ON AN EVEN ORDER NEUTRAL DIFFERENTIAL INEQUALITY

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In this paper, we prove new results related to the nonexistence criteria for eventually positive solutions of certain even order neutral differential inequality with distributed deviating arguments.

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1 Introduction

In order to make this paper self-contained, we introduce the following definition.

Definition 1: The function $f(t)$ is said to be eventually zero if there exists a sufficiently large t_μ such that $f(t) \equiv 0$ holds for $t \geq t_\mu$.

This paper is concerned with nonexistence conditions of eventually positive solutions of the even order neutral differential inequality with distributed deviating arguments

$$[x(t) + c(t)x(t - \tau)]^{(n)} + \int_a^b p(t, \xi)f(x[g(t, \xi)])d\sigma(\xi) \leq 0, \quad t \geq t_0, \quad (1)$$

in which $\tau > 0$ is a constant, n is an even positive integer; $c(t) \in C(I, R)$, $0 \leq c(t) \leq 1$, and $p(t, \xi) \in C(I \times J, R_+)$ is not eventually zero on any $I_\mu \times J$, $I = [t_0, \infty)$, $J = [a, b]$,

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$I_\mu = [t_\mu, \infty)$, $t_\mu \geq t_0$, $R_+ = [0, \infty)$. Furthermore, we assume that $g(t, \xi) \in C(I \times J, R)$ is nondecreasing with respect to t and ξ , respectively, $\frac{d}{dt}g(t, a)$ exists, $g(t, \xi) \leq t$ for $\xi \in J$, and $\liminf_{t \rightarrow \infty, \xi \in J} \{g(t, \xi)\} = \infty$; $f(x) \in C(R, R)$, and $xf'(x) > 0$ ($x \neq 0$); $\sigma(\xi) \in (J, R)$ is nondecreasing; integral of inequality (1) is in Lebesgue-Stieltjes sense.

Recently, Li and Cui [1] have obtained some results dealing with a class of even order neutral differential inequalities with applications. On the other hand, Liu and Fu [2] have studied nonlinear differential inequality with distributed deviating arguments and their applications. These authors provided some results on nonexistence conditions of eventually positive solutions of inequality (1). For example,

Theorem A:(See [1]) *If $0 \leq c(t) \leq 1$, and*

$$\int_{t_0}^t \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds = \infty,$$

then inequality (1) has no eventually positive solutions.

Theorem B:(See [2]) *Assume that $f(-x) = -f(x)$, $x \in (0, \infty)$, and*

$$\frac{f(x)}{x} \geq \lambda, \quad x \in (0, \infty), \quad \text{for some constant } \lambda > 0. \quad (2)$$

If there exists a monotonically increasing function $\varphi(t) \in C'(I, (0, \infty))$ such that

$$\int_{t_0}^t [\lambda \varphi(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - A\varphi'(s)] ds = \infty$$

for any number $A > 0$, then inequality (1) has no eventually positive solutions.

The purpose of this paper is to obtain two new results related to the nonexistence criteria for eventually positive solutions of inequality (1). In the established nonexistence criteria, there is a general class of function $H(t, s)$ as the parameter function. By choosing a different function $H(t, s)$, we are able to derive some useful corollaries.

Definition 2: The solution $x(t) \in C^{(n)}(I, R)$ of inequality (1) is said to be eventually positive if there exists a sufficiently large positive number $T \geq t_0$ such that the inequality $x(t) > 0$ holds for $t \geq T$.

