SOME LIMIT THEOREMS CONNECTED WITH BROWNIAN LOCAL TIME

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Received 26 October 2004; Revised 11 April 2005; Accepted 12 April 2005

Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion and let $(L_t^x; t \geq 0, x \in \mathbb{R})$ be a continuous version of its local time process. We show that the following limit $\lim_{\varepsilon \downarrow 0} (1/2\varepsilon) \int_0^t \{F(s, B_s - \varepsilon) - F(s, B_s + \varepsilon)\} ds$ is well defined for a large class of functions $F(t, x)$, and moreover we connect it with the integration with respect to local time L_t^x . We give an illustrative example of the nonlinearity of the integration with respect to local time in the random case.

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1. Introduction

1.1. The local time of the Brownian motion B at the point a is defined as follows:

$$L_t^a = \mathbb{P} \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(|B_s - a| \leq \varepsilon)} ds, \quad (1.1)$$

which equivalently could be written as follows:

$$L_t^a = \mathbb{P} \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t (1_{(B_s - \varepsilon \leq a)} - 1_{(B_s + \varepsilon \leq a)}) ds. \quad (1.2)$$

Here we are, more generally, interested in the limit in L^1 :

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \{F(s, B_s - \varepsilon) - F(s, B_s + \varepsilon)\} ds \quad (1.3)$$

for some function F .

Our motivation comes from the desire to connect Chitashvili and Mania results [1] with those of Eisenbaum [2].

2 Some limit theorems connected with Brownian local time

1.2. We give an example which illustrates that the integration with respect to $(L_t^x; 0 \leq t \leq 1, x \in \mathbb{R})$ does not admit a linear extension in the random case (see Section 3.2 for details) and in particular local time is not a 1-integrator, which is also proved by Eisenbaum [2].

2. Notation and preliminaries

Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion and let $(L_t^x; t \geq 0, x \in \mathbb{R})$ be a continuous version of its local time process. Let $(\mathcal{F}_t)_{t \geq 0}$ denote the natural filtration generated by B . Without loss of generality, we restrict our attention to functions defined on $[0, 1] \times \mathbb{R}$.

For a measurable function f from $[0, 1] \times \mathbb{R}$ into $\mathbb{R}$, define the norm $\| \cdot \|$ by

$$\|f\| = 2 \left(\int_0^1 \int_{\mathbb{R}} f^2(s, x) e^{-x^2/2s} \frac{ds dx}{\sqrt{2\pi s}} \right)^{1/2} + \int_0^1 \int_{\mathbb{R}} |xf(s, x)| e^{-x^2/2s} \frac{ds dx}{s\sqrt{2\pi s}}. \quad (2.1)$$

Let $\mathcal{H}$ be the set of functions f such that $\|f\| < \infty$.

In Eisenbaum [2], it is shown that the integration with respect to L is possible in the following sense. Let f_{Δ} be an elementary function on $[0, 1] \times \mathbb{R}$, meaning that

$$f_{\Delta}(t, x) = \sum_{(s_i, x_j) \in \Delta} f_{i,j} 1_{(s_i, s_{i+1}]}(t) 1_{(x_j, x_{j+1}]}(x), \quad (2.2)$$

where $\Delta = \{(s_i, x_j), 1 \leq i \leq n, 1 \leq j \leq m\}$ is an $[0, 1] \times \mathbb{R}$ grid, and, for every (i, j) , $f_{i,j}$ is in $\mathbb{R}$. For such a function, integration with respect to L is defined by

$$\int_0^1 \int_{\mathbb{R}} f_{\Delta}(s, x) dL_s^x = \sum_{(s_i, x_j) \in \Delta} f_{i,j} (L_{s_{i+1}}^{x_{j+1}} - L_{s_i}^{x_{j+1}} - L_{s_{i+1}}^{x_j} + L_{s_i}^{x_j}). \quad (2.3)$$

Let f be an element of $\mathcal{H}$. For any sequence of elementary functions $(f_{\Delta_k})_{k \in \mathbb{N}}$ converging to f in $\mathcal{H}$, the sequence $(\int_0^1 \int_{\mathbb{R}} f_{\Delta_k}(s, x) dL_s^x)_{k \in \mathbb{N}}$ converges in L^1 . The limit obtained does not depend on the choice of the sequence (f_{Δ_k}) and represents the integral $\int_0^1 \int_{\mathbb{R}} f(s, x) dL_s^x$.

