{n+1}) \stackrel{d}{=} \left(\frac{E_1}{S_{n+1}}, \dots, \frac{E_{n+1}}{S_{n+1}} \right).$$

Proof of Theorem 1: Using (3), (1) and (2) we see that

$$n^2(I_n - I) = \frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 f''(\tilde{U}_i)$$

for some $\tilde{U}_i \in (U_{i-1}, U_i)$, and hence,

$$n^2(I_n - I) = \frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 f''\left(\frac{i}{n+1}\right) + \frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 \left(\tilde{U}_i - \frac{i}{n+1}\right) f'''(\tilde{\tilde{U}}_i), \quad (9)$$

with $\tilde{\tilde{U}}_i$ between $\tilde{U}_i$ and $\frac{i}{n+1}$. From the boundedness of $|f'''|$ (by M , say) and the weak convergence (to a Brownian bridge) of the uniform quantile process (see, e.g., Shorack and Wellner [3]), it is readily seen that

$$\begin{aligned} & \left| \frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 \left(\tilde{U}_i - \frac{i}{n+1}\right) f'''(\tilde{\tilde{U}}_i) \right| \\ & \leq \frac{n^2}{12} M \sup_{i \in \{1, \dots, n+1\}} \left| \tilde{U}_i - \frac{i}{n+1} \right| \sum_{i=1}^{n+1} D_i^3 = O_p(n^{1/2}) \sum_{i=1}^{n+1} D_i^3. \end{aligned} \quad (10)$$

But

$$\sum_{i=1}^{n+1} D_i^3 \stackrel{d}{=} \frac{1}{S_{n+1}^3} \sum_{i=1}^{n+1} E_i^3, \quad (11)$$

by Lemma 1, and by two applications of the weak law of large numbers, this last expression is $O_p(n^{-2})$. Combining this with (10) and (11) yields that the second term on the right in (9) converges to zero in probability. Hence, it remains to consider the first term

$$\frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 f''\left(\frac{i}{n+1}\right) \stackrel{d}{=} \frac{1}{12n} \left(\frac{n}{S_{n+1}}\right)^3 \sum_{i=1}^{n+1} f''\left(\frac{i}{n+1}\right) E_i^3,$$

or, since $(n/S_{n+1})^3 \xrightarrow{P} 1$,

$$\frac{1}{12n} \sum_{i=1}^{n+1} f''\left(\frac{i}{n+1}\right) E_i^3.$$

By Chebysev's inequality, it follows that

$$\frac{1}{12n} \sum_{i=1}^{n+1} f''\left(\frac{i}{n+1}\right) E_i^3 - \frac{1}{2n} \sum_{i=1}^{n+1} f''\left(\frac{i}{n+1}\right) \xrightarrow{P} 0.$$

The proof is complete by noting that

$$\frac{1}{2n} \sum_{i=1}^{n+1} f''\left(\frac{i}{n+1}\right) \rightarrow \frac{1}{2} \int_0^1 f''(x) dx = \frac{1}{2}(f'(1) - f'(0)). \quad \square$$

The proof of Theorem 2 is heavily based on the following two lemmas.

Lemma 2: Let $E_1, \dots, E_{n+1}$, n odd, be independent exponential random variables with mean 1. Write

$$X_i = (E_{2i-1} + E_{2i})^3 (E_{2i} - E_{2i-1}), \quad i = 1, 2, \dots, \frac{n+1}{2},$$

$$Y_i = X_i \sum_{j=1}^{2i-2} (E_j - 1), \quad i = 2, 3, \dots, \frac{n+1}{2}.$$

Then,

$$\mathbb{E}X_i = 0, \quad \text{Var } X_i = 120960, \quad \mathbb{E}Y_i = 0, \quad \text{Var } Y_i = 120960(2i-2),$$