To develop the nonexistence criteria of eventually positive solutions of inequality (1), we first need the following Lemmas:

Lemma 1: (See [1]) *Assume that $x(t)$ is an eventually positive solution of inequality (1). Let*

$$y(t) = x(t) + c(t)x(t - \tau). \quad (3)$$

Then there exists a $t_1 \geq t_0$ such that

$$y(t) > 0, \quad y'(t) > 0, \quad y^{(n-1)}(t) > 0 \quad \text{and} \quad y^{(n)}(t) \leq 0, \quad t \geq t_1.$$

Lemma 2: (See [3]) *Let $x^{(n)}(t) \in C(I, R_+)$. If $x^{(n)}(t)$ is eventually of one sign for all large t , and $x^{(n)}(t) \times x^{(n-1)}(t) \leq 0$ for $t_1 > t_0$, then there exists a constant $\theta \in (0, 1)$ such that for sufficiently large t , there exists a constant $M_\theta > 0$ satisfying*

$$|u'(t/2)| \geq M_\theta t^{n-2} |u^{(n-1)}(t)|.$$

2 Main Results

The following theorems provide sufficient conditions leading to nonexistence of eventually positive solutions for inequality (1).

Theorem 1: Assume that the condition of Theorem B holds, and there exist functions $H(t, s) \in C'(D; R)$, $h(t, s) \in C(D; R)$, with $D = \{(t, s) | t \geq s \geq t_0\}$ satisfying

$$(H_1) \quad H(t, t) = 0, \quad t \geq t_0; \quad H(t, s) > 0, \quad t > s \geq t_0;$$

$$(H_2) \quad H_t'(t, s) \geq 0, \quad H_s'(t, s) \leq 0, \quad \text{and} \quad -H_s'(t, s) = h(t, s)\sqrt{H(t, s)}, \quad (t, s) \in D.$$

If

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds = \infty, \quad (4)$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{h^2(t, s)}{g^{n-2}(s, a)g'(s, a)} ds < \infty, \quad (5)$$

then inequality (1) has no eventually positive solutions.

Proof: Assume to the contrary that $x(t)$ is an eventually positive solution of inequality (1). Then from $\lim_{t \rightarrow \infty} \inf_{\xi \in J} \{g(t, \xi)\} = \infty$, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(t - \tau) > 0$ and $x[g(t, \xi)] > 0$ for $t \geq t_1$ and $\xi \in J$. From (2) and (3), inequality (1) can be written as

$$\begin{aligned} 0 &\geq y^{(n)}(t) + \int_a^b p(t, \xi) f(x[g(t, \xi)]) d\sigma(\xi) \\ &\geq y^{(n)}(t) + \lambda \int_a^b p(t, \xi) \{y[g(t, \xi)] - c[g(t, \xi)]x[g(t, \xi) - \tau]\} d\sigma(\xi). \end{aligned} \quad (6)$$

From Lemma 1, $y'(t) > 0$ and $y(t) \geq x(t)$, $t \geq t_1$, hence $y[g(t, \xi)] \geq y[g(t, \xi) - \tau] \geq x[g(t, \xi) - \tau]$. Thus

$$y^{(n)}(t) + \lambda \int_a^b p(t, \xi) \{1 - c[g(t, \xi)]\} y[g(t, \xi)] d\sigma(\xi) \leq 0, \quad t \geq t_1. \quad (7)$$

Furthermore, in view of $g(t, \xi)$ being nondecreasing with respect to ξ , we have

$$y^{(n)}(t) + \lambda y[g(t, a)] \int_a^b p(t, \xi) \{1 - c[g(t, \xi)]\} d\sigma(\xi) \leq 0, \quad t \geq t_2. \quad (8)$$

Let

$$z(t) = \frac{y^{(n-1)}(t)}{y[\frac{g(t, a)}{2}]}. \quad (9)$$

Then $z(t) \geq 0$. Since $\frac{d}{dt}g(t, a)$ exists, we obtain $y'[g(t, a)] = \frac{dy}{dg} \frac{dg}{dt} g(t, a)$. Furthermore, from Lemma 1, $y^{(n)}(t) \leq 0$, and in view of $g(t, \xi)$ being nondecreasing with respect to ξ , $g(t, \xi) \leq t$ for $\xi \in J$, we obtain $y^{(n-1)}(t) \leq y^{(n-1)}[g(t, a)] \leq y^{(n-1)}[\frac{g(t, a)}{2}]$. Thus, from

