LINEAR FILTERING WITH FRACTIONAL BROWNIAN MOTION IN THE SIGNAL AND OBSERVATION PROCESSES

M.L. KLEPTSYNA

*Russian Academy of Sciences
Institute of Information Transmission Problems
Moscow 101447, Russia
marina@sci.lpi.ac.ru*

P.E. KLOEDEN

*Johan Wolfgang Goethe Universität
Fachbereich Mathematik
D-60054 Frankfurt am Main Germany
kloeden@math.uni-frankfurt.de*

V.V. ANH

*Queensland University of Technology
School of Mathematical Sciences
Brisbane 4001, Australia
v.anh@fsc.qut.edu.au*

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Integral equations for the mean-square estimate are obtained for the linear filtering problem, in which the noise generating the signal is a fractional Brownian motion with Hurst index $h \in (3/4, 1)$ and the noise in the observation process includes a fractional Brownian motion as well as a Wiener process.

Key words: Linear Filtering, Fractional Brownian Motion, Long-Range Dependence, Optimal Mean-Square Filter.

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1. Introduction

We consider the linear problem with the signal θ_t and the observation ξ_t defined by the linear equations

$$\theta_t = \int_0^t a(s)\theta_s ds + B_t^h, \quad \xi_t = \int_0^t A(s)\theta_s ds + W_t + B_t^h, \quad (1)$$

where the noise generating the signal is a fractional Brownian motion (fBm) B_t^h with Hurst index $h \in (3/4, 1)$ and the noise disturbing the observation of the signal consists of both a standard Wiener process W_t and the fractional Brownian motion B_t^h . The coefficients $a(t)$ and $A(t)$ are bounded measurable functions and the noise processes B_t^h and W_t are independent.

Fractional Brownian motion B_t^h with Hurst index $h \in (1/2, 1)$ is often used to model the long-range dependence in random data commonly encountered in many financial and environmental applications [7, 9]. It is a zero mean Gaussian process having the correlation function

$$\Gamma^h(t, s) = \frac{1}{2} (t^{2h} + s^{2h} - |t - s|^{2h}), \quad 1/2 < h < 1. \quad (2)$$

It is known that B_t^h is not a semimartingale (see e.g. [4, 6]), so neither is the signal process θ_t nor the observation process ξ_t , and the martingale approach to filtering expounded in [6] is not applicable here. In particular, as shown in [8], we cannot uniquely determine an innovation process corresponding to ξ_t . Nevertheless, we can derive an explicit expression for the conditional expectation of the signal

$$\hat{\theta}_t \triangleq \mathbb{E}(\theta_t | \xi_s, 0 \leq s \leq t),$$

using a theorem on normal correlation in [5] provided we restrict the Hurst index h to the interval $(3/4, 1)$. We formulate this results as a theorem in the next section and present its proof in Section 3. Finally, a simple example is provided in Section 4 to illustrate the result.

2. The Optimal Filter

Let $\mathcal{F}_t^\xi$ be the σ -algebra $\sigma(\xi_s, 0 \leq s \leq t)$ and note that $\hat{\theta}_t = \mathbb{E}(\theta_t | \mathcal{F}_t^\xi)$. Define

$$K(t, s) \triangleq \mathbb{E}(\theta_t \theta_s), \quad \tilde{K}(t, s) \triangleq \mathbb{E}(\theta_t B_s^h).$$

Then it follows directly from the first equation of (1) that $K(t, s)$ and $\tilde{K}(t, s)$ satisfy the system of integral equations

$$K(t, s) = \int_0^s a(l) K(t, l) dl + \tilde{K}(t, s), \quad (3)$$

$$\tilde{K}(t, s) = \int_0^t a(l) \tilde{K}(l, s) ds + \Gamma^h(t, s). \quad (4)$$

With these we can obtain an explicit closed-form representation of the optimal mean-square filter for system (1).

Theorem 2.1: *There exists a unique deterministic function $\Phi \in L^2([0, T]^2, \mathbb{R})$ satisfying*

$$\Phi(t, s) = - \int_0^s \Phi(t, \tau) [h(2h-1) |s - \tau|^{2h-2} \quad (5)$$

$$\begin{aligned}
 & + A(s)A(\tau)K(\tau, s) + A(\tau)\frac{\partial \tilde{K}}{\partial s}(\tau, s) + A(s)\frac{\partial \tilde{K}}{\partial \tau}(s, \tau)]d\tau \\
 & + A(s)K(t, s) + \frac{\partial \tilde{K}}{\partial s}(t, s)
 \end{aligned}$$

such that the optimal mean-square filtering estimate $\hat{\theta}_t$ of the linear system (1) satisfies

$$\hat{\theta}_t = \int_0^t \Phi(t, s)d\xi_s, \quad (6)$$

for $t \in [0, T]$, where the integral is understood in the mean-square sense.

