ANALYSIS OF AN N/G/1 FINITE QUEUE WITH THE SUPPLEMENTARY VARIABLE METHOD

GYEMIN LEE and JONGWOO JEON¹

Seoul National University

Department of Statistics

Seoul 151-742, Korea

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In this paper, we suggest a new approach to the analysis of an N/G/1 finite queue with the supplementary variable method. Compared to the conventional approach, our approach yields a simpler formula for the queue length distribution, which in turn gives a more efficient computational algorithm. Also, the new approach enables us to derive the joint density of the queue length and the elapsed service time.

Key words: N-process, Elapsed Service Time, Imbedded Markov Chain, Supplementary Variable Method, Schur-Banachiewicz Formula.

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1. Introduction

Very complex input flows often occur in integrated service communication systems. As an approximation to such a stream, Neuts [4] introduced the N-process. This N-process is analytically tractable and can appropriately represent the correlation and burstiness of the stream. Many familiar arrival processes are special cases of the N-process.

To investigate the performance of the service facility with finite resources, Blondia [1] considered an N/G/1 finite queue, i.e., a single server queue with K waiting rooms in which customers arrive according to an N-process. For the analysis, he used the imbedded Markov chain technique upon service completion epochs. He also gave a computational algorithm for the queue length distribution of the system by using the Schur-Banachiewicz formula [3] for the inverse of the block matrices.

However, the computational algorithm suggested by Blondia [1] needs a large amount of work. This motivated us to study an N/G/1 finite queue. Our aim is to obtain a more efficient computational algorithm. To this end, we employ the supplementary variable method originated by Cox [2].

This paper is organized as follows. In Section 2, we define N-process as originally

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introduced in Neuts [4]. Section 3 consists of the joint density of the queue length and the elapsed service time and the distribution of the queue length of the N/G/1 finite queue.

2. N-Process

Consider a continuous-time Markov process with m transient states and a single absorbing state. Then the infinitesimal generator of this Markov chain has the form

$$Q = \begin{pmatrix} T & T^0 \\ \mathbf{0} & 0 \end{pmatrix},$$

where T is an $m \times m$ non-singular matrix with $T_{i,i} < 0$, $T_{i,j} \geq 0$ for $i \neq j$. The vector T^0 is non-negative and satisfied $Te + T^0 = \mathbf{0}$, with $e = (1, \dots, 1)^t$. Let (α, α_{m+1}) be a vector of initial state probabilities of the Markov process. In what follows, we shall assume that $\alpha_{m+1} = 0$.

Now, construct a continuous-time process by restarting the above Markov process Q instantaneously after each absorption through a multinomial trial with probability α and outcomes $1, \dots, m$. Then this process is also a Markov process with the state space $\{1, 2, \dots, m\}$ and the infinitesimal generator

$$Q^* = T + T^0 A^0,$$

where T^0 is an $m \times m$ matrix whose columns are all T^0 and $A^0 = \text{diag}(\alpha_1, \dots, \alpha_m)$. A transition from the state i to the state j in the Markov process Q^* , which does not involve absorption, will be called an (i, j) -transition, while the others are called (i, j) -renewal transition. Then the N-process is an arrival process defined in the following way [4].

- (1) During any sojourn in the state i , there are Poisson group arrivals of rate λ_i and group size of densities $P\phi_i(k), k \geq 0$. We shall denote $\phi_i^*(z)$ the *p.g.f.* of $\{\phi_i(k), k \geq 0\}$, and define $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$ and $\Phi(z) = \text{diag}(\phi_1^*(z), \dots, \phi_m^*(z))$.
- (2) At (i, j) -renewal transitions, there are group arrivals with probability density $\{\Phi_{i,j}(k): k \geq 0\}$ whose *p.g.f.* is $\Psi_{i,j}(z)$. Let us denote the $m \times m$ matrix $(\Psi_{i,j}(z))$ by $\Phi(z)$.
- (3) At (i, j) -transitions ($i \neq j$), there are group arrivals with probability densities $\{\Omega_{i,j}(k), k \geq 0\}$, whose *p.g.f.* is $\Omega_{i,j}(z)$. For notational convenience, we set $\Omega_{i,i}(z) \equiv 1$ for all i and define $(\Omega_{i,j}(z))_{1 \leq i, j \leq m}$ by $\Omega(z)$.