Lemma 2, we have

$$\begin{aligned} z'(t) &= \frac{y^{(n)}(t)}{y[\frac{g(t,a)}{2}]} - \frac{1}{2} \frac{y^{(n-1)}(t)y'[\frac{g(t,a)}{2}]g'(t,a)}{y^2[\frac{g(t,a)}{2}]} \\ &\leq \frac{y^{(n)}(t)}{y[\frac{g(t,a)}{2}]} - \frac{M_\theta}{2} g^{n-2}(t,a)g'(t,a)z^2(t), \end{aligned} \quad (10)$$

Furthermore, from $y'(t) > 0$ and (8), for $t \geq t_2$, we obtain

$$z'(t) \leq -\lambda \int_a^b p(t, \xi) \{1 - c[g(t, \xi)]\} d\sigma(\xi) - \frac{M_\theta}{2} g^{n-2}(t,a)g'(t,a)z^2(t). \quad (11)$$

Integrating by parts for any $t > T \geq t_1$, and using the properties (H_1) and (H_2) , we have

$$\begin{aligned} &\lambda \int_T^t H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds \\ &\leq - \int_T^t H(t, s) z'(s) ds - \frac{M_\theta}{2} \int_T^t H(t, s) g^{n-2}(s,a)g'(s,a)z^2(s) ds \\ &= - \int_T^t H(t, s) dz(s) - \frac{M_\theta}{2} \int_T^t H(t, s) g^{n-2}(s,a)g'(s,a)z^2(s) ds \\ &= H(t, T)z(T) - \int_T^t h(t, s) \sqrt{H(t, s)} z(s) ds \\ &\quad - \frac{M_\theta}{2} \int_T^t H(t, s) g^{n-2}(s,a)g'(s,a)z^2(s) ds \\ &= H(t, T)z(T) - \frac{1}{2} \int_T^t \left[\sqrt{M_\theta H(t, s) g^{n-2}(s,a)g'(s,a)} z(s) \right. \\ &\quad \left. + \frac{h(t, s)}{\sqrt{M_\theta g^{n-2}(s,a)g'(s,a)}} \right]^2 ds + \int_T^t \frac{h^2(t, s)}{2M_\theta g^{n-2}(s,a)g'(s,a)} ds, \end{aligned}$$

which implies that

$$\begin{aligned} &\int_T^t \left[\lambda H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{h^2(t, s)}{2M_\theta g^{n-2}(s,a)g'(s,a)} \right] ds \\ &\leq H(t, T)z(T) - \frac{1}{2} \int_T^t \left[\sqrt{M_\theta H(t, s) g^{n-2}(s,a)g'(s,a)} z(s) \right. \\ &\quad \left. + \frac{h(t, s)}{\sqrt{M_\theta g^{n-2}(s,a)g'(s,a)}} \right]^2 ds. \end{aligned} \quad (12)$$

Furthermore, in view of (H_2) , for $t_1 \geq t_0$, we have $H(t, t_1) \leq H(t, t_0)$. Thus

$$\begin{aligned} &\int_{t_1}^t \left[\lambda H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{h^2(t, s)}{2M_\theta g^{n-2}(s,a)g'(s,a)} \right] ds \\ &\leq H(t, t_1)z(t_1) \leq H(t, t_0)z(t_1) \end{aligned}$$

$$\begin{aligned}
& \frac{1}{H(t, t_0)} \int_{t_0}^t \left[\lambda H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{h^2(t, s)}{2M_\theta g^{n-2}(s, a)g'(s, a)} \right] ds \\
&= \frac{1}{H(t, t_0)} \left[\int_{t_0}^{t_1} + \int_{t_1}^t \right] \left[\lambda H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) \right. \\
&\quad \left. - \frac{h^2(t, s)}{2M_\theta g^{n-2}(s, a)g'(s, a)} \right] ds \\
&\leq z(t_1) + \int_{t_0}^{t_1} \frac{H(t, s)}{H(t, t_0)} \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds \\
&\leq z(t_1) + \int_{t_0}^{t_1} \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds. \tag{13}
\end{aligned}$$

