LARGE-SCALE STOCHASTIC HEREDITARY SYSTEMS UNDER MARKOVIAN STRUCTURAL PERTURBATIONS. PART I. VARIATIONAL COMPARISON THEOREMS

G. S. LADDE

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This three-part series work investigates the qualitative analysis of large-scale stochastic hereditary systems under random structural perturbations. The random structural perturbations are described by a Markov chain with a finite number of states. The first part of this three-part series work deals with the problem formulation and the development of mathematical tool to undertake this investigation. In this first part, we present variational comparison theorems for isolated subsystems of large-scale stochastic hereditary systems under Markovian structural perturbations. This part lays down the basis for the study of hierarchic systems.

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1. Introduction

A variety of problems that arise in the fields of engineering, medical, social, and physical sciences under hereditary and random environmental perturbations can be modeled by a system of stochastic functional differential equations [5–7, 9, 10, 19]. A feasible model for such a system is an Itô-type stochastic functional differential system perturbed by a finite-state Markov process. In this work, we propose to study the qualitative behavior of a solution of such a large-scale stochastic hereditary system.

The work is decomposed into the three parts, namely, the problem formulation and variational comparison theorems for isolated systems, the convergence and stability analysis of isolated stochastic hereditary subsystems, and large-scale stochastic hereditary systems. In this first part, we develop a very general form of well-known comparison theorems [1–4, 6, 8, 9, 11, 14–16, 18, 19].

In Section 2, the problem formulation and a description of the mathematical model are presented. For isolated subsystems, variational comparison theorems that incorporate the effects of the past history and random structural perturbations are developed in Section 3. Finally, in Section 4, examples are given to illustrate the significance of the

presented results. The presented results extend and generalize the earlier work [1–4, 6–9, 11–16, 18, 19] in a systematic and unified way.

2. Problem formulation

Input-output systems found in the biological, physical, and social sciences can in some settings be modeled by a stochastic system that is governed by random structural dynamics of the system. In these types of systems, the knowledge about the system is limited to output information. For purposes of investigation, it is assumed that measurements $Y(t, x_t, \eta)$ of an unknown functional $f(t, x_t, \eta)$ are of the following form:

$$Y(t, x_t, \eta) = f(t, x_t, \eta) + \sum_{r=1}^m \sigma_r(t, x_t, \eta) \frac{d\xi_r(t)}{dt}, \quad (\text{MOS})$$

where $\sum_{r=1}^m \sigma_r(t, x_t, \eta)(d\xi_r(t)/dt)$ reflects random errors in measurement, $d\xi_r(t)/dt$ is Gaussian white noise, η describes random structural dynamics of the system, and a past-history $x_t(\theta) = x(t + \theta)$, $-\tau \leq \theta \leq 0$ reflects the hereditary effects over the time period $\tau > 0$. Utilizing the continuous version of well-known stochastic approximation procedures [12, 17, 20], a mathematical model of stochastic hereditary dynamic systems under Markovian structural perturbations is described by the following system:

$$dx = \mathbf{F}(t, x_t, \eta(t)) \cdot d\xi(t), \quad x_{t_0} = \varphi_0, \quad (\text{CHS})$$

where $\mathbf{F} = (F_0^T, F_1^T, \dots, F_j^T, \dots, F_m^T)^T$, $F_j \in \mathcal{C}[R_+ \times C^n \times R, \mathcal{R}^n]$ for $j \in I(0, m)$, $t_0 \in [0, \infty)$; $C^n = C[-\tau, 0], R^n$ denotes the space of continuous functions with the domain $[-\tau, 0]$ and range in R^n ; x_t is an element of C^n defined by $x_t(\theta) = x(t + \theta)$, $-\tau \leq \theta \leq 0$; for $-\tau \leq \theta \leq 0$, $\varphi_0(\theta)$ is n -dimensional random vector defined on a complete probability space $(\Omega, \mathcal{F}, P)$ representing an initial past history of the state of the system, and it is sample continuous; $\eta = \eta(t, \omega)$ in (CHS) is a right continuous finite state $I(1, q) = \{1, 2, 3, \dots, q\}$ Markov process with a constant $q \times q$ state intensity matrix $\Pi = (\pi_{jl})$ representing random internal structural changes in the system; $\xi(t) = (\xi_1(t), \xi_2(t), \dots, \xi_m(t))^T$ is an m -dimensional normalized Wiener process with independent increments; $\eta(t)$ and $\xi(t)$ are $\mathcal{F}_t$ -measurable for all $t \geq t_0$ and $\mathcal{F}_t$ is an increasing family of sub- σ -algebras of $\mathcal{F}$; $\varphi_0(\theta)$, $\eta(t)$, and $\xi(t)$ are mutually independent for each $t \geq t_0$; $\xi_0(t) = t$, $\xi(t) = (\xi_0(t), \xi_1(t), \xi_2(t), \dots, \xi_m(t))$. It is assumed that $\mathbf{F}$ is smooth enough to insure the existence of a solution process $x(t) = x(t_0, \varphi_0)(t)$ of (CHS).