It follows from the proof of Theorem 1 that system (5) has a solution. This solution is in fact unique.

Theorem 2.2: *The system of integral equations (3)-(5) has a unique solution.*

3. Proof of Theorem 1

We note that the joint distribution of (ξ_s, θ_t) for all $0 \leq s, t \leq T$ is Gaussian, so Theorem 13.1 of [5] on normal correlation holds here. Let $0 = t_0^{(n)} < t_1^{(n)} < \dots < t_{2^n}^{(n)} = t$ be the dyadic partition of $[0, t]$, that is, with $t_j^{(n)} = \frac{j}{2^n}t$ for $j = 0, 1, \dots, 2^n$, and denote the σ -algebra $\sigma\left(\xi_{t_0^{(n)}}, \xi_{t_1^{(n)}} - \xi_{t_0^{(n)}}, \dots, \xi_{t_{2^n}^{(n)}} - \xi_{t_{2^n-1}^{(n)}}\right)$ by $\mathfrak{F}_t^{\xi, n}$. Then $\mathfrak{F}_t^{\xi, n} \uparrow \mathfrak{F}_t^{\xi}$ as $n \rightarrow \infty$, so

$$\mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{\xi, n}\right) \rightarrow \mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{\xi}\right), \quad n \rightarrow \infty$$

for all $t \in [0, T]$. Furthermore,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{\xi, n}\right) - \mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{\xi}\right) \right]^2 = 0 \quad (7)$$

for all $t \in [0, T]$. Hence using Theorem 13.1 of [5] we obtain

$$\mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{\xi, n}\right) = \mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{\xi}\right) + \sum_{j=1}^{2^n-1} \Phi_n\left(t, t_j^{(n)}\right) \left(\xi_{t_{j+1}^{(n)}} - \xi_{t_j^{(n)}}\right) \quad (8)$$

for all $t \in [0, T]$, where $\Phi_n: [0, T]^2 \rightarrow \mathbb{R}$ is a deterministic function. Denote $\Phi(t, s) = \Phi_n(t, t_j^{(n)})$ for $t_j^{(n)} \leq s < t_{j+1}^{(n)}$. Then we can rewrite (8) as

$$\mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{\xi, n}\right) = \int_0^t \Phi_n(t, s)d\xi_s. \quad (9)$$

But the processes W_t and (B_t^h, θ_t) are independent, so

$$\mathbb{E} \left[\mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{\xi, n}\right) - \mathbb{E}\left(\theta_t \mid \mathfrak{F}_t^{x, m}\right) \right]^2 = \int_0^t |\Phi_n(t, s) - \Phi_m(t, s)|^2 ds$$

$$+ \mathbb{E} \left\{ \int_0^t \Phi_n(t, s)(dB_s^h + A(s)\theta_s ds) - \int_0^t \Phi_m(t, s)(dB_s^h + A(s)\theta_s ds) \right\}^2.$$

(The integration of a deterministic function with respect to an fBm here is understood in the mean-square sense, cf. [4]). Applying (7) we obtain that

$$\lim_{n, m \rightarrow \infty} \int_0^t |\Phi_n(t, s) - \Phi_m(t, s)|^2 ds = 0,$$

so the sequence $\{\Phi_n\}$ is a Cauchy sequence in $L^2[0, t]$. Hence there exist a function $\Phi \in L^2[0, t]$ such that

$$\lim_{n \rightarrow \infty} \int_0^t |\Phi_n(t, s) - \Phi(t, s)|^2 ds = 0.$$

It then follows from [1] that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(\left| \int_0^t \Phi_n(t, s) dB_s^h - \int_0^t \Phi(t, s) dB_s^h \right|^2 \right) = 0$$

and

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(\left| \int_0^t \Phi_n(t, s) A(s) \theta_s ds - \int_0^t \Phi(t, s) A(s) \theta_s ds \right|^2 \right) = 0,$$

