EXISTENCE AND UNIQUENESS OF SOLUTIONS OF A SEMILINEAR FUNCTIONAL-DIFFERENTIAL EVOLUTION NONLOCAL CAUCHY PROBLEM

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(Received December, 1998; Revised April, 1999)

Two theorems about the existence and uniqueness of mild and classical solutions of a semilinear functional-differential evolution nonlocal Cauchy problem in a general Banach space are proved. Methods of semigroups and the Banach contraction theorem are applied.

Key words: Abstract Cauchy Problem, Evolution Equation, Functional-Differential Equation, Nonlocal Condition, Existence and Uniqueness of the Solutions, Mild and Classical Solutions, Banach Contraction Theorem.

AMS subject classifications: 34G20, 34K30, 34K99, 47D03, 47H10.

1. Introduction

In this paper we study the existence and uniqueness of mild and classical solutions of a semilinear functional-differential evolution nonlocal Cauchy problem in a general Banach space. Methods of C_0 semigroups and the Banach theorem about the fixed point are applied. The functional-differential evolution nonlocal Cauchy problem considered here is of the form

$$\begin{aligned} u'(t) + Au(t) &= F_1(t, u(t), u(\sigma_1(t)), \dots, u(\sigma_m(t))) \\ &+ \int_{t_0}^t F_2(t, s, u(s), \int_{t_0}^s f(s, \tau, u(\tau))d\tau)ds, \quad t \in (t_0, t_0 + a] \end{aligned} \quad (1.1)$$

and

$$u(t_0) + G(u) = u_0, \quad (1.2)$$

where $t_0 \geq 0$, $a > 0$, $-A$ is the infinitesimal generator of a C_0 semigroup of operators

on a Banach space, F_i ($i = 1, 2$), G, f, σ_i ($i = 1, \dots, m$) are given functions satisfying some assumptions and u_0 is an element of the Banach space.

The results obtained pertaining to the nonlocal evolution problem are generalizations of those given by Byszewski [2, 4, 5], and by Balasubramaniam and Chandrasekaran [1]. Moreover, the results obtained concerning the evolution problem (1.1)-(1.2), where $F_2 = 0$ and $G = 0$, are generalizations of those given by Winiarska [10] and Pazy [9].

Nonlocal semilinear and nonlinear functional-differential evolution Cauchy problems in general Banach spaces have also been studied by Byszewski [3, 6, 7] and by Lin, Liu [8].

2. Notation and Definitions

Let E be a Banach space with norm $\|\cdot\|$ and let $\{T(t)\}_{t \geq 0}$ be a C_0 semigroup of operators on E .

In this paper we assume that $-A$ is the infinitesimal generator of a C_0 semigroup of operators on E , $D(A)$ is the domain of A , $t_0 \geq 0$, $a > 0$,

$$I = [t_0, t_0 + a], \quad \Delta = \{(t, s) : t_0 \leq s \leq t \leq t_0 + a\}$$

$$M = \sup_{t \in [0, a]} \|T(t)\|_{BL(E, E)}, \quad (2.1)$$

$$X = C(I, E)$$

and

$$F_1 : I \times E^{m+1} \rightarrow E, \quad F_2 : \Delta \times E^2 \rightarrow E, \quad G : X \rightarrow E,$$

$$f : \Delta \times E \rightarrow E, \quad \sigma_i : I \rightarrow I \quad (i = 1, \dots, m)$$

are given functions satisfying some assumptions.

In the sequel, the operator norm $\|\cdot\|_{BL(E, E)}$ will be denoted by $\|\cdot\|$.

We will need the following two definitions of mild and classical solutions of the nonlocal Cauchy problem (1.1)-(1.2):

Definition 2.1: A function $u \in X$ satisfying the integral equation

$$u(t) = T(t - t_0)u_0 - T(t - t_0)G(u)$$

$$+ \int_{t_0}^t T(t - s)F_1(s, u(s), u(\sigma_1(s)), \dots, u(\sigma_m(s)))ds$$

$$+ \int_{t_0}^t T(t - s) \left(\int_{t_0}^s F_2(s, \tau, u(\tau), \int_{t_0}^{\tau} f(\tau, \mu, u(\mu))d\mu)d\tau \right) ds, \quad t \in I,$$
(2.2)

is said to be a *mild solution* of the nonlocal Cauchy problem (1.1)-(1.2) on I .