A system described by (CHS) may involve a large number of state variables. Using a “one shot” technique to study such a system may prove to be both analytically and computationally difficult. A more feasible approach is a decomposition-aggregation approach coupled with Lyapunov’s second method. In this case, the complex stochastic hereditary system (CHS) is decomposed into smaller, simpler, and suitable p -interconnected sub-systems (perturbed) of the following form:

$$dx^i = [\mathbf{a}^i(t, x_t^i, \eta(t)) + \mathbf{c}^i(t, x_t, \eta(t))] \cdot d\xi(t), \quad x_{t_0}^i = \varphi_0^i, \quad (\text{LHS})$$

where $\mathbf{a}^i = (a_0^{iT}, a_1^{iT}, \dots, a_j^{iT}, \dots, a_m^{iT})^T$ and $\mathbf{c}^i = (c_0^{iT}, c_1^{iT}, \dots, c_j^{iT}, \dots, c_m^{iT})^T$, $a_j^i \in C[J \times C^{n_i} \times R, R^{n_i}]$ and $c_j^i \in C[J \times C^n \times R, R^{n_i}]$ for $j \in I(0, m)$; $\varphi_0 = ((\varphi_0^1)^T, (\varphi_0^2)^T, \dots, (\varphi_0^p)^T)^T$, $x = ((x^1)^T, (x^2)^T, \dots, (x^p)^T)^T$, and $\sum_{i=1}^p n_i = n$ for each $i \in I(1, p)$; the interactions (perturbations) among the p -subsystems of system (LHS) are described by $\mathbf{c}^i$.

The qualitative analysis of each decoupled/isolated stochastic hereditary subsystem corresponding to (LHS) under Markovian structural perturbations

$$dx^i = \mathbf{a}^i(t, x_t^i, \eta(t)) \cdot d\xi(t), \quad x_{t_0}^i = \varphi_0^i, \quad (\text{DHS})$$

is aggregated in Part II of this work by an energy/Lyapunov-like function for $i \in I(1, p)$. By employing an energy/Lyapunov-like function associated with each subsystem and the nature of interactions $\mathbf{c}^i$ among the subsystems of (LHS) system, a very general comparison theorem is formed, and then used to study the qualitative properties of large-scale system in Part III.

Let $x(t) = (x^1(t), x^2(t), \dots, x^i(t), \dots, x^p(t))^T = x(t_0, \varphi_0)(t)$ and $x^i(t) = x^i(t_0, \varphi_0^i)(t)$ be the solution processes of large-scale system (LHS) and isolated systems (DHS) through (t_0, φ_0) and (t_0, φ_0^i) , respectively.

3. Variational comparison theorems for isolated stochastic hereditary subsystems

Comparison theorems [3–6, 9, 11–16, 18, 19] provide a very general framework for studying the qualitative properties of dynamic systems. In this section, an attempt is made to generalize and to extend an idea of variational comparison theorems [3, 8, 13] for isolated stochastic hereditary subsystems (DHS). This result connects solution processes of stochastic hereditary subsystems with the maximal solution of comparison systems and a solution of an auxiliary systems of differential equations. As a byproduct of this, several auxiliary comparison results are formulated.

For each $i \in I(1, p)$, let us consider a following auxiliary system:

$$\frac{d}{dt} z^i = f^i(t, z^i), \quad z^i(t_0) = z_0^i, \quad (\text{DAS})$$

where $f^i \in C[J \times R^{n_i}, R^{n_i}]$, and it is twice continuously differentiable with respect to z^i ; the solution $z^i(t, t_0, z_0^i)$ of (DAS) exists for $t \geq t_0$; it is unique and twice continuously differentiable with respect to the initial state z_0^i ; and further assume that its second derivative $(\partial^2/\partial x^i \partial x^i) z^i(t, s, z_0^i)$ is locally Lipschitzian in z_0^i for each (t, s) and $i \in I(1, p)$.