$$\text{Cov}(Y_i, Y_k) = 0, \quad \text{for } i \neq k.$$

Proof: By symmetry, we see that $\mathbb{E}X_i = 0$; a straightforward computation yields $\text{Var } X_i = \mathbb{E}X_i^2 = 120960$. For the Y_i 's we have

$$\begin{aligned} \mathbb{E}Y_i &= \mathbb{E}X_i \mathbb{E} \sum_{j=1}^{2i-2} (E_j - 1) = 0, \\ \text{Var } Y_i &= \mathbb{E}Y_i^2 = \mathbb{E}X_i^2 \mathbb{E} \left(\sum_{j=1}^{2i-2} (E_j - 1) \right)^2 \\ &= \text{Var } X_i \text{Var} \left(\sum_{j=1}^{2i-2} E_j \right) = 120960(2i-2), \end{aligned}$$

and for $i < k$,

$$\begin{aligned} \text{Cov}(Y_i, Y_k) &= \mathbb{E}Y_i Y_k \\ &= \mathbb{E}X_k \left(\sum_{j=1}^{2k-2} (E_j - 1) \right) X_i \left(\sum_{j=1}^{2i-2} (E_j - 1) \right) \\ &= \mathbb{E}X_k \mathbb{E} \left(\sum_{j=1}^{2k-2} (E_j - 1) \right) X_i \left(\sum_{j=1}^{2i-2} (E_j - 1) \right) \\ &= 0. \end{aligned} \quad \square$$

Lemma 3: *Under the conditions of Theorem 2, we have, as $n \rightarrow \infty$,*

$$\left| n^{\frac{3}{2}}(I_n - I) - \frac{n^{\frac{3}{2}}}{72} \sum_{i=1}^{\frac{n+1}{2}} f^{(3)}\left(\frac{2i-2}{n+1}\right) (D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1}) \right| = o_p(1).$$

Proof: By (7), (1) and (6) we have

$$\begin{aligned} n^{\frac{3}{2}}(I_n - I) &= -\frac{n^{\frac{3}{2}}}{6} \sum_{i=1}^{\frac{n+1}{2}} \int_{U_{2i-2}}^{U_{2i}} (x - U_{2i-2})(x - U_{2i-1})(x - U_{2i}) f^{(3)}(\tilde{U}_{2i}) dx, \end{aligned} \quad (12)$$

for some $\tilde{U}_{2i} = \tilde{U}_{2i}(x) \in (U_{2i-2}, U_{2i})$ and hence for some $\tilde{\tilde{U}}_{2i} = \tilde{\tilde{U}}_{2i}(x) \in (U_{2i-2}, \tilde{U}_{2i})$, the right-hand side of (12) is equal to

$$\begin{aligned} & -\frac{n^{\frac{3}{2}}}{6} \sum_{i=1}^{\frac{n+1}{2}} \int_{U_{2i-2}}^{U_{2i}} (x - U_{2i-2})(x - U_{2i-1})(x - U_{2i}) \\ & \quad \times (f^{(3)}(U_{2i-2}) + (\tilde{U}_{2i} - U_{2i-2}) f^{(4)}(\tilde{\tilde{U}}_{2i})) dx \\ &= \frac{n^{\frac{3}{2}}}{72} \sum_{i=1}^{\frac{n+1}{2}} f^{(3)}(U_{2i-2}) (D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1}) \\ & - \frac{n^{\frac{3}{2}}}{6} \sum_{i=1}^{\frac{n+1}{2}} \int_{U_{2i-2}}^{U_{2i}} (x - U_{2i-2})(x - U_{2i-1})(x - U_{2i}) (\tilde{U}_{2i} - U_{2i-2}) f^{(4)}(\tilde{\tilde{U}}_{2i}) dx. \end{aligned}$$

