MIXED PROBLEM WITH INTEGRAL BOUNDARY CONDITION FOR A HIGH ORDER MIXED TYPE PARTIAL DIFFERENTIAL EQUATION

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In this paper, we study a mixed problem with integral boundary conditions for a high order partial differential equation of mixed type. We prove the existence and uniqueness of the solution. The proof is based on energy inequality, and on the density of the range of the operator generated by the considered problem.

Key words: Integral Boundary Condition , Energy Inequalities, Equation of Mixed Type, Sobolev Spaces.

AMS subject classifications: 35B45, 35G10, 35M10.

1 Introduction

In the rectangle $Q = (0,1) \times (0,T)$, we consider the equation

$$\mathcal{L}u = \frac{\partial^2 u}{\partial t^2} + (-1)^\alpha \frac{\partial^\alpha}{\partial x^\alpha}\left(a(x,t)\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right) = f(x,t), \tag{1}$$

where $a(x,t)$ is bounded for $0 < a_0 < a(x,t) \le a_1$, and has bounded partial derivatives such that $0 < a_2 \le \frac{\partial a(x,t)}{\partial t} \le a_3$ and $\left|\frac{\partial^i a(x,t)}{\partial x^i}\right| \le b_i$, $i = \overline{1,\alpha}$, for $(x,t) \in \overline{Q}$.

To equation (1) we add the initial conditions

$$l_1 u = (x,0) = \varphi(x), \quad l_2 u = \frac{\partial u}{\partial t}(x,0) = \psi(x) \quad x \in (0,1), \tag{2}$$

the boundary conditions

$$\frac{\partial^i}{\partial x^i}u(0,t) = 0, \quad \text{for } 0 \le i \le \alpha - 1, t \in (0,T), \tag{3}$$

$$\frac{\partial^i}{\partial x^i}u(1,t) = 0, \quad \text{for } 0 \le i \le \alpha - 2, \ t \in (0,T), \tag{4}$$

and integral condition

$$\int_0^1 u(\xi, t)d\xi = 0, \ \ t \in (0, T), \tag{5}$$

where φ and ψ are known functions which satisfy the compatibility conditions given in equations (3)-(5).

Integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics; see for example Choi and Chan [6], Ewing and Lin [7], Shi [12], and Shi and Shillor [13]. Boundary value problems for parabolic equations with integral boundary conditions are investigated by Batten [1], Bouziani and Benouar [2], Cannon [3, 4], Cannon, Perez Esteva and Van der Hoeck [5], Ionkin [8], Kamynin [9], Kartynnik [10], Shi [12], Yurchuk [14] and many references therein. A method was developed in [2], [10], and [14] based on energy inequalities. In this paper, our objective is to extend this method to high order partial differential equations of mixed type.

2 Preliminaries

In this paper, we prove existence and uniqueness of a strong solution of the problem stated in equations (1)-(5). For this, we consider the problem in equations (1)-(5) as a solution of the operator equation

$$Lu = \mathcal{F}, \tag{6}$$

where $L = (\mathcal{L}, l_1, l_2)$, with domain of definition $D(L)$ consisting of functions $u \in W_2^{2\alpha,2}(Q)$, such that, $\frac{\partial^{i+1}u}{\partial x^i \partial t} \ \frac{\partial^i}{\partial x^i}(\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}) \in L_2(Q)$, $i = \overline{1,\alpha}$ and, u satisfies conditions in problems (3)-(5); the operator L is considered from E to F, where E is the Banach space consisting of all functions $u \in L_2(Q)$, satisfying equations (3)-(5), with the finite norm

$$\|u\|_E^2 = \int_Q (1-x)\left[\left|\frac{\partial^2 u}{\partial t^2}\right|^2 + \left|\frac{\partial^\alpha}{\partial^\alpha}(a\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t})\right|^2\right]dxdt$$

$$+ \sum_{i=1}^{\alpha-1}\int_Q x^{2i}(1-x)^{2i+1}\left|\frac{\partial^i}{\partial x^i}(a\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t})\right|^2 dxdt$$

$$+ \sup_{0 \le t \le T}\int_0^1(1-x)\left[\sum_{i=0}^a\left|\frac{\partial^i u}{\partial x^i}\right|^2 + \left|\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right|^2 + \left|\frac{\partial u}{\partial t}\right|^2\right]dx, \tag{7}$$

and F is the Hilbert space of vector-valued functions $\mathcal{F} = (f, \varphi, \psi)$ obtained by completion of the space $L_2(Q) \times W_2^{2\alpha,2}(0,1) \times W_2^{2\alpha,2}(0,1)$ with respect to the norm

$$\|\mathcal{F}\|_F^2 = \int_Q(1-x)^\nu|f|^2 dxdt + \int_0^1(1-x)\left[\sum_{i=0}^a\left|\frac{\partial^i \varphi}{\partial x^i}\right|^2 + \left|\frac{\partial^\alpha \psi}{\partial x^\alpha}\right|^2 + |\psi|^2\right]dx, \tag{8}$$

where ν is an arbitrary number such that $0 < \nu < 1$.