For each $q_i \geq 1$, $V^i \in C[[-\tau, \infty) \times R^{n_i} \times R, R_+^{q_i}]$; $(\partial/\partial t)V^i(t, z^i, y)$, $(\partial/\partial x)V^i(t, z^i, y)$, and $(\partial^2/\partial x^2)V(t, z^i, y)$ exist and are continuous on $[-\tau, \infty) \times R^{n_i} \times R$; and $(\partial^2/\partial x^2)V^i(t, z^i, y)$ is locally Lipschitzian in z^i for each (t, s) , $t_0 \leq s \leq t$. We define operators $D^+ V_{(\text{DHS})}^i$, $D_{\mathbf{a}^i}^+ V^i$, and $D_{\eta}^+ V^i$ as follows:

$$\begin{aligned} & D_{(\text{DHS})}^+ V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), \eta(s)) \\ &= \limsup_{h \rightarrow 0^+} \frac{1}{h} E \left[V^i(s+h, z^i(t, s+h, \varphi^i(0) + \mathbf{a}^i(s, \varphi^i, \eta(s)) \cdot \Delta \xi(s)), \eta(s+h)) \right. \\ & \quad \left. - V^i(s, z^i(t, s, \varphi^i(0)), \eta(s)) \mid F_s \right], \quad i \in I(1, p), \end{aligned} \quad (3.1)$$

4 Large-scale stochastic systems. Part I. Comparison theorems

where $\Delta\xi(s) = \xi(s+h) - \xi(s)$,

$$\begin{aligned} D_{\mathbf{a}^i}^+ V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), \eta(s)) \\ = \limsup_{h \rightarrow 0^+} \frac{1}{h} E[V^i(s+h, z^i(t, s+h, \varphi^i(0) + \mathbf{a}^i(s, \varphi^i, \eta(s)) \cdot \Delta\xi(s)), \eta(s)) \\ - V^i(s, z^i(t, s, \varphi^i(0)), \eta(s)) \mid F_s], \quad i \in I(1, p), \end{aligned} \quad (3.2)$$

$$\begin{aligned} D_{\eta}^+ V^i(s, z^i(t, s, \varphi^i(0)), j) \\ = \sum_{l=1}^q \pi_{jl} [V^i(s, z^i(t, s, \varphi^i(0)), l) - V^i(s, z^i(t, s, \varphi^i(0)), j)], \end{aligned}$$

where $\eta(s) = j$.

Remark 3.1. Under the assumptions on f^i and V^i , we note that for $t_0 \leq s \leq t$,

$$D_{\mathbf{a}^i}^+ V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), \eta(s)) \leq L_i^D V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), \eta(s)), \quad (3.3)$$

where an operator L_i^D is defined by

$$\begin{aligned} L_i^D V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), \eta(s)) \\ = \frac{\partial}{\partial s} V^i(s, z^i(t, s, \varphi^i(0)), \eta(s)) + \frac{1}{2} \text{tr} \left(\frac{\partial^2}{\partial z_i^2} V^i(s, z^i(t, s, \varphi^i(0)), \eta(s)) \Theta_i^D(t, s, \varphi^i, \eta(s)) \right) \\ + \frac{\partial}{\partial z^i} V^i(s, z^i(t, s, \varphi^i(0)), \eta(s)) \left[\frac{1}{2} \text{tr} \left(\frac{\partial^2}{\partial x^i \partial x^i} z^i(t, s, \varphi^i(0)) \Lambda_i^D(s, \varphi^i, \eta(s)) \right) \right. \\ \left. + \Phi^i(t, s, \varphi^i(0)) M_i^D(s, \varphi^i(0), \varphi^i, \eta(s)) \right], \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} M_i^D(s, \varphi^i(0), \varphi^i, \eta(s)) &= \lim_{h \rightarrow 0^+} \frac{1}{h} E[\mathbf{a}^i(s, \varphi^i, \eta(s)) \cdot \Delta\xi(s) - h f^i(s, \varphi^i(0)) \mid F_s], \\ \Lambda_i^D(s, \varphi^i, \eta(s)) &= \lim_{h \rightarrow 0^+} \frac{1}{h} E[(\mathbf{a}^i(s, \varphi^i, \eta(s)) \cdot \Delta\xi(s)) (\mathbf{a}^i(s, \varphi^i, \eta(s)) \cdot \Delta\xi(s))^T \mid F_s], \end{aligned} \quad (3.5)$$

$\Theta_i^D(t, s, \varphi^i, \eta(s)) = \Phi^i(t, s, \varphi^i(0)) \Lambda_i^D(s, \varphi^i, \eta(s)) \Phi^{iT}(t, s, \varphi^i(0))$, and $\Phi^i(t, s, x^i)$ is as defined in [11, 13, 15, 16].

