UPPER AND LOWER SOLUTIONS METHOD FOR DIFFERENTIAL INCLUSIONS WITH INTEGRAL BOUNDARY CONDITIONS

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A nonlinear alternative of the Leray-Schauder type for multivalued maps combined with upper and lower solutions is used to investigate the existence of solutions for second-order differential inclusions with integral boundary conditions.

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1. Introduction

This paper is concerned with the existence of solutions for the second-order boundary value problem:

$$y''(t) \in F(t, y(t)), \quad \text{a.e. } t \in J := [0, 1], \quad (1.1)$$

$$y(0) - k_1 y'(0) = \int_0^1 h_1(y(s)) ds, \quad (1.2)$$

$$y(1) + k_2 y'(1) = \int_0^1 h_2(y(s)) ds, \quad (1.3)$$

where $F : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a compact convex-valued multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $h_i : \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, 2$) are continuous functions, and k_i are nonnegative constants.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two-, three-, and multipoint and nonlocal boundary value problems as special cases. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers [9, 14, 19] and the references therein. Moreover, boundary value problems with integral boundary conditions have been studied by a number of authors, for example, [4, 6, 16]. The method of upper and lower solutions has been successfully applied to study the existence of multiple solutions for initial and boundary value problems for

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differential equations with nonlinear conditions. This method has been used only in the context of single-valued differential equations. In this regard, we refer the reader to the monographs by Heikkilä and Lakshmikantham [12] and Ladde et al. [17] and to the recent papers by Rahmat [15] and Rahmat and Bashir [1] in which the first- and second-order differential equations with integral boundary conditions have been considered. Recently this method has been used for initial and nonlinear boundary conditions of differential inclusions in the papers by Benchohra [2], Benchohra and Ntouyas [3], Frigon [8], Halidias and Papageorgiou [11], and Palmucci and Papalini [20]. In this paper, we will apply the method of upper and lower solutions combined with the nonlinear alternative of Leray-Schauder type [7] to problem (1.1)–(1.3). These results extend to the multivalued case some results from the literature and complement those related to the application of the method of upper and lower solutions to differential inclusions (see [2, 3, 8, 11, 20]).

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $AC^1(J, \mathbb{R})$ be the space of differentiable functions $y : J \rightarrow \mathbb{R}$, whose first derivative, y' , is absolutely continuous.

The property

$$y \leq \bar{y}, \quad \text{iff } y(t) \leq \bar{y}(t), \quad \forall t \in J \quad (2.1)$$

defines a partial ordering in $AC^1(J, \mathbb{R})$. If $\alpha, \beta \in AC^1(J, \mathbb{R})$ and $\alpha \leq \beta$, we let

$$[\alpha, \beta] = \{y \in AC^1(J, \mathbb{R}) : \alpha \leq y \leq \beta\}. \quad (2.2)$$

We take $C(J, \mathbb{R})$ to be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\| = \sup \{|y(t)| : 0 \leq t \leq 1\}, \quad (2.3)$$

and we let $L^1(J, \mathbb{R})$ denote the Banach space of functions $y : J \rightarrow \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_0^1 |y(t)| dt. \quad (2.4)$$

Let $(X, \|\cdot\|)$ be a Banach space. A multivalued map $G : X \rightarrow \mathcal{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B)$ is bounded in X for each bounded set B of X (i.e., $\sup_{x \in B} \{\sup\{\|y\| : y \in G(x)\}\} < \infty$). The map G is *upper semicontinuous* (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X , and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq N$. Finally, we say that G is *completely continuous* if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \rightarrow x_*$, $y_n \rightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). We say that G has a *fixed point* if there exists $x \in X$ such that $x \in G(x)$.

In what follows, $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathcal{P}_d(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$, and $\mathcal{P}_{c,cp}(X) = \mathcal{P}_c(X) \cap \mathcal{P}_{cp}(X)$. A multivalued map $G : J \rightarrow \mathcal{P}_d(\mathbb{R})$ is said to be *measurable* if for each $x \in \mathbb{R}$ the function $Y : J \rightarrow \mathbb{R}$ defined by

$$Y(t) = d(x, G(t)) = \inf \{|x - z| : z \in G(t)\} \quad (2.5)$$

is measurable. For more details on multivalued maps, see the books of Deimling [5], Górniewicz [10], and Hu and Papageorgiou [13].