so we obtain

$$\hat{\theta}_t = \int_0^t \Phi(t, s) d\xi_s.$$

We shall now show that Φ satisfies equation (5). Let $f: [0, T]^2 \rightarrow \mathbb{R}$ be a bounded and jointly measurable function, so the integral

$$I_t := \int_0^t f(t, s) d\xi_s$$

is well-defined and the process I_t is $\mathfrak{F}_t^\xi$ -measurable with $\mathbb{E}(I_t^2) < \infty$ and $\mathbb{E}((\theta_t - \hat{\theta}_t)I_t) = 0$. Consequently

$$\begin{aligned} \mathbb{E}(\theta_t I_t) &= \mathbb{E}(\hat{\theta}_t I_t) \\ &= \mathbb{E} \left(\int_0^t \Phi(t, s) d\xi_s \int_0^t f(t, s) d\xi_s \right) \end{aligned}$$

$$\begin{aligned}
 &= \int_0^t \Phi(t, s) f(t, s) ds + \int_0^t \int_0^t \Phi(t, \tau) f(t, s) \frac{\partial^2}{\partial s \partial \tau} \Gamma^h(\tau, s) d\tau ds \\
 &\quad + \int_0^t \int_0^t \Phi(t, \tau) f(t, s) A(\tau) A(s) K(\tau, s) d\tau ds \\
 &\quad + \int_0^t \int_0^t \Phi(t, \tau) f(t, s) A(s) \frac{\partial \tilde{K}}{\partial \tau}(s, \tau) d\tau ds \\
 &\quad + \int_0^t \int_0^t \Phi(t, \tau) f(t, s) A(\tau) \frac{\partial \tilde{K}}{\partial s}(\tau, s) d\tau ds
 \end{aligned}$$

and

$$\mathbb{E} \left(\theta_t \int_0^t f(t, s) d\xi_s \right) = \int_0^t f(t, s) \frac{\partial \tilde{K}}{\partial s}(t, s) ds + \int_0^t f(t, s) A(s) K(t, s) ds.$$

Now, f is otherwise arbitrary and the function $\frac{\partial^2}{\partial t \partial s} \Gamma^h(t, s) = h(2h-1) |t-s|^{2h-2}$, so

$$\int_0^T \int_0^T \left(\frac{\partial^2}{\partial t \partial s} \Gamma^h(t, s) \right)^2 dt ds < \infty,$$

which means the process $W_t + B_t^h$ is equivalent in an innovation sense [8] to W_t . Hence equation (5) is valid. Uniqueness follows from the linearity of the equations under consideration.

This completes the proof of Theorem 2.1.

4. Example

Consider the case $a(t) \equiv 0$ and $A(t) \equiv 0$, so system (1) reduces to

$$\theta_t = B_t^h, \quad \xi_t = W_t + B_t^h. \tag{10}$$

By Theorem 2.1, the filtering estimate $\hat{\theta}_t = \int_0^t \Phi(t, s) d\xi_s$ and Φ satisfies the integral equation

$$\begin{aligned}
 \Phi(t, s) &= - \int_0^t h(2h-1) \Phi(t\tau) |s-\tau|^{2h-2} d\tau \\
 &\quad + \frac{1}{2} \frac{\partial}{\partial s} \{ t^{2h} + s^{2h} - |t-s|^{2h} \},
 \end{aligned} \tag{11}$$

which is a Fredholm integral equation with a singular kernel. However, for $h > 3/4$ this kernel is square integrable and equation (11) has a unique solution. In this case,

the filtering error $\gamma_t = \mathbb{E}(|\theta_t - \hat{\theta}_t|^2)$ is given by

$$\begin{aligned}
 \gamma_t &= \mathbb{E}(|\theta_t|^2) - \mathbb{E}(|\hat{\theta}_t|^2) \\
 &= t^{2h} - \mathbb{E}\left(\left|\int_0^t \Phi(t, s) d\xi_s\right|^2\right) \\
 &= t^{2h} - \int_0^t \Phi(t, s) \left[\Phi(t, s) + \int_0^t h(2h-1)\Phi(t, \tau) |s - \tau|^{2h-2} d\tau \right] ds \\
 &= t^{2h} - \frac{1}{2} \int_0^t \Phi(t, s) \frac{\partial}{\partial s} [t^{2h} + s^{2h} - |t-s|^{2h}] ds
 \end{aligned}$$

using (11).

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