Let us first state and provide a sketch of proof of a following auxiliary result that will be used to prove the main comparison theorem.

LEMMA 3.2. *Assume that*

- (i) $V^i \in C[[-\tau, \infty) \times R^{n_i} \times R, R_+^{q_i}]$; $(\partial/\partial t)V^i(s, z^i, y)$, $(\partial/\partial x)V^i(s, z^i, y)$, and $(\partial^2/\partial x^2)V(s, z^i, y)$ exist and are continuous on $[-\tau, \infty) \times R^{n_i} \times R$; and $(\partial^2/\partial x^2)V^i(s, z^i, y)$ is locally Lipschitzian in z^i for each (t, s) , $t_0 \leq s \leq t$;

(ii) $z^i(t, s, z_0^i)$ is the solution process of auxiliary system (DAS) through $(s, \varphi^i(0))$, $t_0 \leq s \leq t$, and its second derivative $(\partial^2/\partial x^i \partial x^i)z^i(t, s, z_0^i)$ is locally Lipschitzian in z_0^i for each (t, s) and $i \in I(1, p)$;

(iii) η is right-continuous finite-state Markov chain as described in (CHS).

Then,

$$\begin{aligned} D_{(\text{DHS})}^+ V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), \eta(s)) \\ = L_i^D V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), \eta(s)) + D_\eta^+ V(s, z^i(t, s, \varphi^i(0)), \eta(s)), \end{aligned} \quad (3.6)$$

where $(t, \varphi^i) \in R_+ \times \mathcal{C}^{n_i}$; $D_{(\text{DHS})}^+ V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), \eta(s))$, $D_\eta^+ V(s, z^i(t, s, \varphi^i(0)), \eta(s))$, and $L_i^D V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), \eta(s))$ are as defined in (3.1), (3.2), and (3.3), respectively.

Proof. For small $h > 0$, we compute

$$\begin{aligned} D_{(\text{DHS})}^+ V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), \eta(s)) \\ = \limsup_{h \rightarrow 0^+} \frac{1}{h} E[V^i(s+h, z^i(t, s+h, \varphi^i(0) + \mathbf{a}^i(s, \varphi^i, \eta(s)) \cdot \Delta \xi(s)), \eta(s+h)) \\ - V^i(s, z^i(t, s, \varphi^i(0)), \eta(s)) \mid F_s], \quad i \in I(1, p), \\ = \limsup_{h \rightarrow 0^+} \frac{1}{h} E \left[P(A_j) [V^i(s+h, z^i(t, s+h, \varphi^i(0) + \mathbf{a}^i(s, \varphi^i, \eta(s)) \cdot \Delta \xi(s)), \eta(s)) \right. \\ \left. - V^i(s, z^i(t, s, \varphi^i(0)), \eta(s)) \mid F_s], i \in I(1, p), \right. \\ \left. + \sum_{l \neq j}^q P(A_{jl}) [V^i(s+h, z^i(t, s+h, \varphi^i(0) + \mathbf{a}^i(s, \varphi^i, \eta(s)) \cdot \Delta \xi(s)), l) \right. \\ \left. - V^i(s, z^i(t, s, \varphi^i(0)), \eta(s)) \mid F_s] \right], \quad i \in I(1, p), \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} P(A_j) &= P(\eta(s+h) = j \mid \eta(s) = j) = 1 - \pi_{jj}h + o(h), \\ P(A_{jl}) &= P(\eta(s+h) = l \mid \eta(s) = j) = \pi_{jl}h + o(h), \\ \pi_{jj} &= \sum_{l \neq j}^s \pi_{jl}. \end{aligned} \quad (3.8)$$

From these expressions, (3.7) can be written as

$$\begin{aligned} \limsup_{h \rightarrow 0^+} \frac{1}{h} E[V^i(s+h, z^i(t, s+h, \varphi^i(0) + \mathbf{a}^i(s, \varphi^i, \eta(s)) \cdot \Delta \xi(s)), \eta(s)) \\ - V^i(s, z^i(t, s, \varphi^i(0)), \eta(s)) \mid F_s] \end{aligned}$$

