GENERIC STABILITY OF THE SPECTRA FOR BOUNDED LINEAR OPERATORS ON BANACH SPACES

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In this paper, we study the stability of the spectra of bounded linear operators $B(X)$ in a Banach space X , and obtain that their spectra are stable on a dense residual subset of $B(X)$.

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1. Introduction

Spectral theory is an important part of functional analysis, which attracted many authors, e.g. [1, 3]. It is known (see Kreyszig [3]) that the spectra of a bounded linear operator is a nonempty compact subset of complex plane C . When the operator is perturbed, how does its spectrum change? After Rayleigh and Schrödinger created perturbation theory, stability of spectra has been intensively developed. In a finite-dimensional space, the eigenvalues of a linear operator T depend on T continuous [1], but it does not apply to a general Banach space. Kato [1, pp. 210] gives an example, in which he shows that the set of spectrum of a bounded linear operator in Banach space is not stable.

In this paper, by using Lemma 2.3 of K.K. Tan, J. Yu, and X.Z. Yuan [4], we obtain that the spectra of a bounded linear operator is stable on a dense residual subset of $B(X)$.

2. Preliminaries

If X and Y are two topological spaces, we shall denote by $K(X)$ and $P_0(Y)$ the space of all nonempty compact subsets of X and the space all nonempty subsets of Y , respectively, both endowed with the Vietoris topology (see Klein and Thompson [2]). Then a mapping $T: X \rightarrow P_0()$ is said to be (i) *upper* (resp. *lower*) *semicontinuous* at

$x \in X$ if, for each open set G in Y with $G \supset T(x)$ (resp. $G \cap T(x) \neq \emptyset$), there exists an open neighborhood $O(x)$ of x in X such that $G \supset T(x')$ (resp. $G \cap T(x') \neq \emptyset$) for each $x' \in O(x)$; (ii) T is *upper* (resp. *lower*) *semicontinuous on X* if, T is upper (resp. lower) semicontinuous at each $x \in X$; (iii) T is an *usco mapping* if, T is upper semicontinuous with nonempty compact values.

The following result is due to K.K. Tan, J. Yu, and X.Z. Yuan [4, Lemma 2.3].

Theorem 2.1: *If X is (completely) metrizable, Y is a Baire space and $T: Y \rightarrow K(X)$ is a usco mapping, then the set of points where T is lower semicontinuous is a (dense) residual set in Y .*

Let C denotes the whole complex plane, X denotes a complex Banach space, $T: X \rightarrow X$ linear operator. Then, $\sigma(T) = \{\lambda \in C: T - \lambda I \text{ is not invertible}\}$ is called the *spectra* of T , the complementary set $\rho(T) = C \setminus \sigma(T)$ is called the *resolvent set* of T . Here I is the identity mapping.

The following theorems are from Kreyszig [3].

Theorem 2.2: *Let X and Y be complex (or real) topological vector spaces, $T: D(T) \rightarrow Y$ be a linear operator, and $D(T) \subset X$, $R(T) \subset Y$. Then*

- (1) $T^{-1}: R(T) \rightarrow D(T)$ exists if and only if $Tx = 0$ implies that $x = 0$;
- (2) if T^{-1} exists, then T^{-1} is a linear operator.

Theorem 2.3: *The spectra $\sigma(T)$ of a bounded linear operator T in Banach space is a nonempty compact subset of C .*

Let $B(X, Y)$ be the set of all bounded linear operators from X to Y and let $CO(X, Y)$ be the set of closed linear operators from X to Y .

The following theorem is due to T. Kato [1, Theorem 2.23].

Theorem 2.4: *Let $T, T_n \in CO(X, Y)$, $n = 1, 2, \dots$,*

- (1) *if $T \in B(X, Y)$, then $T_n \rightarrow T$ in the generalized sense if and only if $T_n \in B(X, Y)$ for sufficiently large n and $\|T_n - T\| \rightarrow 0$;*
- (2) *if T^{-1} exists and belongs to $B(X, Y)$, then $T_n \rightarrow T$ in the generalized sense if and only if T_n^{-1} exists and belongs to $B(X, Y)$ for sufficiently large n and $\|T_n^{-1} - T^{-1}\| \rightarrow 0$.*

3. Main Results

Let $X \neq \{0\}$ be a complex Banach space, and let $B(X)$ denote the set of all bounded linear operators in X . Then $B(X)$ is a Banach space.