Definition 2.1. The multivalued map $F : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is an L^1 -Carathéodory if

- (i) $t \mapsto F(t, y)$ is measurable for each $y \in \mathbb{R}$;
- (ii) $y \mapsto F(t, y)$ is u.s.c. for a.a. $t \in J$;
- (iii) for each $q > 0$, there exists $\phi_q \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, y)\|_{\mathcal{P}} = \sup \{|v| : v \in F(t, y)\} \leq \phi_q(t) \quad \forall |y| \leq q \text{ and for a.a. } t \in J. \quad (2.6)$$

Definition 2.2. A function $y \in AC^1([0, 1], \mathbb{R})$ is said to be a solution of (1.1)–(1.3) if there exists $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, y(t))$ for a.e. $t \in J$ such that y satisfies the differential equation $y''(t) = v(t)$ a.e. on J and the conditions (1.2) and (1.3).

For any $y \in C([0, 1], \mathbb{R})$, we define the set

$$S_{F(y)} = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}. \quad (2.7)$$

This is known as the set of *selection functions*.

Definition 2.3. A function $\alpha \in AC^1([0, 1], \mathbb{R})$ is said to be a lower solution of (1.1)–(1.3) if there exists $v_1 \in L^1(J, \mathbb{R})$ such that $v_1(t) \in F(t, \alpha(t))$ a.e. on J , $\alpha''(t) \geq v_1(t)$ a.e. on J , $\alpha(0) - k_1\alpha'(0) \leq \int_0^1 h_1(\alpha(s))ds$, and $\alpha(1) + k_2\alpha'(1) \leq \int_0^1 h_2(\alpha(s))ds$. Similarly, a function $\beta \in AC^1([0, 1], \mathbb{R})$ is said to be an upper solution of (1.1)–(1.3) if there exists $v_2 \in L^1(J, \mathbb{R})$ such that $v_2(t) \in F(t, \beta(t))$ a.e. on J , $\beta''(t) \leq v_2(t)$ a.e. on J , $\beta(0) - k_1\beta'(0) \geq \int_0^1 h_1(\beta(s))ds$, and $\beta(1) + k_2\beta'(1) \geq \int_0^1 h_2(\beta(s))ds$.

The following lemma is crucial in the proof of our main theorem.

LEMMA 2.4 [18]. *Let X be a Banach space. Let $F : [0, b] \times X \rightarrow \mathcal{P}_{cp,c}(X)$ be an L^1 -Carathéodory multivalued map and let Γ be a linear continuous mapping from $L^1([0, b], X)$ to $C([0, b], X)$, then the operator*

$$\Gamma \circ S_F : C([0, b], X) \longrightarrow \mathcal{P}_{cp,c}(C([0, b], X)), y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F(y)}) \quad (2.8)$$

is a closed graph operator in $C([0, b], X) \times C([0, b], X)$.

3. Main result

We are now in a position to state and prove our existence result for the problem (1.1)–(1.3). We first list the following hypotheses:

- (H1) $F : J \times \mathbb{R} \rightarrow \mathcal{P}_{cp,c}(\mathbb{R})$ is an L^1 -Carathéodory multivalued map;

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(H2) there exist α and $\beta \in AC^1([0, 1], \mathbb{R})$, lower and upper solutions, respectively, for the problem (1.1)–(1.3) such that $\alpha \leq \beta$;

(H3) h_i are continuous and nondecreasing functions, $i = 1, 2$.