We then establish an energy inequality:

$$\|u\|_E \leq C_1 \|Lu\|_F, \tag{9}$$

and we show that the operator L has the closure $\overline{L}$.

Definition 1: A solution of the operator equation $\overline{L}u = \mathcal{F}$ is called a strong solution of the problem in equations (1)-(5).

Inequality (9) can be extended to $u \in D(\overline{L})$, i.e.,

$$\|u\|_E \leq C_1 \|\overline{L}u\|_F, \quad \forall u \in D(\overline{L}).$$

From this inequality, we obtain the uniqueness of a strong solution if it exists, and the equality of sets $R(\overline{L})$ and $\overline{R(L)}$. Thus, to prove the existence of a strong solution of the problem in equations (1)-(5) for any $\mathcal{F} \in F$, it remains to prove that the set $R(L)$ is dense in F.

Lemma 1: *For any function $u \in E$ satisfying the condition* (2), *for $c \geq 8$*

$$\int_0^1 \exp(-c\tau)(1-x)\left|\tfrac{\partial u}{\partial t}(x,\tau)\right|^2 dx \leq \int_0^1 (1-x)|\psi|^2 dx$$

$$+ \tfrac{1}{8}\int_0^\tau \int_0^1 \exp(-ct)(1-x)\left|\tfrac{\partial^2 u}{\partial t^2}\right|^2 dxdt. \tag{10}$$

Proof: Starting from $\int_0^\tau \int_0^1 \exp(-ct)(1-x)\tfrac{\partial^2 u}{\partial t^2}\tfrac{\partial \overline{u}}{\partial t}dxdt$ and integrating by parts, we obtain equation (10).

Remark 1: We note the above lemma holds for weaker conditions on u.

Lemma 2: *For $u \in E$,*

$$\int_0^1 (1-x)\left|\tfrac{\partial^\alpha u}{\partial x^{\alpha-1}\partial t}\right|^2 dx \leq \int_0^1 (1-x)\left|\tfrac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right|^2 dx. \tag{11}$$

Proof: Starting from $\int_0^1 (1-x)\tfrac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\tfrac{\partial^\alpha \overline{u}}{\partial x^{\alpha-1}\partial t}dx$ and using elementary inequalities yields (11).

Lemma 3: *For $u \in E$ satisfying the condition in equation* (2),

$$\sum_{i=0}^\alpha \int_0^1 \exp(-c\tau)(1-x)\left|\tfrac{\partial^i u(x,\tau)}{\partial x^i}\right|^2 dx \leq \sum_{i=0}^\alpha \int_0^1 (1-x)\left|\tfrac{\partial^i \varphi}{\partial x^i}\right|^2 dx.$$

$$+ (\alpha+1)\int_0^\tau \int_0^1 \exp(-ct)(1-x)\left|\tfrac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right|^2 dxdt. \tag{12}$$

Proof: Integrating the expression $\int_0^\tau \exp(-ct)\tfrac{\partial^{i+1}u}{\partial x^i \partial t}\tfrac{\partial^i u}{\partial x^i}dt$ by parts for $i = \overline{0, \alpha}$, using elementary inequalities and Lemma 2, we obtain expression (12).

3. An Energy Inequality and Its Consequences

Theorem 1: *For any function $u \in D(L)$ we have the inequality*

$$\|u\|_E \leq C_1 \|Lu\|_F, \tag{13}$$

where constant $C_1 = \frac{\exp(cT)\max((4\alpha^2/(1-\nu)^2)+\frac{9}{8},a_1)}{\min(\frac{1}{64},a_0)}$, *with the constant c satisfying*

$$c \geq 8 \text{ and } ca_0 - (\alpha + 1) \geq a_3 + \frac{a_1}{16}(3\beta_1 - 6). \tag{14}$$

Proof: Denote

$$Mu = (1 - x)\frac{\partial^2 u}{\partial t^2} + \alpha J\frac{\partial^2 u}{\partial t^2} \text{ where } Ju = \int_0^x u(\xi, t)d\xi.$$