THEOREM 2.1 (see [2]). *Let $(A(x, t); x \in \mathbb{R}, 0 \leq t \leq 1)$ be a continuous random process taking values in $\mathbb{R}$, such that for any t in $[0, 1]$ and any ω , $A(\cdot, t)$ is absolutely continuous with respect to dx . Note $\partial A / \partial x$ its derivative and ask $\partial A / \partial x$ to be continuous. Then $\int_0^1 \int_{\mathbb{R}} A(x, s) dL_s^x$ exists and the following hold:*

(i) *for any couple (a, b) in $\mathbb{R}^2$ with $a < b$*

$$\int_0^t \int_b^a A(x, s) dL_s^x = - \int_0^t \frac{\partial A}{\partial x}(B_s, s) ds + \int_0^t A(b, s) d_s L_s^b - \int_0^t A(a, s) d_s L_s^a; \quad (2.4)$$

(ii)

$$\int_0^1 \int_{\mathbb{R}} A(x, s) dL_s^x = - \int_0^1 \frac{\partial A}{\partial x}(B_s, s) ds; \quad (2.5)$$

(iii)

$$\left(\int_0^t \int_b^a A(x, s) dL_s^x \right)(\omega) = \int_0^t \int_b^a A(x, s)(\omega) dL_s^x(\omega). \quad (2.6)$$

3. Main results

3.1. Deterministic case

THEOREM 3.1. *Let F be a bounded element of $\mathcal{H}$. The following equalities hold in L^1 :*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \{F(s, B_s) - F(s, B_s - \varepsilon)\} ds = - \int_0^t \int_{\mathbb{R}} F(s, x) dL_s^x; \quad (3.1)$$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \{F(s, B_s + \varepsilon) - F(s, B_s)\} ds = - \int_0^t \int_{\mathbb{R}} F(s, x) dL_s^x; \quad (3.2)$$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \{F(s, B_s - \varepsilon) - F(s, B_s + \varepsilon)\} ds = \int_0^t \int_{\mathbb{R}} F(s, x) dL_s^x. \quad (3.3)$$

Remark 3.2. (1) If we take $F(t, x) = 1_{(x \leq a)}$ in (3.1), we have the very definition of L_t^a .

(2) Eisenbaum [2] has shown that for any Borelian function $b(t)$,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(|B_s - b(s)| < \varepsilon)} ds = \int_0^t \int_{\mathbb{R}} 1_{(-\infty, b(s))}(x) dL_s^x \quad \text{in } L^1, \quad (3.4)$$

which corresponds to (3.3) with $F(t, x) = 1_{(x \leq b(t))}$.

Proof. Define $H_\varepsilon(t, x) = (1/\varepsilon) \int_{x-\varepsilon}^x F(t, y) dy$. Then $H_\varepsilon \rightarrow F$ in $\mathcal{H}$ as $\varepsilon \downarrow 0$. On the one hand, $(\partial/\partial x)H_\varepsilon(t, x) = (1/\varepsilon)\{F(t, x) - F(t, x - \varepsilon)\}$. It follows that (see Eisenbaum [2, Theorem 5.1(ii)]) $\int_0^t \int_{\mathbb{R}} H_\varepsilon(s, x) dL_s^x = -(1/\varepsilon) \int_0^t \{F(s, B_s) - F(s, B_s - \varepsilon)\} ds$. On the other hand, $\int_0^t \int_{\mathbb{R}} H_\varepsilon(s, x) dL_s^x \rightarrow \int_0^t \int_{\mathbb{R}} F(s, x) dL_s^x$ in L^1 . $\square$

COROLLARY 3.3 (see [3]). *The following relation holds in L^1 :*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t g(s) I(b(s) - \varepsilon < B_s < b(s) + \varepsilon) ds = \int_0^t g(s) dL_s^b \quad (3.5)$$

for a continuous function $g : [0, t] \rightarrow \mathbb{R}$ and a continuous curve $b(\cdot)$ with bounded variation on $[0, t]$.