It follows from (13) that

$$\begin{aligned}
& \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \lambda H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) \\
&\leq \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[\lambda H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) \right. \\
&\quad \left. - \frac{h^2(t, s)}{2M_\theta g^{n-2}(s, a)g'(s, a)} \right] ds \\
&\quad + \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{h^2(t, s)}{g^{n-2}(s, a)g'(s, a)} ds \\
&\leq z(t_1) + \int_{t_0}^{t_1} \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds \\
&\quad + \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{h^2(t, s)}{g^{n-2}(s, a)g'(s, a)} ds \\
&< \infty,
\end{aligned}$$

which contradicts (4). Therefore, the proof of Theorem 1 is complete.

Remark 1: From Theorem 1, we can establish various sufficient conditions by means of the choices of parameter function $H(t, s)$. For example, choosing $H(t, s) = (t-s)^{m-1}$, $t \geq s \geq t_0$, in which $m > 2$ is an integer, we obtain $h(t, s) = (m-1)(t-s)^{\frac{m-3}{2}}$, $t \geq s \geq t_0$. From Theorem 1, we have

Corollary 1: *If there exists an integer $m > 2$ such that*

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{m-1}} \int_{t_0}^t (t-s)^{m-1} \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds = \infty, \tag{14}$$

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{m-1}} \int_{t_0}^t \frac{(m-1)^2(t-s)^{m-3}}{g^{n-2}(s, a)g'(s, a)} ds < \infty, \tag{15}$$

then inequality (1) has no eventually positive solutions.

If

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds < \infty, \tag{16}$$

we have the following result:

Theorem 2: Assume that the conditions of Theorem 1 and (16) hold. If $H'_t(t, s)$ is nondecreasing, and there exists a function $\varphi(t) \in C(I, R)$ satisfying

$$\liminf_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_u^t \left[\lambda H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{h^2(t, s)}{2M_\theta g^{n-2}(s, a)g'(s, a)} \right] ds \geq \varphi(u), u \geq t_0, \quad (17)$$

$$\lim_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s) g^{n-2}(s, a) g'(s, a) \varphi_+^2(s) ds = \infty, \quad \varphi_+(s) = \max_{s \geq t_0} \{\varphi(s), 0\}, \quad (18)$$

then inequality (1) has no eventually positive solutions.

Proof: Assume to the contrary that $x(t)$ is an eventually positive solution of inequality (1). Then from the proof of Theorem 1, there exists a $t_1 \geq t_0$ such that

$$z'(t) \leq -\lambda \int_a^b p(t, \xi) \{1 - c[g(t, \xi)]\} d\sigma(\xi) - \frac{M_\theta}{2} g^{n-2}(t, a) g'(t, a) z^2(t). \quad (19)$$

Thus

$$\begin{aligned} & \lambda \int_{t_1}^t H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds \\ & \leq - \int_{t_1}^t H(t, s) z'(s) ds - \frac{M_\theta}{2} \int_{t_1}^t H(t, s) g^{n-2}(s, a) g'(s, a) z^2(s) ds \\ & = H(t, t_1) z(t_1) - \int_{t_1}^t \sqrt{H(t, s)} h(t, s) z(s) ds \\ & \quad - \frac{M_\theta}{2} \int_{t_1}^t H(t, s) g^{n-2}(s, a) g'(s, a) z^2(s) ds \end{aligned} \quad (20)$$

and

$$\begin{aligned} & \lambda \int_{t_1}^t H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds \\ & \leq H(t, t_1) z(t_1) - \frac{1}{2} \int_{t_1}^t \left\{ \sqrt{M_\theta g^{n-2}(s, a) g'(s, a) H(t, s)} z(s) \right. \\ & \quad \left. + \frac{h(t, s)}{\sqrt{M_\theta g^{n-2}(s, a) g'(s, a)}} \right\}^2 ds \\ & \quad + \int_{t_1}^t \frac{h^2(t, s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} ds \\ & \leq H(t, t_1) z(t_1) + \int_{t_1}^t \frac{h^2(t, s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} ds. \end{aligned} \quad (21)$$