Definition 2.2: A function $u : I \rightarrow E$ is said to be a *classical solution* of the nonlocal Cauchy problem (1.1)-(1.2) on I if:

- (i) u is continuous on I and continuously differentiable on $I \setminus \{t_0\}$,
 (ii) $u'(t) + Au(t) = F_1(t, u(t), u(\sigma_1(t)), \dots, u(\sigma_m(t)))$

$$+ \int_{t_0}^t F_2(t, s, u(s), \int_{t_0}^s f(s, \tau, u(\tau)) d\tau) ds, \quad t \in I \setminus \{t_0\},$$

- (iii) $u(t_0) + G(u) = u_0$.

3. Theorem about a Mild Solution

Theorem 3.1: Assume that

- (i) for all $z_i \in E$ ($i = 0, 1, \dots, m$), the function $I \ni t \rightarrow F_1(t, z_0, z_1, \dots, z_m) \in E$ is continuous on I , for all $z_i \in E$ ($i = 1, 2$) the function $\Delta \ni (t, s) \rightarrow F_2(t, s, z_1, z_2) \in E$ is continuous on Δ , for all $z \in E$ the function $\Delta \ni (t, s) \rightarrow f(t, s, z) \in E$ is continuous on Δ , $G: X \rightarrow E$, $\sigma_i \in C(I, I)$ ($i = 1, \dots, m$) and $u_0 \in E$;
 (ii) there are constants $L_i > 0$ ($i = 1, 2, 3, 4$) such that

$$\|F_1(t, z_0, z_1, \dots, z_m) - F_1(t, \tilde{z}_0, \tilde{z}_1, \dots, \tilde{z}_m)\| \leq L_1 \sum_{i=0}^m \|z_i - \tilde{z}_i\|$$

$$\text{for } t \in I, z_i, \tilde{z}_i \in E \quad (i = 0, 1, \dots, m); \quad (3.1)$$

$$\|F_2(t, s, z_1, z_2) - F_2(t, s, \tilde{z}_1, \tilde{z}_2)\| \leq L_2 \sum_{i=1}^2 \|z_i - \tilde{z}_i\|$$

$$\text{for } (t, s) \in \Delta, z_i, \tilde{z}_i \in E \quad (i = 1, 2); \quad (3.2)$$

$$\|f(t, s, z) - f(t, s, \tilde{z})\| \leq L_3 \|z - \tilde{z}\|$$

$$\text{for } (t, s) \in \Delta, z, \tilde{z} \in E; \quad (3.3)$$

$$\|G(w) - G(\tilde{w})\| \leq L_4 \|w - \tilde{w}\|_X \text{ for } w, \tilde{w} \in X; \quad (3.4)$$

- (iii) $M[L_1 a(m+1) + L_2 a^2(1 + L_3 a) + L_4] < 1$.

Then the nonlocal Cauchy problem (1.1)-(1.2) has a unique mild solution on I .

Proof: Introduce an operator $\mathfrak{F}$ on X by the formula

$$\begin{aligned} (\mathfrak{F}w)(t) &= T(t - t_0)u_0 - T(t - t_0)G(w) \\ &+ \int_{t_0}^t T(t - s)F_1(s, w(s), w(\sigma_1(s)), \dots, w(\sigma_m(s)))ds \end{aligned}$$

$$+ \int_{t_0}^t T(t-s) \left(\int_{t_0}^s F_2(s, \tau, w(\tau), \int_{t_0}^{\tau} f(\tau, \mu, w(\mu)) d\mu) d\tau \right) ds$$

for $w \in X$ and $t \in I$.