We consider the quadratic formula

$$\text{Re}\int_0^\tau \int_0^1 \exp(-ct)\mathcal{L}u\overline{Mu}dxdt, \tag{15}$$

with the constant c satisfying condition (14); obtained by multiplying equation (1) by $\exp(-ct)\overline{Mu}$; integrating over Q^τ, where $Q^\tau = (0,1) \times (0,\tau)$, with $0 \leq \tau \leq T$; and by taking the real part. Integrating by parts α times in formula (15) with the use of boundary conditions in equations (3), (4), and (5), we obtain

$$\text{Re}\int_0^\tau \int_0^1 \exp(-ct)\mathcal{L}u\overline{Mu}dxdt = \int_0^\tau \int_0^1 \exp(-ct)(1-x)\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dxdt$$

$$+ \text{Re}\int_0^\tau \int_0^1 \exp(-ct)(1-x)a\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\frac{\partial^{\alpha+2}\overline{u}}{\partial x^\alpha \partial t^2}dxdt. \tag{16}$$

By substituting the expression of Mu in formula (15), using elementary inequalities and the inequality

$$\int_0^1 \frac{\left|J\frac{\partial^2 u}{\partial t^2}\right|^2}{(1-x)^\nu}dx \leq \frac{4}{(1-\nu)^2}\int_0^1 (1-x)\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dx, \text{ where } 0 < \nu < 1,$$

yields:

$$\text{Re}\int_0^\tau \int_0^1 \exp(-ct)\mathcal{L}u\overline{Mu}dxdt \leq (\frac{4\alpha^2}{(1-\nu)^2} + 1)\int_0^\tau \int_0^1 \exp(-ct)(1-x)^\nu|\mathcal{L}u|^2 dxdt$$

$$+ \frac{1}{2}\int_0^\tau \int_0^1 \exp(-ct)(1-x)\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dxdt. \tag{17}$$

From equation (1) we have:

$$\frac{1}{2}\left|\frac{\partial^\alpha}{\partial x^\alpha}(a\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t})\right|^2 \leq |\mathcal{L}u|^2 + \left|\frac{\partial^2 u}{\partial t^2}\right|^2. \tag{18}$$

Using elementary inequalities, we obtain

$$\frac{1}{4}\sum_{i=1}^{\alpha-1}\int_0^\tau \int_0^1 \exp(-ct)x^{2i}(1-x)^{2i+1}\left|\frac{\partial^i}{\partial x^i}(a\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t})\right|^2 dxdt$$

$$\leq \frac{1}{2}\int_0^\tau \int_0^1 \exp(-ct)(1-x)\left|\frac{\partial^\alpha}{\partial x^\alpha}\left(a\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)\right|^2 dxdt$$

$$+ (3\beta_1 - 6)\int_0^\tau \int_0^1 a\exp(-ct)(1-x)\left|\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right|^2 dxdt, \tag{19}$$

where $\beta_1 > 0$.

By integrating the last term on the right-hand side of expression (16) and combining the obtained results with Lemmas 1 and 3, and the inequalities in equations (17), (18), and (19), we get:

$$\left(\frac{4\alpha^2}{(1-\nu)^2} + \frac{9}{8}\right)\int_0^\tau \int_0^1 \exp(-ct)(1-x)^\nu |\mathcal{L}u|^2 dxdt + a_1\int_0^1 (1-x)\left|\frac{\partial^\alpha \psi}{\partial x^\alpha}\right|^2 dx + \int_0^1 (1-x)|\psi|^2 dx$$

$$+ \sum_{i=1}^{\alpha}\int_0^1 (1-x)\left|\frac{\partial^i \varphi}{\partial x^i}\right|^2 dx \geq \sum_{i=0}^{\alpha}\int_0^1 \exp(-c\tau)(1-x)\left|\frac{\partial^i u(x,\tau)}{\partial x^i}\right|^2 dx$$

$$+ \frac{2}{8}\int_0^\tau \int_0^1 \exp(-ct)(1-x)\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dxdt$$

$$+ a_0\int_0^1 \exp(-c\tau)(1-x)\left|\frac{\partial^{\alpha+1}u(x,\tau)}{\partial x^\alpha \partial t}\right|^2 dx + \frac{1}{32}\int_0^\tau \int_0^1 \exp(-ct)(1-x)\left|\frac{\partial^\alpha}{\partial x^\alpha}\left(a\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)\right|^2 dxdt$$

$$+ \frac{1}{64}\sum_{i=1}^{\alpha-1}\int_0^\tau \int_0^1 \exp(-ct)x^{2i}(1-x)^{2i+1}\left|\frac{\partial^i}{\partial x^i}\left(a\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)\right|^2 dxdt$$

$$+ \int_0^1 \exp(-c\tau)(1-x)\left|\frac{\partial u(x,\tau)}{\partial t}\right|^2 dx$$

$$+ \left(ca_0 - a_3 - (\alpha+1) - a_1\left(\frac{3\beta_1-6}{16}\right)\right)\int_0^\tau \int_0^1 \exp(-ct)(1-x)\left|\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right|^2 dxdt. \tag{20}$$