Let M be a bound on $|f^{(5)}|$ and all lower order derivatives of f . Then the absolute value of this last term is bounded from above by

$$\begin{aligned} & M \frac{n^{\frac{3}{2}}}{6} \sum_{i=1}^{\frac{n+1}{2}} \int_{U_{2i-2}}^{U_{2i}} (x - U_{2i-2}) |x - U_{2i-1}| (U_{2i} - x) (\tilde{U}_{2i} - U_{2i-2}) dx \\ & \leq \frac{M}{6} n^{\frac{3}{2}} \sum_{i=1}^{\frac{n+1}{2}} (D_{2i-1} + D_{2i})^5 \stackrel{d}{=} \frac{M}{6} n^{\frac{3}{2}} \frac{1}{S_{n+1}^5} \sum_{i=1}^{\frac{n+1}{2}} (E_{2i-1} + E_{2i})^5 = o_p(1), \end{aligned}$$

due to Lemma 1 and two applications of the weak law of large numbers.

So, it suffices to show the convergence to zero in probability of

$$\begin{aligned} & \frac{n^{\frac{3}{2}}}{72} \sum_{i=1}^{\frac{n+1}{2}} \left(f^{(3)}(U_{2i-2}) - f^{(3)}\left(\frac{2i-2}{n+1}\right) \right) (D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1}) \\ &= \frac{n^{\frac{3}{2}}}{72} \sum_{i=2}^{\frac{n+1}{2}} \left(U_{2i-2} - \frac{2i-2}{n+1} \right) f^{(4)}\left(\frac{2i-2}{n+1}\right) (D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1}) \end{aligned}$$

$$+ \frac{n^{3\frac{1}{2}}}{144} \sum_{i=2}^{\frac{n+1}{2}} \left(U_{2i-2} - \frac{2i-2}{n+1} \right)^2 f^{(5)}(\bar{U}_{2i-2})(D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1})$$

$$=: T_{1,n} + T_{2,n},$$

for some $\bar{U}_{2i-2}$ between U_{2i-2} and $\frac{2i-2}{n+1}$. By the weak convergence of the uniform quantile process,

$$\begin{aligned} |T_{2,n}| &\leq \frac{n^{3\frac{1}{2}}}{144} M \sup_{i \in \{2, 3, \dots, \frac{n+1}{2}\}} \left(U_{2i-2} - \frac{2i-2}{n+1} \right)^2 \sum_{i=2}^{\frac{n+1}{2}} (D_{2i-1} + D_{2i})^4 \\ &= O_p(n^{\frac{1}{2}}) \sum_{i=2}^{\frac{n+1}{2}} (D_{2i-1} + D_{2i})^4. \end{aligned}$$

By Lemma 1 and twice the weak law of large numbers, this last expression is easily seen to be $o_p(1)$. Hence, the proof of Lemma 3 is complete if we show $T_{1,n} = o_p(1)$.

From Lemma 1 we obtain

$$\begin{aligned} T_{1,n} &\stackrel{d}{=} \frac{n^{3\frac{1}{2}}}{72} \sum_{i=2}^{\frac{n+1}{2}} \left(\frac{\sum_{j=1}^{2i-2} E_j}{S_{n+1}} - \frac{2i-2}{n+1} \right) f^{(4)}\left(\frac{2i-2}{n+1}\right) \left(\frac{E_{2i-1} + E_{2i}}{S_{n+1}} \right)^3 \left(\frac{E_{2i} - E_{2i-1}}{S_{n+1}} \right) \\ &= \frac{n^{3\frac{1}{2}}}{72} S_{n+1}^{-5} \sum_{i=2}^{\frac{n+1}{2}} \left(\sum_{j=1}^{2i-2} (E_j - 1) \right) f^{(4)}\left(\frac{2i-2}{n+1}\right) (E_{2i-1} + E_{2i})^3 (E_{2i} - E_{2i-1}) \\ &\quad + \frac{n^{3\frac{1}{2}}}{72} S_{n+1}^{-5} \left(1 - \frac{S_{n+1}}{n+1} \right) \sum_{i=2}^{\frac{n+1}{2}} (2i-2) f^{(4)}\left(\frac{2i-2}{n+1}\right) (E_{2i-1} + E_{2i})^3 (E_{2i} - E_{2i-1}) \\ &=: T_{3,n} + T_{4,n}. \end{aligned}$$