Define the conditional probabilities

$$P_{i,j}(n, t) = \Pr(J(t) = j, N(t) = n \mid J(0) = i, N(0) = 0),$$

where $N(t)$ and $J(t)$ denote the number of arrivals during $(0, t]$ and the state of the underlying Markov process Q^* at time t , respectively. We also define conditional probability matrices $P(n, t) = (P_{i,j}(n, t))_{1 \leq i, j \leq m, n \geq 0}$. It was shown in [4] that

$$\sum_{n=0}^{\infty} z^n P(n, t) = \exp(R(z)t), 0 \leq z \leq 1, \quad (1)$$

with $R(z) = \sum_{n=0}^{\infty} z^n R_n$ and

$$R_0 = \Lambda \Phi(0) - \Lambda + T^0 A^0 \circ \Psi(0) + T \circ \Omega(0)$$

$$R_n = \Lambda \Phi(n) + T^0 A^0 \circ \Psi(n) + T \circ \Omega(n), n \geq 1,$$

where $\circ$ denotes the Schur (entrywise) product of two matrices. For the upcoming analysis, we shall assume that the matrix R_0^{-1} exists.

3. Analysis of an N/G/1 Finite Queue

In this section, we will analyze the N/G/1 finite queue with the supplementary variable method. The queue size is assumed to be K . When describing the N-process, we will use the same notations as in Section 2. The successive service times are independent and identically distributed according to $H(x)$. Also the hazard rate function and the mean of $H(x)$ are denoted by $r(x)$ and μ respectively.

3.1 Supplementary Variable Method

Let $X(t)$ denote the number of customers in the system at time t . We define the *elapsed service time* $S(t)$ as follows: If $X(t) > 0$, $S(t)$ denotes the amount of service already received by a customer in service. Otherwise, $S(t)$ denotes the amount of time elapsed after the last service completion. Then, the triplet $(J(t), X(t), S(t))$ is a three-dimensional Markov process with state space $\{1, \dots, m\} \times \{0, \dots, K\} \times [0, \infty)$.

Suppose that

$$\pi(i, n, x) dx = \lim_{t \rightarrow \infty} Pr(J(t) = i, X(t) = n, x \leq S(t) < x + dx)$$

exists for all states and define $\pi(n, x) = (\pi(1, n, x), \dots, \pi(m, n, x))$. Then the Kolmogorov differential equations of the joint density $\pi(n, x)$ can be written down as follows:

$$\frac{d}{dx} \pi(0, x) = \pi(0, x) R_0, \quad (2)$$

$$\frac{d}{dx} \pi(n, x) = -\pi(n, x) r(x) + \sum_{k=1}^n \pi(k, x) R_{n-k}, \quad 0 < n < K, \quad (3)$$

$$\frac{d}{dx} \pi(K, x) = -\pi(K, x) r(x) + \sum_{k=1}^K \sum_{l=K-k}^{\infty} \pi(k, x) R_l. \quad (4)$$

The joint density $\pi(n, x)$ should satisfy the boundary conditions

$$\pi(0, 0) = \int_0^{\infty} \pi(1, x) r(x) dx, \quad (5)$$

$$\pi(n, 0) = \int_0^\infty \pi(n+1, x)r(x)dx + \int_0^\infty \pi(0, x)\mathbf{R}_n dx, \quad 0 < n < K, \quad (6)$$

$$\pi(K, 0) = \sum_{i=0}^\infty \int_0^\infty \pi(0, x)\mathbf{R}_i dx, \quad (7)$$

and the normalization condition

$$\sum_{n=0}^K \int_0^\infty \pi(n, x)dx \mathbf{e} = 1, \quad (8)$$

where $\mathbf{e} = (1, \dots, 1)^t$.

Now, we shall find the joint density $\pi(n, x)$ of the queue length and the elapsed service time. From equation (1), we obtain

$$\frac{d}{dx}P(n, x) = \sum_{k=0}^n P(k, x)\mathbf{R}_{n-k}, \quad n \geq 0.$$

With this and equations (2)-(8), we get

$$\pi(0, x) = \pi(0, 0)P(0, x), \quad (9)$$

$$\pi(n, x) = \sum_{k=1}^n \pi(k, 0)P(n-k, x)(1-H(x)), \quad 0 < n < K, \quad (10)$$

$$\pi(K, x) = \sum_{k=1}^K \sum_{i=K-k}^\infty \pi(k, 0)P(i, x)(1-H(x)). \quad (11)$$

We may also derive the above solutions by conditioning on the state of the system time x back.

Before finding the coefficients $\pi(n, 0)$, we consider the embedded Markov chain $\{J(\tau_n), X(\tau_n)\}$, where $\{\tau_n, n \geq 0\}$ is the n^{th} epoch of service or idle completion. Then the transition probability matrix of $\{J(\tau_n), X(\tau_n)\}$ is

$$\mathbf{Q}_E = \begin{pmatrix} \mathbf{0} & \mathbf{U}_1 & \cdots & \mathbf{U}_{K-2} & \mathbf{U}_{K-1} & \sum_{n=K}^\infty \mathbf{U}_n \\ \mathbf{A}_0 & \mathbf{A}_1 & \cdots & \mathbf{A}_{K-2} & \sum_{n=K-1}^\infty \mathbf{A}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_0 & \cdots & \mathbf{A}_{K-3} & \sum_{n=K-2}^\infty \mathbf{A}_n & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \sum_{n=0}^\infty \mathbf{A}_n & \mathbf{0} \end{pmatrix},$$

where

$$\mathbf{U}_n = \int_0^\infty P(0, x)\mathbf{R}_n dx = [-\mathbf{R}_0^{-1}]\mathbf{R}_n, \quad n \geq 1,$$

$$A_n = \int_0^\infty P(n, x) dH(x), \quad n \geq 0.$$

The matrix U_n (or A_n) is the probability that n customers arrive during an idle time (or a service time).