It is easy to see that

$$\mathfrak{F}: X \rightarrow X. \quad (3.5)$$

Now, we shall show that $\mathfrak{F}$ is a contraction on X . For this purpose, consider the difference

$$\begin{aligned} (\mathfrak{F}w)(t) - (\mathfrak{F}\tilde{w})(t) &= -T(t-t_0)[G(w) - G(\tilde{w})] \\ &+ \int_{t_0}^t T(t-s)[F_1(s, w(s), w(\sigma_1(s)), \dots, w(\sigma_m(s))) \\ &\quad - F_1(s, \tilde{w}(s), \tilde{w}(\sigma_1(s)), \dots, \tilde{w}(\sigma_m(s)))] ds \\ &+ \int_{t_0}^t T(t-s) \left(\int_{t_0}^s \left[F_2(s, \tau, w(\tau), \int_{t_0}^{\tau} f(\tau, \mu, w(\mu)) d\mu \right. \right. \\ &\quad \left. \left. - F_2(s, \tau, \tilde{w}(\tau), \int_{t_0}^{\tau} f(\tau, \mu, \tilde{w}(\mu)) d\mu) \right] d\tau \right) ds \end{aligned} \quad (3.6)$$

for $w, \tilde{w} \in X$ and $t \in I$.

From (3.6), (2.1) and (3.1)-(3.4),

$$\begin{aligned} \|(\mathfrak{F}w)(t) - (\mathfrak{F}\tilde{w})(t)\| &\leq \|T(t-t_0)\| \|G(w) - G(\tilde{w})\| \\ &+ \int_{t_0}^t \|T(t-s)\| \|F_1(s, w(s), w(\sigma_1(s)), \dots, w(\sigma_m(s))) \\ &\quad - F_1(s, \tilde{w}(s), \tilde{w}(\sigma_1(s)), \dots, \tilde{w}(\sigma_m(s)))\| ds \\ &+ \int_{t_0}^t \|T(t-s)\| \left(\int_{t_0}^s \|F_2(s, \tau, w(\tau), \int_{t_0}^{\tau} f(\tau, \mu, w(\mu)) d\mu \right. \\ &\quad \left. - F_2(s, \tau, \tilde{w}(\tau), \int_{t_0}^{\tau} f(\tau, \mu, \tilde{w}(\mu)) d\mu)\| d\tau \right) ds \\ &\leq ML_4 \|w - \tilde{w}\|_X + ML_1 \int_{t_0}^t \left(\|w(s) - \tilde{w}(s)\| + \sum_{i=1}^m \|w(\sigma_i(s)) - \tilde{w}(\sigma_i(s))\| \right) ds \end{aligned} \quad (3.7)$$

$$\begin{aligned}
 & + ML_2 \int_{t_0}^t \left(\int_{t_0}^s \left[\|w(\tau) - \tilde{w}(\tau)\| + \int_{t_0}^{\tau} \|f(\tau, \mu, w(\mu)) - f(\tau, \mu, \tilde{w}(\mu))\| d\mu \right] d\tau \right) ds \\
 & \leq ML_4 \|w - \tilde{w}\|_X + ML_1 a(m+1) \|w - \tilde{w}\|_X \\
 & + ML_2 \int_{t_0}^t \left(\int_{t_0}^s \left[\|w(\tau) - \tilde{w}(\tau)\| + L_3 \int_{t_0}^{\tau} \|w(\mu) - \tilde{w}(\mu)\| d\mu \right] d\tau \right) ds \\
 & = M[L_1 a(m+1) + L_2 a^2(1 + L_3 a) + L_4] \|w - \tilde{w}\|_X
 \end{aligned}$$

for $w, \tilde{w} \in X$ and $t \in I$.

Let

$$q := M[L_1 a(m+1) + L_2 a^2(1 + L_3 a) + L_4].$$

Then, by (3.7) and by assumption (iii),

$$\|\mathfrak{F}w - \mathfrak{F}\tilde{w}\|_X \leq q \|w - \tilde{w}\|_X \text{ for } w, \tilde{w} \in X \quad (3.8)$$

with $0 < q < 1$. This shows that operator $\mathfrak{F}$ is a contraction on X .

Consequently, from (3.5) and (3.8), operator $\mathfrak{F}$ satisfies all the assumptions of the Banach contraction theorem. Therefore, in space X there is only one fixed point of $\mathfrak{F}$ and this point is the mild solution of the nonlocal Cauchy problem (1.1)-(1.2). So the proof of Theorem 3.1 is complete.