$$\begin{aligned}
& + \liminf_{h \rightarrow 0^+} \frac{1}{h} \sum_{l \neq j}^q h \pi_{\eta(s)l} [V^i(s+h, z^i(t, s+h, \varphi^i(0) + \mathbf{a}^i(s, \varphi^i, \eta(s)) \cdot \Delta \xi(s)), l) \\
& \quad - V^i(s, z^i(t, s, \varphi^i(0)), \eta(s)) \mid F_s] \\
& \leq D_{(\text{DHS})}^+ V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), \eta(s)) \\
& \leq \limsup_{h \rightarrow 0^+} \frac{1}{h} E[V^i(s+h, z^i(t, s+h, \varphi^i(0) + \mathbf{a}^i(s, \varphi^i, \eta(s)) \cdot \Delta \xi(s)), \eta(s)) \\
& \quad - V^i(s, z^i(t, s, \varphi^i(0)), \eta(s)) \mid F_s] \\
& + \limsup_{h \rightarrow 0^+} \frac{1}{h} \sum_{l \neq j}^q h \pi_{\eta(s)l} [V^i(s+h, z^i(t, s+h, \varphi^i(0) + \mathbf{a}^i(s, \varphi^i, \eta(s)) \cdot \Delta \xi(s)), l) \\
& \quad - V^i(s, z^i(t, s, \varphi^i(0)), \eta(s)) \mid F_s]. \tag{3.9}
\end{aligned}$$

Because of the continuity of V^i , the above expression establishes relation (3.6) in the context of the definitions of $D_{(\text{DHS})}^+ V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), \eta(s))$, $D_{\eta}^+ V^i(s, z^i(t, s, \varphi^i(0)), j)$, and $L_i^D V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), \eta(s))$. This completes the proof of the lemma. $\square$

Now we are ready to present a very general comparison theorem in the context of vector Lyapunov-like functions and the system of functional differential inequalities [4, 6, 10, 18]. The presented result relates the solutions of (DHS), (DAS), and a system of comparison differential equations (CDE). We state a result with a brief sketch of its proof. The detailed proof can be formulated by utilizing the arguments in the proofs of comparison theorems for systems of deterministic functional differential equations [4, 15, 16, 18], systems of stochastic differential equations without past history [11, 13, 19], and systems of stochastic functional differential equations [1, 2, 6–9].

THEOREM 3.3. *Assume that*

- (a) $g^{ij} \in C[R_+ \times R_+^{q_i} \times C_+^{q_i}]$, $g^{ij}(t, u^i, \sigma^i)$ is concave in (u^i, σ^i) for each t , quasimonotone nondecreasing in u^i for each $(t, \sigma^i) \in R_+ \times \mathcal{C}_+^{q_i}$, and increasing in σ^i for each $(t, u^i) \in R_+ \times R_+^{q_i}$ for each $i \in I(1, p)$ and $j \in I(1, q)$;
- (b) $w^i \in C[R_+ \times (R_+^{q_i})^q \times (\mathcal{C}_+^{q_i})^q, (R^{q_i})^q]$; it is defined by

$$w^i(t, u^i, \sigma^i) = (w^{i1}(t, u^i, \sigma^i)^T, w^{i2}(t, u^i, \sigma^i)^T, \dots, w^{iq}(t, u^i, \sigma^i)^T)^T, \tag{3.10}$$

where $w^{ij} \in C[R_+ \times (R_+^{q_i})^q \times (\mathcal{C}_+^{q_i})^q, R^{q_i}]$ and

$$\begin{aligned}
w^{ij}(t, u^i, \sigma^i) = & \left(g_1^{ij}(t, u^{ij}, \sigma^{ij}) + \sum_{l=1}^q \lambda_{jl} u_1^{il}, g_2^{ij}(t, u^{ij}, \sigma^{ij}) \right. \\
& \left. + \sum_{l=1}^q \lambda_{jl} u_2^{il}, \dots, g_{q_i}^{ij}(t, u^{ij}, \sigma^{ij}) + \sum_{l=1}^q \lambda_{jl} u_{q_i}^{il} \right)^T, \tag{3.11}
\end{aligned}$$

where $u^{ij}, u^{il} \in R_+^{q_i}$ for $j, l \in I(1, q)$, and the $q \times q$ matrix Λ is defined by $\Lambda = (\lambda_{jl})_{q \times q}$ such that $\lambda_{jl} = \pi_{jl}$ for $j \neq l$, and $\lambda_{jj} = -\pi_{jj}$ for $j = l$ for all $j, l \in I(1, q)$;

(c) $r^i(t_0, \sigma_0^i)(t) = r^i(t)$ is the maximal solution of the block comparison system (BCS),

$$\frac{d}{dt}u^i = w^i(t, u^i, u_t^i), \quad u_{t_0}^i = \sigma_0^i, \quad (\text{BCS})$$

$(t_0, \sigma_0^i) \in R_+ \times (\mathcal{C}_+^{q_i})^q$ existing for $t \geq t_0$ for each $i \in I(1, p)$;