THEOREM 3.1. *Assume that hypotheses (H1)–(H3) hold. Then the problem (1.1)–(1.3) has at least one solution y such that*

$$\alpha(t) \leq y(t) \leq \beta(t) \quad \forall t \in J. \quad (3.1)$$

Proof. We will transform the problem (1.1)–(1.3) into a fixed point problem. Consider the modified problem

$$y''(t) \in F_1(t, y(t)), \quad \text{a.e. } t \in J, \quad (3.2)$$

$$y(0) - k_1 y'(0) = \int_0^1 L_1(y(s)) ds, \quad (3.3)$$

$$y(1) + k_2 y'(1) = \int_0^1 L_2(y(s)) ds, \quad (3.4)$$

where

$$F_1(t, y) = \begin{cases} F(t, \beta(t)) + \frac{y - \beta(t)}{1 + |y - \beta(t)|}, & \text{if } y > \beta(t), \\ F(t, y), & \text{if } \alpha(t) \leq y \leq \beta(t), \\ F(t, \alpha(t)) + \frac{y - \alpha(t)}{1 + |y - \alpha(t)|}, & \text{if } y < \alpha(t), \end{cases} \quad (3.5)$$

$$L_i(y) = \begin{cases} h_i(\beta(t)), & \text{if } y > \beta(t), \\ h_i(y), & \text{if } \alpha(t) \leq y \leq \beta(t), \\ h_i(\alpha(t)), & \text{if } y < \alpha(t), \end{cases}$$

for every fixed $t \in J$. A solution to (3.2)–(3.4) is a fixed point of the operator $N : C(J, \mathbb{R}) \rightarrow \mathcal{P}(C(J, \mathbb{R}))$ defined by

$$N(y) = \left\{ h \in C(J, \mathbb{R}) : h(t) = p(t) + \int_0^1 G(t, s)g(s)ds, g \in \tilde{S}_{F_1(y)}^1 \right\}, \quad (3.6)$$

where

$$\begin{aligned} \tilde{S}_{F_1(y)}^1 &= \{v \in S_{F_1(y)}^1 : v(t) \leq v_1(t) \text{ a.e. on } A_1, v(t) \geq v_2(t) \text{ a.e. on } A_2\}, \\ S_{F_1(y)}^1 &= \{v \in L^1(J, \mathbb{R}) : v(t) \in F_1(t, y(t)) \text{ for a.e. } t \in J\}, \\ A_1 &= \{t \in J : y(t) < \alpha(t) \leq \beta(t)\}, \quad A_2 = \{t \in J : \alpha(t) \leq \beta(t) < y(t)\}, \\ p(t) &= \frac{1}{1 + k_1 + k_2} \left\{ (1 - t + k_2) \int_0^1 L_1(y(s)) ds + (k_1 + t) \int_0^1 L_2(y(s)) ds \right\}, \end{aligned} \quad (3.7)$$

and $G(\cdot, \cdot)$ is the Green function of the problem

$$\begin{aligned} x''(t) &= 0, \quad t \in J, \\ x(0) - k_1 x'(0) &= 0, \\ x(1) + k_2 x'(1) &= 0, \end{aligned} \tag{3.8}$$

where

$$G(t, s) = \frac{-1}{k_1 + k_2 + 1} \begin{cases} (k_1 + t)(1 - s + k_2), & 0 \leq t < s \leq 1, \\ (k_1 + s)(1 - t + k_2), & 0 \leq s < t \leq 1. \end{cases} \tag{3.9}$$

□

Remark 3.2. (i) For each $y \in C(J, \mathbb{R})$, the set $S_{F(y)}^1$ is nonempty (see Lasota and Opial [18]).

(ii) For each $y \in C(J, \mathbb{R})$, the set $\tilde{S}_{F_1(y)}^1$ is nonempty. In fact, (i) implies that there exists $v \in S_{F(y)}^1$, so we set

$$w = v_1 \chi_{A_1} + v_2 \chi_{A_2} + v \chi_{A_3}, \tag{3.10}$$

where

$$A_3 = \{t \in J : \alpha(t) \leq y(t) \leq \beta(t)\}. \tag{3.11}$$

Then, by the decomposability, $w \in \tilde{S}_{F_1(y)}^1$.