4. Theorem about a Classical Solution

Theorem 4.1: Suppose that assumptions (i)-(iii) of Theorem 3.1 are satisfied. Then the nonlocal Cauchy problem (1.1)-(1.2) has a unique mild solution on I . Assume, additionally, that:

- (i) E is a reflexive Banach space, $u_0 \in D(A)$ and $G(u) \in D(A)$, where u denotes the unique mild solution of problem (1.1)-(1.2);
- (ii) there are constants $C_i > 0$ ($i = 1, 2$) such that

$$\|F_1(t, z_0, z_1, \dots, z_m) - F_1(\tilde{t}, z_0, z_1, \dots, z_m)\| \leq C_1 |t - \tilde{t}| \quad (4.1)$$

$$\text{for } t, \tilde{t} \in I, z_i \in E \text{ (} i = 0, 1, \dots, m \text{)}$$

and

$$\|F_2(t, s, z_1, z_2) - F_2(\tilde{t}, s, z_1, z_2)\| \leq C_2 |t - \tilde{t}| \quad (4.2)$$

$$\text{for } (t, s) \in \Delta, (\tilde{t}, s) \in \Delta, z_i \in E \text{ (} i = 1, 2 \text{);}$$

(iii) there is a constant $c > 0$ such that

$$\|u(\sigma_i(t)) - u(\sigma_i(\tilde{t}))\| \leq c \|u(t) - u(\tilde{t})\| \quad (4.3)$$

for $t, \tilde{t} \in I$ ($i = 0, 1, \dots, m$).

Then u is the unique classical solution of the nonlocal Cauchy problem (1.1)-(1.2) on I .

Proof: Since all the assumptions of Theorem 3.1 are satisfied, then the nonlocal Cauchy problem (1.1)-(1.2) possesses a unique mild solution which, according to assumption (i), is denoted by u .

Now, we shall show that u is the unique classical solution of problem (1.1)-(1.2) on I . To this end, introduce

$$N_1 := \max_{s \in I} \|F_1(s, u(s), u(\sigma_1(s)), \dots, u(\sigma_m(s)))\| \quad (4.4)$$

and

$$N_2 := \max_{(\xi, \eta) \in \Delta} \|F_2(\xi, \eta, u(\eta), \int_{t_0}^{\eta} f(\eta, \mu, u(\mu))d\mu)\|, \quad (4.5)$$

and observe that

$$\begin{aligned} u(t+h) - u(t) &= [T(t+h-t_0)u_0 - T(t-t_0)u_0] \\ &\quad - [T(t+h-t_0)G(u) - T(t-t_0)G(u)] \\ &\quad + \int_{t_0}^{t_0+h} T(t+h-s)F_1(s, u(s), u(\sigma_1(s)), \dots, u(\sigma_m(s)))ds \\ &\quad + \int_{t_0+h}^{t+h} T(t+h-s)F_1(s, u(s), u(\sigma_1(s)), \dots, u(\sigma_m(s)))ds \\ &\quad - \int_{t_0}^t T(t-s)F_1(s, u(s), u(\sigma_1(s)), \dots, u(\sigma_m(s)))ds \\ &\quad + \int_{t_0}^{t_0+h} T(t+h-s) \left(\int_{t_0}^s F_2(s, \tau, u(\tau), \int_{t_0}^{\tau} f(\tau, \mu, u(\mu))d\mu)d\tau \right) ds \\ &\quad + \int_{t_0+h}^{t+h} T(t+h-s) \left(\int_{t_0}^s F_2(s, \tau, u(\tau), \int_{t_0}^{\tau} f(\tau, \mu, u(\mu))d\mu)d\tau \right) ds \end{aligned} \quad (4.6)$$