(d) hypotheses of Lemma 3.2 are satisfied, and $L_i^D V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), j)$ satisfies the following relation:

$$L_i^D V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), j) \leq g^{ij}(s, V^i(s, z^i(t, s, \varphi^i(0)), j), V_s^i), \quad (3.12)$$

where $V_s^i = V^i(s + \theta, z^i(t, s + \theta, \varphi^i(s + \theta)), j)$ for $\theta \in [-\tau, 0]$;

(e) for any solution process $x^i(t_0, \varphi_0, \eta(t))(t) = x^i(t)$ of (DHS), $E[V^i(t, x^i(t), \eta(t))]$ exists for $t \geq t_0$ and

$$E[V^i(t_0 + \theta, z^i(t, s + \theta, \varphi^i(\theta)), \eta(t_0)) \mid \eta(t_0) = j_0] \leq \sigma_0^{ij_0}(\theta), \quad \forall \theta \in [-\tau, 0]. \quad (3.13)$$

Then,

$$E[V^i(t, x^i(t), j) \mid x(t_0) = \varphi(0)] \leq r^{ij}(t_0, \sigma_0^i)(t), \quad t \geq t_0, \quad (3.14)$$

for all $t \geq t_0$, $i \in I(1, p)$, and $j \in I(1, q)$.

Proof. Let $x^i(s) = x^i(t_0, \varphi^i(0))(s)$ be a solution process of (DHS) through $(t_0, \varphi^i(0))$. We set

$$m^i(s) = V^i(s, z^i(t, s, x^i(s)), \eta(s)), \quad \text{for } t_0 \leq s \leq t. \quad (3.15)$$

For small $h > 0$, we define

$$m^{ij}(s+h) = E[V^i(s+h, z^i(t, s+h, x^i(s+h)), \eta(s+h)) \mid x(s) = x, \eta(s) = j]. \quad (3.16)$$

We note that

$$m^{ij}(s) = E[V^i(s, z^i(t, s, x^i(s)), \eta(s)) \mid x(s) = x, \eta(s) = j]. \quad (3.17)$$

By using this notation, for each $i \in I(1, p)$, we consider

$$\begin{aligned} & m^{ij}(s+h) - m^{ij}(s) \\ &= E[V^i(s+h, z^i(t, s+h, \varphi^i(0) + \mathbf{a}^i(s, \varphi^i, \eta(s)) \cdot \Delta \xi(s)), \eta(s+h)) \\ &\quad - V^i(s, z^i(t, s, \varphi^i(0)), \eta(s)) \mid F_s] \\ &\quad + E[V^i(s+h, z^i(t, s+h, x^i(s+h)), \eta(s+h)) \\ &\quad - V^i(s+h, z^i(t, s+h, \varphi^i(0) + \mathbf{a}^i(s, \varphi^i, \eta(s))) \cdot \Delta \xi(s)), \eta(s+h)) \mid F_s], \end{aligned} \quad (3.18)$$

where $\varphi^i = x_s$, $\varphi^i(0) = x^i(s)$, $\eta(s) = j$, $\Delta \xi(s) = \xi(s+h) - \xi(s)$, $i \in I(1, p)$, and $j \in I(1, q)$.

8 Large-scale stochastic systems. Part I. Comparison theorems

From this, hypotheses on V^i and $z^i(t, s, \varphi^i(s))$, Lebesgue convergence theorem, and Lemma 3.2, we arrive at

$$D^+ m^{ij}(s) \leq L_i^D V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), j) + D_{\eta}^+ V^i(s, z^i(t, s, \varphi^i(0)), j). \quad (3.19)$$

From (3.2), (3.12), (3.15), (3.17), (3.19), and assumptions (a) and (b), we have

$$D^+ m^i(s) \leq w^i(s, m^i(s), m_s^i), \quad \text{for } t_0 \leq s \leq t, \quad (3.20)$$

and from the assumption (e), we have

$$m^i(\theta) \leq \sigma_0^i(\theta), \quad \theta \in [-\tau, 0]. \quad (3.21)$$

Now, the rest of the proof can be completed by imitating the standard argument [4, 6, 9, 15, 16]. $\square$

To illustrate the scope and the significance of Theorem 3.3, we first present a corollary. Furthermore, this provides a tool to study the behavior of block comparison system (BCS).