Furthermore, for $t > u \geq t_0$, we have

$$\begin{aligned} & \frac{1}{H(t, t_0)} \int_u^t \left[\lambda H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{h^2(t, s)}{2M_\theta g^{n-2}(s, a)g'(s, a)} \right] ds \\ & \leq \frac{H(t, u)}{H(t, t_0)} z(u). \end{aligned} \quad (22)$$

From (17) and (H_2) , we conclude that

$$\begin{aligned} \varphi(u) & \leq \frac{1}{H(t, t_0)} \int_u^t \left[\lambda H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) \right. \\ & \quad \left. - \frac{h^2(t, s)}{2M_\theta g^{n-2}(s, a)g'(s, a)} \right] ds \\ & \leq \frac{H(t, u)}{H(t, t_0)} z(u) \leq z(u), \end{aligned} \quad (23)$$

which implies that

$$\varphi_+^2(u) \leq z^2(u). \quad (24)$$

Let

$$\begin{aligned} v(t) & = \frac{1}{H(t, t_0)} \int_{t_1}^t \sqrt{H(t, s)} h(t, s) z(s) ds \\ w(t) & = \frac{1}{H(t, t_0)} \int_{t_1}^t \frac{M_\theta}{2} H(t, s) g^{n-2}(s, a) g'(s, a) z^2(s) ds. \end{aligned}$$

Then, from (20), we find

$$v(t) + w(t) \leq \frac{H(t, t_1)}{H(t, t_0)} z(t_1) - \frac{1}{H(t, t_0)} \int_{t_1}^t \lambda H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds. \quad (25)$$

It follows from (17) that

$$\liminf_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_u^t \lambda H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds \geq \varphi(u).$$

Furthermore, we obtain

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_1}^t \lambda H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds \\ & - \liminf_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_1}^t \frac{h^2(t, s)}{2M_\theta g^{n-2}(s, a)g'(s, a)} ds \geq \varphi(t_1). \end{aligned} \quad (26)$$

It turns out from (26) and (16)

$$\liminf_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_1}^t \frac{h^2(t, s)}{2M_\theta g^{n-2}(s, a)g'(s, a)} ds < \infty.$$

Thus, there exists a sequence $\{t_n\}_1^\infty$ in $[t_1, \infty)$ such that $\lim_{n \rightarrow \infty} t_n = \infty$ that satisfies

$$\lim_{n \rightarrow \infty} \frac{1}{H(t_n, t_0)} \int_{t_1}^{t_n} \frac{h^2(t_n, s)}{2M_\theta g^{n-2}(s, a)g'(s, a)} ds < \infty. \quad (27)$$

Result (27) implies that

$$\begin{aligned} \limsup_{t \rightarrow \infty} \{v(t) + w(t)\} \\ \leq z(t_1) - \liminf_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_1}^t \lambda H(t, s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds \\ \leq z(t_1) - \varphi(t_1) \triangleq M. \end{aligned} \quad (28)$$

Then, for any sufficiently large n , we have

$$v(t_n) + w(t_n) < M_1, \quad (29)$$

where $M_1 > M$, M and M_1 are constant. According to the definition of $w(t)$, we have

$$w'(t) = \int_{t_1}^t \frac{M_\theta (H'_t(t, s)H(t, t_0) - H'_t(t, t_0)H(t, s))}{2H^2(t, t_0)} g^{n-2}(s, a)g'(s, a)z^2(s) ds.$$