Using elementary inequalities and condition (14), we obtain:

$$\left(\frac{4\alpha^2}{(1-\nu)^2} + \frac{9}{8}\right)\int_Q (1-x)^\nu |\mathcal{L}u|^2 dxdt + a_1\int_0^1 (1-x)\left|\frac{\partial^\alpha \psi}{\partial x^\alpha}\right|^2 dx + \int_0^1 (1-x)|\psi|^2 dx$$

$$+ \sum_{i=0}^{\alpha}\int_0^1 (1-x)\left|\frac{\partial^i \varphi}{\partial x^i}\right|^2 dx$$

$$\geq \exp(-cT)\left[\sum_{i=0}^{\alpha}\int_0^1 (1-x)\left|\frac{\partial^i u(x,\tau)}{\partial x^i}\right|^2 dx + \frac{2}{8}\int_0^\tau \int_0^1 (1-x)\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dxdt\right.$$

$$+ a_0\int_0^1 (1-x)\left|\frac{\partial^{\alpha+1}u(x,\tau)}{\partial x^\alpha \partial t}\right|^2 dx + \frac{1}{32}\int_0^\tau \int_0^1 (1-x)\left|\frac{\partial^\alpha}{\partial x^\alpha}\left(a\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)\right|^2 dxdt$$

$$+ \frac{1}{64}\sum_{i=0}^{\alpha-1}\int_0^\tau\int_0^1 x^{2i}(1-x)^{2i+1}\left|\frac{\partial^i}{\partial x^i}\left(a\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)\right|^2 dxdt + \int_0^1(1-x)\left|\frac{\partial u(x,\tau)}{\partial t}\right|^2 dx\right]. \quad (21)$$

As the left-hand side of equation (21) is independent of τ, by replacing the right-hand side by its upper bound with respect to τ in the interval $[0, T]$, we obtain the desired inequality. $\qquad\square$

Lemma 4: *The operator L from E to F admits a closure.*

Proof: Suppose that $(u_n) \in D(L)$ is a sequence such that

$$u_n \to 0 \text{ in } E, \qquad (22)$$

and

$$Lu_n \to \mathcal{F} \text{ in } F. \qquad (23)$$

We must show that $\mathcal{F} = (f, \varphi_1, \varphi_2) = 0$. The fact that $\varphi_i = 0$; $i = 1, 2$ results directly from the continuity of the trace operators l_i.

Introducing the operator

$$\mathcal{L}_0 v = \frac{\partial^2 v}{\partial t^2} + (-1)^{\alpha+1}\frac{\partial^{\alpha+1}}{\partial x^\alpha\partial t}\left(a(x,t)\frac{\partial^\alpha v}{\partial x^\alpha}\right),$$

defined on the domain $D(\mathcal{L}_0)$ of functions $v \in W_2^{2\alpha,2}(Q)$ verifying

$$v|_{t=T} = 0, \ \frac{\partial v}{\partial t}\big|_{t=T} = 0, \ \frac{\partial^i v}{\partial x^i}\big|_{x=0} = 0, \ i = \overline{0, \alpha-1}, \ \frac{\partial^i v}{\partial x^i}\big|_{x=1} = 0, \ i = \overline{0, \alpha},$$

we note that $D(\mathcal{L}_0)$ is dense in the Hilbert space obtained from the completion of $L_2(Q)$ with respect to the norm

$$\|v\|^2 = \int_Q (1-x)^\nu |v|^2 dxdt.$$

Additionally, since

$$\int_Q (1-x)^\nu f\overline{v}dxdt = \lim_{n\to\infty}\int_Q \mathcal{L}u_n(1-x)^\nu\overline{v}dxdt = \lim_{n\to\infty}\int_Q u_n\mathcal{L}_0((1-x)^\nu\overline{v})dxdt = 0,$$

this holds for every function $v \in D(\mathcal{L}_0)$, and yields $f = 0$.