COROLLARY 3.4. *Assume that the hypotheses of Theorem 3.3 are satisfied. In addition, assume that g^{ij} satisfies*

$$\begin{aligned} & \sum_{k=1}^{q_i} d_k g_k^{ij}(s, V^i(s, z^i(t, s, \varphi^i(0)), j), V_s^i) \\ & \leq \alpha^{ij}(s) \nu^i(s, z^i(t, s, \varphi^i(0)), j) + \beta^{ij}(s) \int_{-\tau}^0 \nu^i(s + \theta, z^i(t, s + \theta, \varphi^i(\theta)), j) d_{\theta} \kappa^i(s, \theta), \end{aligned} \quad (3.22)$$

where $\eta(s) = j$, $d_i > 0$, $\alpha^{ij}, \beta^{ij} \in C[J, R]$, $\beta^{ij} \geq 0$, $\kappa^i(s, \theta)$ is a function of bounded variation and $d_{\theta} \kappa^i(s, \theta)$ is a nonnegative Stieltjes measure on $[-\tau, 0]$ for $t_0 \leq s \leq t$, and

$$\nu^i(s, z^i(t, s, \varphi^i(0)), \eta(s)) = \sum_{k=1}^{q_i} d_k V_k^i(s, z^i(t, s, \varphi^i(0)), \eta(s)). \quad (3.23)$$

Then,

$$D_{(\text{DHS})}^+ \nu^i(s, \varphi^i, z^i(t, s, \varphi^i(0))) \leq -\bar{A}^i(s) \nu^i(s, z^i(t, s, \varphi^i(0))) + L^i(\nu_s^i), \quad (3.24)$$

where $\nu^i(s, z^i(t, s, \varphi^i(0))) = (\nu^i(t, x, 1), \nu(t, x, 2), \dots, \nu(t, x, q))^T$,

$$\bar{A}^i(s) = (a_{kl}^i(s))_{q \times q}; \quad \bar{B}^i(s) = \text{diag}(\beta^{i1}(s), \beta^{i2}(s), \dots, \beta^{iq}(s)), \quad (3.25)$$

$$L^i(\nu_s^i) = \bar{B}^i(s) \int_{-\tau}^0 \nu^i(s + \theta, z^i(t, s + \theta, \varphi^i(\theta)), j) d_{\theta} \kappa^i(s, \theta),$$

$$a_{jl}^i(s) = \begin{cases} \pi_{jj} - \alpha^{ij}(s), & l = j, \\ -\pi_{jl}, & l \neq j. \end{cases} \quad (3.26)$$

Moreover,

$$E[\nu^i(t, x^i(t)) \mid \varphi_0^i(t_0) = x_0] \leq r^i(t_0, \sigma_0^i)(t), \quad (3.27)$$

whenever $\nu_{t_0}^i \leq \sigma_0^i$, where $r^i(t_0, \sigma_0^i)(t)$ is the maximal solution of the following linear system of functional differential equations:

$$\frac{d}{dt}u^i = -\bar{A}^i(t)u^i + L^i(u_t^i), \quad u_{t_0}^i = \sigma_0^i. \quad (\text{LCS})$$

Proof. From (3.6) and the proof of Theorem 3.3, for each $j \in I(1, q)$, we have

$$\begin{aligned} D_{(\text{DHS})}^+ \nu^i(s, \varphi(0), \varphi, j) &\leq \sum_{k=1}^{q_i} d_k g_k^{ij}(s, V^i(s, z^i(t, s, \varphi^i(0)), j), V_s^i) \\ &\quad + \sum_{l \neq j}^q \pi_{kl} \sum_{k=1}^{q_l} d_k V_k^i(s, z^i(t, s, \varphi^i(0)), l) \\ &\quad - \pi_{jj} \sum_{k=1}^{q_i} d_k V_k^i(s, z^i(t, s, \varphi^i(0)), j). \end{aligned} \quad (3.28)$$

Using the definition of ν^i and (3.22), (3.28) reduces to

$$\begin{aligned} D_{(\text{DHS})}^+ \nu^i(s, \varphi(0), \varphi^i, j) &\leq \sum_{k=1}^{q_i} d_k (\alpha^{ij}(s) - \pi_{jj}) V_k^i(s, z^i(t, s, \varphi^i(0)), j) \\ &\quad + \beta^{ij}(s) \int_{-\tau}^0 \nu^i(s + \theta, z^i(t, s + \theta, \varphi^i(\theta)), j) d_\theta \kappa^i(s, \theta) \\ &\quad + \sum_{k \neq j}^q \pi_{jk} \nu^i(s, z^i(t, s, \varphi^i(0)), k). \end{aligned} \quad (3.29)$$

Using the definitions of $\bar{A}^i(s)$, $L^i(\nu_s^i)$, ν^i , and the nature of the elements of $\bar{A}^i$, (3.29) reduces to the desired inequality (3.24):

$$D_{(\text{DHS})}^+ \nu^i(s, \varphi^i, z^i(t, s, \varphi^i(0))) \leq -\bar{A}^i(s) \nu^i(s, \varphi^i(0)) + L^i(\nu_s^i). \quad (3.30)$$

The proof of inequality (3.27) follows from the uniqueness of solution of (LCS) and the application of Theorem 3.3. This completes the proof of the corollary. $\square$

Remark 3.5. If $f^i \equiv 0$, then Theorem 3.3 reduces to the usual comparison theorems [2, 6, 8, 9, 11] as special cases.