Since $H'_t(t, s)$ is nondecreasing and (H_2) holds, we have $w'(t) \geq 0$, thus, $w(t)$ is increasing, and $\lim_{t \rightarrow \infty} w(t) = l$ exists, where l is finite or infinite. In the case of $\lim_{n \rightarrow \infty} w(t_n) = \infty$. Consequently, it follows from (29) that

$$\lim_{n \rightarrow \infty} v(t_n) = -\infty, \quad (30)$$

and

$$\frac{v(t_n)}{w(t_n)} + 1 < \frac{M_1}{w(t_n)}.$$

Thus, for any $0 < \varepsilon < 1$ and sufficiently large n , we have

$$\frac{v(t_n)}{w(t_n)} < \varepsilon - 1 < 0. \quad (31)$$

On the other hand, by using the Schwartz inequality, for $t \geq t_1$, we obtain

$$\begin{aligned} 0 &\leq v^2(t_n) = \frac{1}{H^2(t_n, t_0)} \left\{ \int_{t_1}^{t_n} \sqrt{H(t_n, s)} h(t_n, s) z(s) ds \right\}^2 \\ &\leq \left\{ \frac{1}{H(t_n, t_0)} \int_{t_1}^{t_n} \frac{M_\theta}{2} H(t_n, s) g^{n-2}(s, a) g'(s, a) z^2(s) ds \right\} \\ &\quad \times \left\{ \frac{1}{H(t_n, t_0)} \int_{t_1}^{t_n} \frac{2h^2(t_n, s)}{M_\theta g^{n-2}(s, a) g'(s, a)} ds \right\} \\ &= w(t_n) \frac{1}{H(t_n, t_0)} \int_{t_1}^{t_n} \frac{2h^2(t_n, s)}{M_\theta g^{n-2}(s, a) g'(s, a)} ds. \end{aligned}$$

Then,

$$0 \leq \frac{v^2(t_n)}{w(t_n)} \leq \frac{1}{H(t_n, t_0)} \int_{t_1}^{t_n} \frac{2h^2(t_n, s)}{M_\theta g^{n-2}(s, a) g'(s, a)} ds. \quad (32)$$

It follows from (27) that

$$0 \leq \lim_{n \rightarrow \infty} \frac{v^2(t_n)}{w(t_n)} < \infty. \quad (33)$$

In view of (31), we obtain

$$\lim_{n \rightarrow \infty} \frac{v(t_n)}{w(t_n)} = \lim_{n \rightarrow \infty} \frac{v'(t_n)}{w'(t_n)} \leq \varepsilon - 1 < 0,$$

and then,

$$\lim_{n \rightarrow \infty} \frac{v^2(t_n)}{w(t_n)} = \lim_{n \rightarrow \infty} \frac{2v(t_n)v'(t_n)}{w'(t_n)} = 2 \lim_{n \rightarrow \infty} v(t_n) \lim_{n \rightarrow \infty} \frac{v'(t_n)}{w'(t_n)} = \infty,$$

which contradicts (33). Thus, we have $\lim_{t \rightarrow \infty} w(t) = l < \infty$. Furthermore, according to (24), we conclude that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_1}^t H(t, s) g^{n-2}(s, a) g'(s, a) \varphi_+^2(s) ds \\ & \leq \lim_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_1}^t H(t, s) g^{n-2}(s, a) g'(s, a) z^2(s) ds = \frac{2}{M_\theta} \lim_{t \rightarrow \infty} w(t) < \infty, \end{aligned} \quad (34)$$

which implies that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s) g^{n-2}(s, a) g'(s, a) \varphi_+^2(s) ds \\ & = \lim_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \left[\int_{t_0}^{t_1} + \int_{t_1}^t \right] H(t, s) g^{n-2}(s, a) g'(s, a) \varphi_+^2(s) ds \\ & \leq \int_{t_0}^{t_1} H(t, s) g^{n-2}(s, a) g'(s, a) \varphi_+^2(s) ds + \frac{2}{M_\theta} \lim_{t \rightarrow \infty} w(t) < \infty. \end{aligned}$$

The latter contradicts (18). Therefore, the proof of Theorem 2 is complete.

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