Theorem 1 is valid for strong solutions, i.e., we have the inequality

$$\|u\|_E \le C_1\|\overline{L}u\|_F, \ \ \forall\, u \in D(\overline{L}),$$

hence we obtain

Corollary 1: *A strong solution of the problem in equations* (1)-(5) *is unique if it exists, and depends continuously on $\mathcal{F} = (f, \varphi, \psi) \in F$.*

Corollary 2: *The range $R(L)$ of the operator $\overline{L}$ is closed in F, and $R(\overline{L}) = \overline{R(L)}$.*

4 Solvability of the Problem

To prove the solvability of the problem in equations (1)-(5), it is sufficient to show that $R(L)$ is dense in F. The proof is based on the following lemma.

Lemma 5: *Suppose that* $\left| \frac{\partial^{i+1}a(x,t)}{\partial x^i \partial t} \right| \leq b_i$, $i = \overline{1, \alpha}$, *is also bounded. Let* $D_0(L) = \{u \in D(L) : l_1 u = 0, l_2 u = 0\}$. *If, for* $u \in D_0(L)$ *and some* $\omega \in L_2(Q)$, *we have*

$$\int_Q (1-x)^{\alpha+\nu+1} \mathcal{L}u\varpi \, dx \, dt = 0, \tag{24}$$

where ν *is an arbitrary number such that* $0 < \nu < 1$, *then* $\omega = 0$.

Proof: The equality (24) can be written as follows:

$$-\int_Q (1-x)^{\alpha+\nu+1} \frac{\partial^2 u}{\partial t^2} \varpi \, dx \, dt = (-1)^\alpha \int_Q (1-x)^{\alpha+\nu+1} \frac{\partial^\alpha}{\partial x^\alpha} \left(a \frac{\partial^{\alpha+1} u}{\partial x^\alpha \partial t} \right) \varpi \, dx \, dt. \tag{25}$$

If we introduce the smoothing operators with respect to t [14] $J_\epsilon^{-1} = (I + \epsilon \frac{\partial}{\partial t})^{-1}$ and $(J_\epsilon^{-1})^*$, then these operators provide the solutions of the problems

$$\epsilon \frac{dg_\epsilon(t)}{dt} + g_\epsilon(t) = g(t),$$

$$g_\epsilon(t)|_{t=0} = 0 \tag{26}$$

and

$$-\epsilon \frac{dg_\epsilon^*(t)}{dt} + g_\epsilon^*(t) = g(t),$$

$$g_\epsilon^*(t)|_{t=T} = 0, \tag{27}$$

and also have the following properties: for any $g \in L_2(0, T)$, the functions $g_\epsilon = (J_\epsilon^{-1})g$ and $g_\epsilon^* = (J_\epsilon^{-1})^* g$ are in $W_2^1(0, T)$ such that $g_\epsilon|_{t=0} = 0$ and $g_\epsilon^*|_{t=T} = 0$. Moreover, J_ϵ^{-1} commutes with $\frac{\partial}{\partial t}$, so $\int_0^T |g_\epsilon - g|^2 dt \to 0$ and $\int_0^T |g_\epsilon^* - g|^2 dt \to 0$, for $\epsilon \to 0$.

Now, for given $\omega(x, t)$, we introduce the function

$$v(x,t) = (1-x)^{\alpha-1} \int_0^x (1-\xi)^{\nu+1} \frac{\partial \omega(\xi,t)}{\partial \xi} d\xi - (\alpha+\nu+1)(1-x)^{\alpha-1} \int_0^x (1-\xi)^\nu \omega(\xi,t) d\xi.$$

Integrating by parts, we obtain

$$(1-x)v + \alpha Jv = (1-x)^{\alpha+\nu+1}\omega, \text{ and } \int_0^1 v(x,t)dx = 0. \tag{28}$$

Then from equality (25), we have

$$-\int_Q \frac{\partial^2 u}{\partial t^2} N\overline{v} \, dx \, dt = \int_Q A(t) \frac{\partial u}{\partial t} \overline{v} \, dx \, dt, \tag{29}$$

where $Nv = (1-x)v + \alpha Jv$, and $A(t)u = (-1)^\alpha \frac{\partial^\alpha}{\partial x^\alpha}((1-x)a(x,t)\frac{\partial^\alpha u}{\partial x^\alpha})$. Replacing $\frac{\partial u}{\partial t}$ in equation (29) by the smoothed function $J_\epsilon^{-1}\frac{\partial u}{\partial t}$, and using the relation

$$A(t)J_\epsilon^{-1} = J_\epsilon^{-1}A(\tau) + \epsilon J_\epsilon^{-1}\frac{\partial A(\tau)}{\partial \tau}J_\epsilon^{-1},$$

we get

$$\int_Q \frac{\partial u}{\partial t}\overline{N\frac{\partial v_\epsilon^*}{\partial t}}dxdt = \int_Q A(t)\frac{\partial u}{\partial t}\overline{v_\epsilon^*}dxdt + \epsilon\int_Q \frac{\partial A}{\partial t}\left(\frac{\partial u}{\partial t}\right)_\epsilon\overline{v_\epsilon^*}dxdt. \tag{30}$$

By passing to the limit, we satisfy the equality (30) for all functions satisfying the conditions in equations (2)-(5), such that $\frac{\partial^i}{\partial x^i}(a\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}) \in L_2(Q)$ for $0 \le i \le \alpha$.