4. Examples

In this section, a few examples are given to illustrate the scope of the results.

Example 4.1. Assume that the hypotheses of Theorem 3.3 are satisfied in the context of the following considerations: suppose that for each $i \in I(1, p)$, $f^i(t, z_i) = A^i(t)z_i$ in (DAS),

where A^i is a continuous $n_i \times n_i$ matrix function. $\Phi^i(t, s)$ is a fundamental matrix solution of $z_i^i = A^i(t)z^i$, $\Phi^i(s, s) = I_{n_i \times n_i}$ (identity matrix). Further assume that $g^i(t, u^i, \sigma^i) \equiv 0$. In this case $r^i(t_0, \sigma_0^i)(t) = \Psi(t, t_0)\sigma_0^i(0)$, where $\Psi(t, t_0)$ is fundamental matrix solution of ordinary linear systems of differential equations corresponding to the comparison system, for each $z^i(t, s, x^i) = \Phi^i(t, s)x^i$ for each $i \in I(1, p)$. Moreover, one can choose $\sigma_0^i(\theta) = V^i(t_0 + \theta, \Phi^i(t, t_0 + \theta)\varphi_0^i(\theta), \eta(t_0))$.

Example 4.2. For $i \in I(1, p)$, $V^i(s, z^i) = z^{iT}P^i(\eta(s))z^i$ are admissible in Theorem 3.3 with $q_i = 1$. Here P_i is a nonnegative definite $n_i \times n_i$ matrix for each $i \in I(1, p)$. Further assume the conditions as described in Example 4.1. In this case, V^i and $z^i(t, s, x^i) = \Phi^i(t, s)x^i$ are continuously differentiable with respect to (s, x_i) for each t , $t_0 \leq s \leq t$. In this case, $L_i^D V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), \eta(s))$, defined in (3.3), reduces to

$$\begin{aligned} & L_i^D V^i(s, \varphi^i, z^i(t, s, \varphi^i(0)), \eta(s)) \\ &= (\Phi^i(t, s)\varphi^i(0))^T P^i(\eta(s))\Phi^i(t, s)M_i^D(s, \varphi^i(0), \varphi^i, \eta(s)) \\ &+ (\Phi^i(t, s)M_i^D(s, \varphi^i(0), \varphi^i, \eta(s)))^T P^i(\eta(s))\Phi^i(t, s)\varphi^i(0) \\ &+ \text{tr}(P^i(\eta(s))\Phi^i(t, s)\Lambda_i^D(s, \varphi^i, \eta(s))\Phi^i(t, s)). \end{aligned} \quad (4.1)$$

It is assumed that

$$\begin{aligned} & (\Phi^i(t, s)\varphi^i(0))^T P^i(j)\Phi^i(t, s)M_i^D(s, \varphi^i(0), \varphi^i, j) \\ &+ (\Phi^i(t, s)M_i^D(s, \varphi^i(0), \varphi^i, j))^T P^i(j)\Phi^i(t, s)\varphi^i(0) \\ &+ \text{tr}(P^i(j)\Phi^i(t, s)\Lambda_i^D(s, \varphi^i, j)\Phi^i(t, s)) \\ &\leq \mu^{ij}(s)(\Phi^i(t, s)\varphi^i(0))^T P^i(j)\Phi^i(t, s)\varphi^i(0) \\ &+ \beta^{ij}(s) \int_{-\tau}^0 (\Phi^i(t, s+\theta)\varphi^i(\theta))^T P^i(j)\Phi^i(t, s+\theta)\varphi^i(\theta)d\theta. \end{aligned} \quad (4.2)$$

A comparison system of differential equation is constructed similar to (LCS).

Remark 4.3. Examples 4.1 and 4.2 exhibit the connection between the solution processes of (DHS), (DAS), and (BCS)/(LCS).

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G. S. Ladde: Department of Mathematics, The University of Texas at Arlington, Arlington, TX 76019, USA
E-mail address: ladde@uta.edu