Remark 3.3. Notice that F_1 is an L^1 -Carathéodory multivalued map with compact convex values and there exists $\phi \in L^1(J, \mathbb{R})$ such that

$$||F_1(t, y)|| \leq \phi(t) + 1. \tag{3.12}$$

In order to apply the nonlinear alternative of Leray-Schauder type, we first show that N is completely continuous with convex values. The proof will be given in several steps.

Step 3.4. $N(y)$ is convex for each $y \in C(J, \mathbb{R})$.

Indeed, if h_1, h_2 belong to $N(y)$, then there exist $g_1, g_2 \in \tilde{S}_{F_1(y)}^1$ such that for each $t \in J$, we have

$$h_i(t) = p(t) + \int_0^1 G(t, s) g_i(s) ds, \quad i = 1, 2. \tag{3.13}$$

Let $0 \leq d \leq 1$. Then, for each $t \in J$, we have

$$(dh_1 + (1 - d)h_2)(t) = p(t) + \int_0^t G(t, s) [dg_1(s) + (1 - d)g_2(s)] ds. \tag{3.14}$$

Since $\tilde{S}_{F_1(y)}^1$ is convex (because F_1 has convex values), we see that

$$dh_1 + (1 - d)h_2 \in N(y). \tag{3.15}$$

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Step 3.5. N maps bounded sets into bounded sets in $C(J, \mathbb{R})$.

It suffices to show that for each $q > 0$, there exists a positive constant ℓ such that for each $y \in B_q = \{y \in C(J, \mathbb{R}) : \|y\|_\infty \leq q\}$, we have

$$\|N(y)\|_{\mathcal{P}} := \sup \{\|h\|_\infty : h \in N(y)\} \leq \ell. \quad (3.16)$$

Let $y \in B_q$ and $h \in N(y)$; then there exists $g \in \tilde{S}_{F_1(y)}^1$ such that for each $t \in J$, we have

$$h(t) = p(t) + \int_0^1 G(t, s)g(s)ds. \quad (3.17)$$

By (H1), for each $t \in J$, we have

$$\begin{aligned} |h(t)| &\leq |p(t)| + \int_0^1 |G(t, s)| |g(s)| ds \\ &\leq 2\bar{p} + \frac{(k_1 + 1)(1 + k_2)}{k_1 + k_2 + 1} \int_0^1 (\phi(s) + 1) ds, \end{aligned} \quad (3.18)$$

where

$$\bar{p} = \frac{\max(1 + k_2, k_1 + 1)}{1 + k_1 + k_2} \left(\max \left(h_i \left(\min_{t \in [0, 1]} |\alpha(t)| \right), h_i \left(\max_{t \in [0, 1]} |\beta(t)| \right), i = 1, 2 \right) \right). \quad (3.19)$$

Step 3.6. N maps bounded sets into equicontinuous sets in $C(J, \mathbb{R})$.

Let $u_1, u_2 \in J$, $u_1 < u_2$, and let B_q be a bounded set in $C(J, \mathbb{R})$ as in Step 3.5. Let $y \in B_q$ and $h \in N(y)$; then there exists $g \in \tilde{S}_{F_1(y)}^1$ such that for each $t \in J$, we have

$$\begin{aligned} |h(u_2) - h(u_1)| &\leq |p(u_2) - p(u_1)| + \int_0^1 |G(u_2, s) - G(u_1, s)| |g(s)| ds \\ &\leq |p(u_2) - p(u_1)| + \int_0^1 |G(u_2, s) - G(u_1, s)| \phi(s) ds \\ &\quad + \int_0^1 |G(u_2, s) - G(u_1, s)| ds, \end{aligned} \quad (3.20)$$

where

$$\begin{aligned} |p(u_2) - p(u_1)| &\leq \frac{|u_2 - u_1|}{1 + k_1 + k_2} \int_0^1 |L_1(y(s))| ds + \frac{|u_2 - u_1|}{1 + k_1 + k_2} \int_0^1 |L_2(y(s))| ds \\ &\leq 2 \frac{|u_2 - u_1|}{1 + k_1 + k_2} \left(\max \left(h_i \left(\min_{t \in [0, 1]} |\alpha(t)| \right), h_i \left(\max_{t \in [0, 1]} |\beta(t)| \right), i = 1, 2 \right) \right). \end{aligned} \quad (3.21)$$

As $u_2 \rightarrow u_1$, $|h(u_2) - h(u_1)|$ tends to zero.