$$\begin{aligned}
 & - \int_{t_0}^t T(t-s) \left(\int_{t_0}^s F_2(s, \tau, u(\tau), \int_{t_0}^{\tau} f(\tau, \mu, u(\mu)) d\mu) d\tau \right) ds \\
 & = T(t-t_0)[T(h)-I]u_0 - T(t-t_0)[T(h)-I]G(u) \\
 & + \int_{t_0}^{t_0+h} T(t+h-s) F_1(s, u(s), u(\sigma_1(s)), \dots, u(\sigma_m(s))) ds \\
 & + \int_{t_0}^t T(t-s) [F_1(s+h, u(s+h), u(\sigma_1(s+h)), \dots, u(\sigma_m(s+h))) \\
 & \quad - F_1(s, u(s), u(\sigma_1(s)), \dots, u(\sigma_m(s)))] ds \\
 & + \int_{t_0}^{t_0+h} T(t+h-s) \left(\int_{t_0}^s F_2(s, \tau, u(\tau), \int_{t_0}^{\tau} f(\tau, \mu, u(\mu)) d\mu) d\tau \right) ds \\
 & + \int_{t_0}^t T(t-s) \left(\int_{t_0}^s \left[F_2(s+h, \tau, u(\tau), \int_{t_0}^{\tau} f(\tau, \mu, u(\mu)) d\mu \right. \right. \\
 & \quad \left. \left. - F_2(s, \tau, u(\tau), \int_{t_0}^{\tau} f(\tau, \mu, u(\mu)) d\mu) \right] d\tau \right) ds \\
 & + \int_{t_0}^t T(t-s) \left(\int_s^{s+h} F_2(s+h, \tau, u(\tau), \int_{t_0}^{\tau} f(\tau, \mu, u(\mu)) d\mu) d\tau \right) ds
 \end{aligned}$$

for $t \in [t_0, t_0 + a)$, $h > 0$ and $t + h \in (t_0, t_0 + a]$.

Consequently, by (4.6), (2.1) and (4.1)-(4.5),

$$\begin{aligned}
 & \|u(t+h) - u(t)\| \leq hM \|Au_0\| + hM \|AG(u)\| + hMN_1 + ahML_1 \\
 & + ML_1 \int_{t_0}^t \left(\|u(s+h) - u(s)\| + \sum_{i=1}^m \|u(\sigma_i(s+h)) - u(\sigma_i(s))\| \right) ds \quad (4.7) \\
 & + a^2ML_2h + 2aMN_2h \leq Ch + ML_1(1+mc) \int_{t_0}^t \|u(s+h) - u(s)\| ds
 \end{aligned}$$

for $t \in [t_0, t_0 + a)$, $h > 0$ and $t + h \in (t_0, t_0 + a]$, where

$$C := M \left[\|Au_0\| + \|AG(u)\| + N_1 + aL_1 + a^2L_2 + 2aN_2 \right].$$

From (4.7) and Gronwall's inequality,

$$\|u(t+h) - u(t)\| \leq Ce^{aML_1(1+mc)}h$$

for $t \in [t_0, t_0 + a)$, $h > 0$ and $t+h \in (t_0, t_0 + a]$. Hence u is Lipschitz continuous on I .

The Lipschitz continuity of u on I and inequalities (4.1), (3.1), (4.2) imply that the function

$$\begin{aligned} I \ni t \mapsto k(t) &:= F_1(t, u(t), u(\sigma_1(t)), \dots, u(\sigma_m(t))) \\ &+ \int_{t_0}^t F_2(t, s, u(s), \int_{t_0}^s f(s, \tau, u(\tau))d\tau)ds \in E \end{aligned}$$

is Lipschitz continuous on I . This property of $t \mapsto k(t)$ together with assumptions of Theorem 4.1 imply by Theorem 1 from [10], by Theorem 3.1 from this paper and by (2.2), that the linear Cauchy problem

$$v'(t) + Av(t) = k(t), \quad t \in I \setminus \{t_0\},$$

$$v(t_0) = u_0 - G(u)$$

has a unique classical solution v such that

$$\begin{aligned} v(t) &= T(t-t_0)u_0 - T(t-t_0)G(u) + \int_{t_0}^t T(t-s)k(s)ds \\ &= T(t-t_0)u_0 - T(t-t_0)G(u) \\ &+ \int_{t_0}^t T(t-s)F_1(s, u(s), u(\sigma_1(s)), \dots, u(\sigma_m(s)))ds \\ &+ \int_{t_0}^t T(t-s) \left(\int_{t_0}^s F_2(s, \tau, u(\tau), \int_{t_0}^{\tau} f(\tau, \mu, u(\mu))d\mu)d\tau \right) ds = u(t), \quad t \in I. \end{aligned}$$

Consequently, u is the unique classical solution of the nonlocal Cauchy problem (1.1)-(1.2) on I . Therefore, the proof of Theorem 4.1 is complete.

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