The operator $A(t)$ has, on $L_2(0,1)$, a continuous inverse defined by the relation

$$A^{-1}(t)g = (-1)^\alpha\int_0^x\int_0^{\eta_{\alpha-1}}\cdots\int_0^{\eta_1}\frac{1}{a(\eta,t)(1-\eta)}\left[\int_0^\eta\int_0^{\xi_{\alpha-1}}\cdots\int_0^{\xi_1}g(\xi)d\xi d\xi_1\ldots d\xi_{\alpha-1}\right.$$

$$\left.+\sum_{i=1}^\alpha C_i\frac{\eta^{i-1}}{(i-1)!}\right]d\eta d\eta_1\ldots d\eta_{\alpha-1}, \tag{31}$$

where

$$\int_0^1 A^{-1}(t)gdx = 0. \tag{32}$$

Hence the function $\left(\frac{\partial u}{\partial t}\right)_\epsilon$ can be represented in the form $\left(\frac{\partial u}{\partial t}\right)_\epsilon = J_\epsilon^{-1}A^{-1}A\frac{\partial u}{\partial t}$. Then, $\frac{\partial A}{\partial t}\left(\frac{\partial u}{\partial t}\right)_\epsilon = A_\epsilon(t)A(t)\frac{\partial u}{\partial t}$, where

$$A_\epsilon(t)g = \sum_{k=0}^\alpha \mathcal{C}_\alpha^k\frac{\partial^{k+1}a}{\partial x^k\partial t}\left[\sum_{p=0}^{\alpha-k}\frac{\partial^p\left(\frac{1}{a}\right)}{\partial x^p}\frac{\partial^{\alpha-k-p}}{\partial x^{\alpha-k-p}}\left(\int_0^x\int_0^{\xi_{\alpha-1}}\cdots\int_0^{\xi_1}g(\xi,t)d\xi d\xi_1\ldots d\xi_{\alpha-1}\right)\right]_\epsilon$$

$$+\left[\sum_{k=1}^\alpha C_k\sum_{\ell=0}^\alpha\left\{\mathcal{C}_\alpha^\ell\frac{\partial^{\ell+1}a}{\partial x^\ell\partial t}\sum_{p=0}^{\alpha-\ell}\left(\frac{\partial^p\left(\frac{1}{a}\right)}{\partial x^p}\frac{\partial^{\alpha-\ell-p}}{\partial x^{\alpha-\ell-p}}\frac{x^{k-1}}{(k-1)!}\right)_\epsilon\right\}\right]. \tag{33}$$

Consequently, equation (30) becomes

$$\int_Q \frac{\partial u}{\partial t}\overline{N\frac{\partial v_\epsilon^*}{\partial t}}dxdt = \int_Q A(t)\frac{\partial u}{\partial t}\overline{(v_\epsilon^* + \epsilon A_\epsilon^* v_\epsilon^*)}dxdt, \tag{34}$$

in which $A_\epsilon^*(t)$ is the adjoint of the operator $A_\epsilon(t)$.

The left-hand side of equation (34) is a continuous linear functional of $\frac{\partial u}{\partial t}$. Hence the function $h_\epsilon = v_\epsilon^* + \epsilon A_\epsilon^* v_\epsilon^*$ has the derivatives $\frac{\partial^i h_\epsilon}{\partial x^i} \in L_2(Q)$, $\frac{\partial^i}{\partial x^i}(a(x,t)\frac{\partial^\alpha h_\epsilon}{\partial x^\alpha}) \in L_2(Q)$, $i = \overline{1,\alpha}$, and the following conditions are satisfied

$$\frac{\partial^i h_\epsilon}{\partial x^i}\big|_{x=0} = 0, \ i = \overline{0,\alpha-1}, \ \frac{\partial^i h_\epsilon}{\partial x^i}\big|_{x=1} = 0, \ i = \overline{1,\alpha}. \tag{35}$$