Theorem 3.1: *$\sigma: B(X) \rightarrow K(C)$ is an usco mapping.*

Proof: By Theorem 2.3, $\sigma(T)$ is a nonempty compact subset of C for each $T \in B(X)$. Suppose that σ is not upper semicontinuous at some $T_0 \in B(X)$, i.e., that for any $\varepsilon_0 > 0$ there is a $\delta > 0$, such that for all $S \in B(X)$ with $\|X - T_0\| < \delta$,

$$H_+(\sigma(T_0), \sigma(S)) = \sup_{\lambda \in (S)} \{\text{dist}(\lambda, \sigma(T_0))\} \geq \varepsilon_0$$

where H is the Hausdorff metric and $H(\cdot, \cdot) = \max\{H_+(\cdot, \cdot), H_-(\cdot, \cdot)\}$. Then there exists a $\lambda_0 \in \sigma(S)$ such that $\text{dist}(\lambda_0, \sigma(T_0)) \geq \varepsilon_0 > 0$. Thus

$$\lambda_0 \notin \sigma(T_0), \quad \lambda_0 \in \rho(T_0), \quad (T_0 - \lambda_0 I)^{-1} \in B(X),$$

and, by Theorem 2.4,

$$(S - \lambda_0 I)^{-1} \in B(X), \lambda_0 \notin \sigma(S),$$

which contradicts that $\lambda_0 \in \sigma(S)$. Therefore, σ is an usco mapping.

Definition 3.1: For each $T \in B(X)$,

- (i) $\lambda \in (T)$ is an *essential spectrum value relative to $B(X)$* if, for each open neighborhood $N(\lambda)$ of λ in C , there exists an open neighborhood $O(T)$ of T in $B(X)$ such that $\sigma(T') \cap N(\lambda) \neq \emptyset$ for each $T' \in O(T)$;
- (ii) T is *essential relative to $B(X)$* if, every $\lambda \in \sigma(T)$ is an essential spectrum value relative to $B(X)$.

Theorem 3.2: (1) σ is lower semicontinuous at $T \in B(X)$ if and only if T is essential relative to $B(X)$;

- (2) σ is continuous at $T \in B(X)$ if and only if T is essential relative to $B(X)$.

Proof: (1) σ is lower semicontinuous at $T \in B(X)$ if and only if each $\lambda \in \sigma(T)$ is an essential spectrum value relative to $B(X)$ and T is essential relative to $B(X)$.

- (2) The proof follows from (1) and Theorem 3.1.

Theorem 3.3: If $T \in B(X)$ such that $\sigma(T)$ is a singleton set, then T is essential relative to $B(X)$.

Proof: Suppose $\sigma(T) = \{\lambda\}$, and let G be any open set in C such that $\sigma(T) \cap G \neq \emptyset$. Then $\lambda \in G$, so that $\sigma(T) \subset G$. Since σ is upper semicontinuous at T , by Theorem 3.1, there exists an open neighborhood $O(T)$ of T in $B(X)$ such that $\sigma(T') \subset G$ for each $T' \in O(T)$. In particular, $G \cap \sigma(T') \neq \emptyset$ for each $T' \in O(T)$. Thus σ is lower semicontinuous at T , and by Theorem 3.2 (1), T is essential relative to $B(X)$.

Theorem 3.4: Let C be complex plane, and $X \neq \{0\}$ be a complex Banach space. Then there exists a dense residual subset Q of $B(X)$ such that T is essential relative to $B(X)$ for each $T \in Q$.

Proof: By Theorem 3.1 and Theorem 2.1, σ is lower semicontinuous on some dense residual subset Q of $B(X)$. Consequently, by Theorem 3.2 (1), T is essential relative to $B(X)$ for each $T \in Q$.

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