It is immediate from the central limit theorem for $S_{n+1}/(n+1)$ that

$$T_{4,n} = O_p(n^{-2}) \sum_{i=2}^{\frac{n+1}{2}} (2i-2) f^{(4)}\left(\frac{2i-2}{n+1}\right) X_i,$$

where the X_i 's are as in Lemma 2. Now using that lemma in conjunction with Chebysev's inequality, it readily follows that $T_{4,n} = o_p(1)$. Finally, in the notation of Lemma 2,

$$T_{3,n} = \frac{n^{3\frac{1}{2}}}{72} S_{n+1}^{-5} \sum_{i=2}^{\frac{n+1}{2}} f^{(4)}\left(\frac{2i-2}{n+1}\right) Y_i = O_p(n^{-\frac{1}{2}}) \sum_{i=2}^{\frac{n+1}{2}} f^{(4)}\left(\frac{2i-2}{n+1}\right) Y_i.$$

From Lemma 2, we have

$$\mathbb{E} \sum_{i=2}^{\frac{n+1}{2}} f^{(4)}\left(\frac{2i-2}{n+1}\right) Y_i = 0,$$

$$\text{Var} \sum_{i=2}^{\frac{n+1}{2}} f^{(4)}\left(\frac{2i-2}{n+1}\right) Y_i = \sum_{i=2}^{\frac{n+1}{2}} \left(f^{(4)}\left(\frac{2i-2}{n+1}\right) \right)^2 \text{Var} Y_i = O(n^2).$$

Now, Chebysev's inequality yields $T_{3,n} = o_p(1)$ and hence $T_{1,n} = o_p(1)$. $\square$

Proof of Theorem 2: Given the lemmas, especially Lemma 3, the proof of Theorem 2 is rather easy. If $\int_0^1 (f^{(3)}(x))^2 dx = 0$, then $f^{(3)}(x) = 0$ for all $x \in [0, 1]$ and hence trivially $I_n = I$, because f is a polynomial of second degree. Therefore, we assume now $\int_0^1 (f^{(3)}(x))^2 dx > 0$. Using Lemma 1 we have

$$\begin{aligned} & \frac{n^{\frac{3}{2}}}{72} \sum_{i=1}^{\frac{n+1}{2}} f^{(3)}\left(\frac{2i-2}{n+1}\right) (D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1}) \\ & \stackrel{d}{=} \frac{1}{72\sqrt{2}} \left(\frac{n}{S_{n+1}}\right)^4 \frac{1}{(n/2)^{1/2}} \sum_{i=1}^{\frac{n+1}{2}} f^{(3)}\left(\frac{2i-2}{n+1}\right) (E_{2i-1} + E_{2i})^3 (E_{2i} - E_{2i-1}) \\ & =: \left(\frac{n}{S_{n+1}}\right)^4 W_n. \end{aligned}$$

By the weak law of large numbers and Lemma 3, it now remains to show Theorem 2 with $n^{\frac{3}{2}}(I_n - I)$ replaced by W_n . By Lemma 2, we see that $\mathbb{E}W_n = 0$ and

$$\text{Var } W_n = \frac{1}{2(72)^2} \frac{2}{n} \sum_{i=1}^{\frac{n+1}{2}} \left(f^{(3)}\left(\frac{2i-2}{n+1}\right)\right)^2 120960 \rightarrow \frac{35}{3} \int_0^1 \left(f^{(3)}(x)\right)^2 dx.$$

Now, the Lindeberg central limit theorem applies, because of the boundedness of $|f^{(3)}|$, and it yields the result. $\square$

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