Theorem 1: The coefficients $\pi(n, 0)$ of the joint density $\pi(n, x)$ are given by

$$(\pi(0, 0), \dots, \pi(K, 0)) = \frac{1}{\mu - \mathbf{x}_0[\mu \mathbf{I} + \mathbf{R}^{-1}(0)]\mathbf{e}} (\mathbf{x}_0, \dots, \mathbf{x}_K),$$

where $(\mathbf{x}_0, \dots, \mathbf{x}_K)$ is the stationary vector of the transition probability matrix $\mathbf{Q}_E$. $\mathbf{I}$ is an identity matrix of size m .

Proof: By inserting (9)-(11) into the boundary conditions, we show that $(\pi(0, 0), \dots, \pi(K, 0))$ is a positive invariant vector of $\mathbf{Q}_E$, that is,

$$(\pi(0, 0), \dots, \pi(K, 0)) = c(\mathbf{x}_0, \mathbf{x}_K) \text{ for some constant } c > 0.$$

Applying (9)-(11) to the normalization condition, we have

$$\pi(0, 0)[-\mathbf{R}_0^{-1}]\mathbf{e} + \mu \sum_{n=1}^{\infty} \pi(n, 0)\mathbf{e} = 1.$$

Therefore, we have

$$c = \frac{1}{\mu - \mathbf{x}_0[\mu \mathbf{I} + \mathbf{R}_0^{-1}]\mathbf{e}}.$$

So the proof is complete.

The matrices $P(n, x)$ and A_n can be efficiently evaluated by means of an iterative procedure in [3]. Therefore, we can compute $\pi(n, x)$ by deriving the stationary vector of the transition probability matrix $\mathbf{Q}_E$. As Blondia [1] did, we can also reduce the complexity of the computation for the stationary vector with the Schur-Banachiewicz formula for the inverse of block matrices.

3.2 Queue Length Distribution

In this subsection, we shall consider two computational algorithms to obtain the queue length distribution $\pi(n)$ using the coefficients $\pi(n, 0)$ derived in the previous subsection.

Let us define

$$\mathbf{M}_n = \int_0^\infty P(n, x)(1 - H(x))dx, \quad n \geq 0$$

and let $\mathbf{M}(z)$ be the generating function of $\{\mathbf{M}_n, n \geq 0\}$. Then equations (9)-(10) yield

$$\pi(0) = \pi(0, 0)[-\mathbf{R}_0^{-1}], \tag{12}$$

$$\pi(n) = \sum_{k=1}^n \pi(k, 0)\mathbf{M}_{n-k}, \quad 1 \leq n \leq K-1.$$

Since $\sum_{n=0}^K \pi(n)$ is the stationary vector θ of the underlying Markov process $\mathbf{Q}^*$, we get

$$\pi(K) = \theta - \sum_{n=0}^{K-1} \pi(n). \quad (13)$$

Using the fact that $(\pi(0,0), \dots, \pi(K,0))$ is an invariant vector of the transition probability matrix $\mathbf{Q}_E$ and $\mathbf{A}(z) = \mathbf{M}(z)\mathbf{R}(z) + \mathbf{I}$, we have

$$\begin{aligned} \pi(n) = & \left(\sum_{k=1}^{n-1} \pi(k) \mathbf{R}_{n-k} = \pi(0) \mathbf{R}_{n-1} \right) [-\mathbf{R}_0^{-1}] \\ & - (\mathbf{1}_{(n \geq 2)} \pi(n-1, 0) - \pi(n, 0)) [-\mathbf{R}_0^{-1}], \quad 1 \leq n \leq K-1, \end{aligned} \quad (14)$$

where $\mathbf{1}_{(\cdot)}$ is the indicator function.

The above equation (14) for the queue length distribution has a simpler form than on derived by Blondia [1], since Blondia's formulas require additional computation of matrices $\{\mathbf{R}_n(s), n \geq 0\}$ satisfying $[\mathbf{R}(z) + s\mathbf{I}]^{-1} = \sum_{n=0}^{\infty} \mathbf{R}_n(s) z^n$. Consequently, we can obtain a more efficient computational algorithm for the queue length distribution.

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