As a consequence of Steps 3.4–3.6, together with the Arzela-Ascoli theorem, we can conclude that $N : C(J, \mathbb{R}) \rightarrow \mathcal{P}_{cp}(C(J, \mathbb{R}))$ is a completely continuous multivalued map.

Step 3.7. N has a closed graph.

Let $y_n \rightarrow y_*$, $h_n \in N(y_n)$, and $h_n \rightarrow h_*$. We will prove that $h_* \in N(y_*)$. Now $h_n \in N(y_n)$ implies that there exists $g_n \in \tilde{S}_{F_1(y_n)}^1$ such that for each $t \in J$,

$$h_n(t) = p_n(t) + \int_0^1 G(t,s)g_n(s)ds, \quad (3.22)$$

where

$$p_n(t) = \frac{1}{1+k_1+k_2} \left[(1-t+k_2) \int_0^1 L_1(y_n(s))ds + (k_1+t) \int_0^1 L_2(y_n(s))ds \right]. \quad (3.23)$$

We must prove that there exists $g_* \in \tilde{S}_{F_1(y_*)}^1$ such that for each $t \in J$,

$$h_*(t) = p_*(t) + \int_0^1 G(t,s)g_*(s)ds, \quad (3.24)$$

where

$$p_*(t) = \frac{1}{1+k_1+k_2} \left[(1-t+k_2) \int_0^1 L_1(y_*(s))ds + (k_1+t) \int_0^1 L_2(y_*(s))ds \right]. \quad (3.25)$$

Since p is continuous, we have

$$\|(h_n - p_n) - (h_* - p_*)\|_\infty \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.26)$$

Consider the continuous linear operator

$$\begin{aligned} \Gamma : L^1(J, \mathbb{R}) &\longrightarrow C(J, \mathbb{R}), \\ g &\longmapsto (\Gamma g)(t) = \int_0^1 G(t,s)g(s)ds. \end{aligned} \quad (3.27)$$

From Lemma 2.4, it follows that $\Gamma \circ \tilde{S}_{F_1(y)}^1$ is a closed graph operator. Moreover, we have that

$$h_n(t) - p_n(t) \in \Gamma(\tilde{S}_{F_1(y_n)}^1). \quad (3.28)$$

Since $y_n \rightarrow y_*$, it follows from Lemma 2.4 that

$$h_*(t) - p_*(t) = \int_0^1 G(t,s)g_*(s)ds \quad (3.29)$$

for some $g_* \in \tilde{S}_{F_1(y_*)}^1$.

Step 3.8. A priori bounds on solutions exist.

Let $y \in \lambda N(y)$ for some $\lambda \in (0, 1)$. Then there exists $g \in \tilde{S}_{F_1, y}^1$ such that

$$y(t) = \lambda \left[p(t) + \int_0^1 G(t,s)g(s)ds \right]. \quad (3.30)$$

By Remark 3.3, for each $t \in J$, we have

$$\begin{aligned} |y(t)| &\leq |p(t)| + \int_0^1 |G(t,s)| |g(s)| ds \\ &\leq 2\bar{p} + \frac{(k_1+1)(1+k_2)}{1+k_1+k_2} \int_0^1 \phi(t) ds + \frac{(k_1+1)(1+k_2)}{1+k_1+k_2}. \end{aligned} \quad (3.31)$$

Thus

$$\|y\|_\infty \leq 2\bar{p} + \frac{(k_1+1)(1+k_2)}{1+k_1+k_2} \int_0^1 \phi(t) ds + \frac{(k_1+1)(1+k_2)}{1+k_1+k_2} := \ell_*. \quad (3.32)$$

Set

$$U = \{y \in C([0,1], \mathbb{R}) : \|y\|_\infty < \ell_* + 1\} \quad (3.33)$$

and consider the operator N defined on $\bar{U}$. From the choice of U , there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [7], we deduce that N has a fixed point y in $\bar{U}$, that is, a solution of the problem (3.2)–(3.4).

