A-MONOTONICITY AND APPLICATIONS TO NONLINEAR VARIATIONAL INCLUSION PROBLEMS

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A new notion of the A -monotonicity is introduced, which generalizes the H -monotonicity. Since the A -monotonicity originates from hemivariational inequalities, and hemivariational inequalities are connected with nonconvex energy functions, it turns out to be a useful tool proving the existence of solutions of nonconvex constrained problems as well.

Recently, Fang and Huang [1] introduced a new class of mappings— h -monotone mappings—in the context of solving a system of variational inclusions involving a combination of h -monotone and strongly monotone mappings based on the resolvent operator technique. The notion of the h -monotonicity has revitalized the theory of maximal monotone mappings in several directions, especially in the domain of applications. Here we announce the notion of the A -monotone mappings and its applications to the solvability of systems of nonlinear variational inclusions. The class of the A -monotone mappings generalizes the h -monotonicity. On the top of that, A -monotonicity originates from hemivariational inequalities, and emerges as a major contributor to the solvability of nonlinear variational problems on nonconvex settings. As a matter of fact, some nice examples on A -monotone (or generalized maximal monotone) mappings can be found in Naniewicz and Panagiotopoulos [2] and Verma [4]. Hemivariational inequalities—initiated and developed by Panagiotopoulos [3]—are connected with nonconvex energy functions and turned out to be useful tools proving the existence of solutions of nonconvex constrained problems. We note that the A -monotonicity is defined in terms of relaxed monotone mappings—a more general notion than the monotonicity/strong monotonicity—which gives a significant edge over the h -monotonicity.

Definition 1 [1]. Let $h : H \rightarrow H$ and $M : H \rightarrow 2^H$ be any two mappings on H . The map M is said to be h -monotone if M is monotone and $(h + \rho M)(H) = H$ holds for $\rho > 0$. This is equivalent to stating that M is h -monotone if M is monotone and $(h + \rho M)$ is maximal monotone.

Let X denote a reflexive Banach space and X^* its dual. Inspired by [2, 4], we introduce the notion of the A -monotonicity as follows.

Definition 2. Let $A : X \rightarrow X^*$ and $M : X \rightarrow 2^{X^*}$ be any mappings on X . The map M is said to be A -monotone if M is m -relaxed monotone and $(A + \rho M)$ is maximal monotone for $\rho > 0$.

LEMMA 3. Let $A : H \rightarrow H$ be r -strongly monotone and $M : H \rightarrow 2^H$ be A -monotone. Then the resolvent operator $J_{A,M}^\rho : H \rightarrow H$ is $(1/(r - \rho m))$ -Lipschitz continuous for $0 < \rho < r/m$.

Example 4 [2, Lemma 7.11]. Let $A : X \rightarrow X^*$ be (m) -strongly monotone and $f : X \rightarrow R$ be locally Lipschitz such that ∂f is (α) -relaxed monotone. Then ∂f is A -monotone, that is, $A + \partial f$ is maximal monotone for $m - \alpha > 0$, where $m, \alpha > 0$.

Example 5 [4, Theorem 4.1]. Let $A : X \rightarrow X^*$ be (m) -strongly monotone and let $B : X \rightarrow X^*$ be (c) -strongly Lipschitz continuous. Let $f : X \rightarrow R$ be locally Lipschitz such that ∂f is (α) -relaxed monotone. Then ∂f is $(A - B)$ -monotone.

Let H_1 and H_2 be two real Hilbert spaces and K_1 and K_2 , respectively, be nonempty closed convex subsets of H_1 and H_2 . Let $A : H_1 \rightarrow H_1$, $B : H_2 \rightarrow H_2$, $M : H_1 \rightarrow 2^{H_1}$, and $N : H_2 \rightarrow 2^{H_2}$ be nonlinear mappings. Let $S : H_1 \times H_2 \rightarrow H_1$ and $T : H_1 \times H_2 \rightarrow H_2$ be any two multivalued mappings. Then the problem of finding $(a, b) \in H_1 \times H_2$ such that

$$0 \in S(a, b) + M(a), \quad 0 \in T(a, b) + N(b) \quad (1)$$

is called the system of nonlinear variational inclusion (SNVI) problems.

When $M(x) = \partial_{K_1}(x)$ and $N(y) = \partial_{K_2}(y)$ for all $x \in K_1$ and $y \in K_2$, where K_1 and K_2 , respectively, are nonempty closed convex subsets of H_1 and H_2 , and ∂_{K_1} and ∂_{K_2} denote indicator functions of K_1 and K_2 , respectively, the SNVI (1) reduces to determine an element $(a, b) \in K_1 \times K_2$ such that

$$\langle S(a, b), x - a \rangle \geq 0 \quad \forall x \in K_1, \quad (2)$$

$$\langle T(a, b), y - b \rangle \geq 0 \quad \forall y \in K_2. \quad (3)$$

LEMMA 6. Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \rightarrow H_1$ and $B : H_2 \rightarrow H_2$ be strictly monotone, let $M : H_1 \rightarrow 2^{H_1}$ be A -monotone, and let $N : H_2 \rightarrow 2^{H_2}$ be B -monotone. Let $S : H_1 \times H_2 \rightarrow H_1$ and $T : H_1 \times H_2 \rightarrow H_2$ be any two multivalued mappings. Then a given element $(a, b) \in H_1 \times H_2$ is a solution to the SNVI (1) problem if and only if (a, b) satisfies

$$a = J_{A,M}^\rho(A(a) - \rho S(a, b)), \quad b = J_{B,N}^\eta(B(b) - \eta T(a, b)). \quad (4)$$

THEOREM 7. Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \rightarrow H_1$ be (r_1) -strongly monotone and (α_1) -Lipschitz continuous, and let $B : H_2 \rightarrow H_2$ be (r_2) -strongly monotone and (α_2) -Lipschitz continuous. Let $M : H_1 \rightarrow 2^{H_1}$ be A -monotone and let $N : H_2 \rightarrow 2^{H_2}$ be B -monotone. Let $S : H_1 \times H_2 \rightarrow H_1$ be such that $S(\cdot, y)$ is (γ, r) -relaxed cocoercive and (μ) -Lipschitz continuous in the first variable and $S(x, \cdot)$ is (ν) -Lipschitz continuous in the second variable for all $(x, y) \in H_1 \times H_2$. Let $T : H_1 \times H_2 \rightarrow H_2$ be such that $T(u, \cdot)$ is (λ, s) -relaxed

cocoercive and (β) -Lipschitz continuous in the second variable and $T(\cdot, v)$ is (τ) -Lipschitz continuous in the first variable for all $(u, v) \in H_1 \times H_2$. If, in addition, there exist positive constants ρ and η such that

$$\begin{aligned}\sqrt{\alpha_1 - 2\rho r + 2\rho\gamma\mu^2 + \rho^2\mu^2} + \eta\tau &< r_1, \\ \sqrt{\alpha_2 - 2\eta s + 2\eta\lambda\beta^2 + \eta^2\beta^2} + \rho v &< r_2,\end{aligned}\tag{5}$$

then the SNVI (1) problem has a unique solution.

References

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