The operators $A_\epsilon^*(t)$ are bounded in $L_2(Q)$. For ϵ sufficiently small, we have $\|\epsilon A_\epsilon^*(t)\|_{L_2(Q)} < 1$, hence the operator, $I + \epsilon A_\epsilon^*(t)$ has a bounded inverse in $L_2(Q)$. In addition, $\frac{\partial^i A_\epsilon^*(t)}{\partial x^i}$, $i = \overline{1,\alpha}$, are bounded operators in $L_2(Q)$. So from the equality,

$$\frac{\partial^i h_\epsilon}{\partial x^i} = (I + \epsilon A_\epsilon^*(t))\frac{\partial^i v_\epsilon^*}{\partial x^i} + \epsilon \sum_{k=1}^{i} \mathcal{C}_i^k \frac{\partial^k A_\epsilon^*(t)}{\partial x^k} \frac{\partial^{i-k} v_\epsilon^*}{\partial x^{i-k}}, \quad i = \overline{0,\alpha}. \tag{36}$$

We conclude that the function v_ϵ^* has the derivatives $\frac{\partial^i v_\epsilon^*}{\partial x^i} \in L_2(Q)$, $i = \overline{0,\alpha}$. Taking into account equations (36) and (35), we have

$$\left[(I + \epsilon A_\epsilon^*(t))\frac{\partial^i v_\epsilon^*}{\partial x^i} + \epsilon \sum_{k=1}^{i} \mathcal{C}_i^k \frac{\partial^k A_\epsilon^*(t)}{\partial x^k} \frac{\partial^{i-k} v_\epsilon^*}{\partial x^{i-k}} \right]_{x=0} = 0, \quad i = \overline{0,\alpha-1}, \tag{37}$$

and

$$\left[(I + \epsilon A_\epsilon^*(t))\frac{\partial^i v_\epsilon^*}{\partial x^i} + \epsilon \sum_{k=1}^{i} \mathcal{C}_i^k \frac{\partial^k A_\epsilon^*(t)}{\partial x^k} \frac{\partial^{i-k} v_\epsilon^*}{\partial x^{i-k}} \right]_{x=0} = 0, \quad i = \overline{0,\alpha}. \tag{38}$$

Similarly, for ϵ sufficiently small, in each fixed point $x \in [0,1]$, the operators $\frac{\partial^i A_\epsilon^*(t)}{\partial x^i}$, $i = \overline{0,\alpha}$ are bounded in $L_2(Q)$, and the operator $I + \epsilon A_\epsilon^*(t)$ is continuous inversible in $L_2(Q)$. And so from equations (37) and (38), v_ϵ^* satisfies the conditions

$$\frac{\partial^i v_\epsilon^*}{\partial x^i}\Big|_{x=0} = 0, \quad i = \overline{0,\alpha-1},$$

$$\frac{\partial^i v_\epsilon^*}{\partial x^i}\Big|_{x=0} = 0, \quad i = \overline{0,\alpha}. \tag{39}$$

So, for ϵ sufficiently small, the function v_ϵ^* has the same properties as h_ϵ. In addition, v_ϵ^* satisfies the integral condition in equation (28).

Putting $u = \int_0^t \int_0^\tau \exp(c\eta)v_\epsilon^*(\eta,\tau)d\eta d\tau$ in equation (29), where the constant c satisfies $ca_0 - a_3 - \frac{a_3^2}{a_0} \geq 0$ and using equation (27), we obtain

$$\int_Q \exp(ct)v_\epsilon^* \overline{Nv}dxdt = -\int_Q A(t)\frac{\partial u}{\partial t}\exp(-ct)\frac{\partial^2 \overline{u}}{\partial t^2}dxdt + \epsilon \int_Q A(t)\frac{\partial u}{\partial t} \frac{\partial \overline{v_\epsilon^*}}{\partial t}dxdt. \tag{40}$$

Integrating each term in the left-hand side of equation (40) by parts and taking the real parts, we have

$$\mathrm{Re}\int_Q A(t)\frac{\partial u}{\partial t}\exp(-ct)\frac{\partial^2 \overline{u}}{\partial t^2}dxdt \geq \frac{c}{2}\int_Q a(x,t)\exp(-ct)(1-x)\left|\frac{\partial^{\alpha+1} u}{\partial x^\alpha \partial t}\right|^2 dxdt$$

$$-\frac{1}{2}\int_Q \frac{\partial a}{\partial t}\exp(-ct)(1-x)\left|\frac{\partial^{\alpha+1} u}{\partial x^\alpha \partial t}\right|^2 dxdt. \tag{41}$$

$$\mathrm{Re}\left(-\epsilon\int_Q A(t)\frac{\partial u}{\partial t} \frac{\partial \overline{v_\epsilon^*}}{\partial t}dxdt\right) \geq \frac{-\epsilon a_3^2}{2a_0}\int_Q \exp(-ct)(1-x)\left|\frac{\partial^{\alpha+1} u}{\partial x^\alpha \partial t}\right|^2 dxdt. \tag{42}$$