Step 3.9. The solution y of (3.2)–(3.4) satisfies

$$\alpha(t) \leq y(t) \leq \beta(t) \quad \forall t \in J. \quad (3.34)$$

Let y be a solution to (3.2)–(3.4). We will show that

$$\alpha(t) \leq y(t) \quad \forall t \in J. \quad (3.35)$$

Suppose this is not the case. Let $r(t) = \alpha(t) - y(t)$, then the function r has positive maximum at some $t_0 \in [0,1]$, such that $r(t_0) = \max r(t)$, then

$$r(t_0) > 0, \quad r'(t_0) = 0, \quad r''(t_0) \leq 0. \quad (3.36)$$

If $t_0 \in (0,1)$, then there exists $c_1 \in [0,1]$ such that

$$r(t) > 0 \quad \forall t \in [c_1, t_0]. \quad (3.37)$$

In view of the definition of F_1 , we have

$$y''(t) \in F(t, \alpha(t)) + \frac{y(t) - \alpha(t)}{1 + |y(t) - \alpha(t)|} \quad \text{on } [c_1, t_0]. \quad (3.38)$$

Thus, there exists $v(t) \in F(t, \alpha(t))$ a.e. on $[c_1, t_0]$ with $v(t) \geq v_1(t)$ a.e. on $[c_1, t_0]$ such that

$$y''(t) = v(t) + \frac{y(t) - \alpha(t)}{1 + |y(t) - \alpha(t)|}, \quad \text{a.e. on } [c_1, t_0]. \quad (3.39)$$

Using the fact that α is a lower solution to (1.1)–(1.3), we have

$$\alpha''(t) - y''(t) = \alpha''(t) - v(t) - \frac{y(t) - \alpha(t)}{1 + |y(t) - \alpha(t)|} > 0 \quad \text{on } [c_1, t_0]. \quad (3.40)$$

Hence

$$0 \geq \alpha''(t_0) - y''(t_0) > 0, \quad (3.41)$$

which is a contradiction. If $t_0 = 0$,

$$0 < r(0) = \alpha(0) - k_1 y'(0) - \int_0^1 L_1(y(s)) ds. \quad (3.42)$$

By definition of the lower solution, we have

$$\begin{aligned} 0 &< k_1 \alpha'(0) - k_1 y'(0) + \int_0^1 [h_1(\alpha(s)) - L_1(y(s))] ds \\ &= k_1 r'(0) + \int_0^1 [h_1(\alpha(s)) - L_1(y(s))] ds \\ &\leq \int_0^1 [h_1(\alpha(s)) - L_1(y(s))] ds \\ &\leq \max_{t \in [0,1]} (h_1(\alpha(t)) - L_1(y(t))). \end{aligned} \quad (3.43)$$

Let $t_* \in [0, 1]$ such that

$$\max_{t \in [0,1]} (h_1(\alpha(t)) - L_1(y(t))) = h_1(\alpha(t_*)) - L_1(y(t_*)). \quad (3.44)$$

If $\alpha(t_*) > y(t_*)$, we have $L_1(y(t_*)) = h_1(\alpha(t_*))$, which implies $0 < r(0) \leq 0$; this is a contradiction.

If $y(t_*) > \beta(t_*)$, then $L_1(y(t_*)) = h_1(\beta(t_*)) \geq h_1(\alpha(t_*))$ and again $0 < r(0) \leq 0$; another contradiction. Similarly, if $t_0 = 1$, we get a contradiction. Thus $\alpha(t) \leq y(t)$, $t \in [0, 1]$. Similarly, we can show that $y(t) \leq \beta(t)$ $t \in [0, 1]$.

This shows that the problem (3.2)–(3.4) has a solution in the interval $[\alpha, \beta]$, which is a solution of the problem (1.1)–(1.3).

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