Now, using inequalities (41) and (42) in equation (40) with the choice of c indicated above, we have $2\text{Re}\int_Q \exp(ct)v_\epsilon^* \overline{N}v dx dt \leq 0$; then for $\epsilon \to 0$, we obtain $2\text{Re}\int_Q \exp(ct)v\overline{N}v dx dt \leq 0$, i.e., $2\text{Re}\int_Q \exp(ct)(1-x)|v|^2 dx dt + 2\text{Re}\int_Q \exp(ct)vJ\overline{v}dx dt \leq 0$.

Since $\text{Re}\int_Q \exp(ct)vJ\overline{v}dx dt = 0$, we conclude that $v = 0$; hence $\omega = 0$, which ends the proof of this lemma. □

Theorem 2: *The range $R(\overline{L})$ of $\overline{L}$ coincides with F.*

Proof: Since F is a Hilbert space, we have $R(\overline{L}) = F$ if and only if the relation

$$\int_Q (1-x)^\nu \mathcal{L}u\overline{f}dx dt + \int_0^1 (1-x)\left(\sum_{i=0}^\alpha \frac{\partial^i l_1 u}{\partial x^i}\frac{\partial^i \overline{\varphi}}{\partial x^i} + \frac{\partial^\alpha l_2 u}{\partial x^\alpha}\frac{\partial^\alpha \overline{\psi}}{\partial x^\alpha} + l_2 u\overline{\psi}\right)dx = 0. \quad (43)$$

for arbitrary $u \in E$ and $\mathcal{F} = (f, \varphi, \psi) \in F$, implies that $f = 0$, $\varphi = 0$ and $\psi = 0$.

Putting $u \in D_0(L)$ in relation (43), we obtain $\int_Q (1-x)^\nu \mathcal{L}u\overline{f}dx dt = 0$. Taking $\omega = \frac{f}{(1-x)^{\alpha+1}}$, and using Lemma 5, we obtain $\omega = \frac{f}{(1-x)^{\alpha+1}} = 0$, then $f = 0$.

Consequently, $\forall u \in D(L)$ we have

$$\int_0^1 (1-x)\left(\sum_{i=0}^\alpha \frac{\partial^i l_1 u}{\partial x^i}\frac{\partial^i \overline{\varphi}}{\partial x^i} + \frac{\partial^\alpha l_2 u}{\partial x^\alpha}\frac{\partial^\alpha \overline{\psi}}{\partial x^\alpha} + l_2 u\overline{\psi}\right)dx = 0. \quad (44)$$

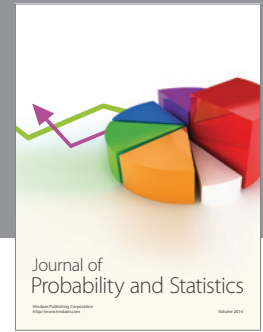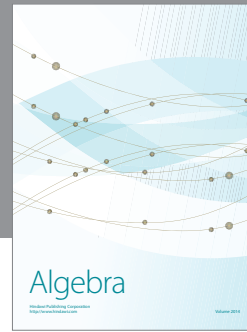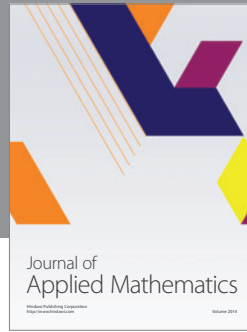The range of the trace operator (l_1, l_2) is everywhere dense in Hilbert space with the norm
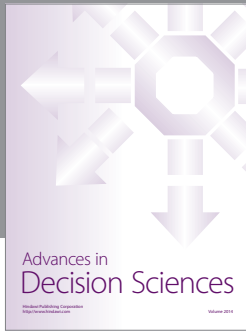
$$\left[\int_0^1 (1-x)\left(\sum_{i=0}^\alpha \left|\frac{\partial^i \varphi}{\partial x^i}\right|^2 + \left|\frac{\partial^\alpha \psi}{\partial x^\alpha}\right|^2 + |\psi|^2\right)dx\right]^{\frac{1}{2}},$$

hence $(\varphi, \psi) = (0, 0)$.   □

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