Proof. We apply Theorem 3.1 to the function $F(t, x) = g(t)I(x < b(t))$. It follows that $(1/2\varepsilon) \int_0^t g(s) I(b(s) - \varepsilon < B_s < b(s) + \varepsilon) ds \rightarrow \int_0^t \int_{\mathbb{R}} g(s) I(x < b(s)) dL_s^x$ in L^1 as $\varepsilon \downarrow 0$. We conclude using (see [4, Corollary 2.9]) that for the continuous function g , we have $\int_0^t g(s) \partial_s L_s^{b(s)} = \int_0^t g(s) dL_s^b$. $\square$

3.2. Random function case. Let a, b be in $\mathbb{R}$ with $a < b$. Let $\mathcal{M}$ be the set of elementary processes A such that

$$A(s, x) = \sum_{(s_i, x_j) \in \Delta} A_{ij} 1_{[s_i, s_{i+1}]}(s) 1_{(x_j, x_{j+1}]}(x), \quad (3.6)$$

where $(s_i)_{1 \leq i \leq n}$ is a subdivision of $(0, 1]$, $(x_j)_{1 \leq j \leq m}$ is a finite sequence of real numbers in $(a, b]$, $\Delta = \{(s_i, x_j), 1 \leq i \leq n, 1 \leq j \leq m\}$, and, is A_{ij} an $\mathcal{F}_{s_j}$ -measurable random variable such that $|A_{ij}| \leq 1$ for every (i, j) .

4 Some limit theorems connected with Brownian local time

Eisenbaum [2] asked the following question: does integration with respect to $(L_t^x; 0 \leq t \leq 1, x \in \mathbb{R})$ admit a *linear* extension to $\mathcal{P}$ the field generated by $\mathcal{M}$, verifying the following property?

If $(A_n)_{n \geq 0}$ converges a.e. to $A(t, x)$, then $(\int_0^1 \int_a^b A_n(s, x) dL_s^x)_{n \geq 0}$ converges in L^1 to $\int_0^1 \int_a^b A(s, x) dL_s^x$.

She only obtained a negative answer to the following *weaker* question:

$$\text{Is the set } \left\{ \int_0^1 \int_a^b A(s, x) dL_s^x, A \in \mathcal{M} \right\} \text{ bounded in } L^1? \quad (3.7)$$

Consequently, integration with respect to $(L_t^x; 0 \leq t \leq 1, x \in \mathbb{R})$ does not admit a *continuous* extension in L^1 .

Here we give an *illustrative example*, thanks to a result obtained by Walsh, which shows the lack of a *linear* extension.

Let us define $A_\varepsilon(t, x) = (1/\varepsilon) \int_{x-\varepsilon}^x L_t^y dy$ and $\tilde{A}_\varepsilon(t, x) = (1/\varepsilon) \int_x^{x+\varepsilon} L_t^y dy$. We see easily that $A_\varepsilon(t, x)$ (resp., $\tilde{A}_\varepsilon(t, x)$) converges a.e. to L_t^x , nevertheless we have

$$\lim_{\varepsilon \downarrow 0} \int_0^t \int_{\mathbb{R}} A_\varepsilon(s, x) dL_s^x \neq \lim_{\varepsilon \downarrow 0} \int_0^t \int_{\mathbb{R}} \tilde{A}_\varepsilon(s, x) dL_s^x. \quad (3.8)$$

Remark 3.4. The integrals $\int_0^t \int_{\mathbb{R}} A_\varepsilon(s, x) dL_s^x$ and $\int_0^t \int_{\mathbb{R}} \tilde{A}_\varepsilon(s, x) dL_s^x$ are well defined thanks to Theorem 2.1, however, one does not know whether $\int_0^t \int_{\mathbb{R}} L_s^x dL_s^x$ is well defined or not.

Let us recall, for the convenience of the reader, Walsh's theorem about the decomposition of $A(t, B_t) := \int_0^t 1_{\{B_s \leq B_t\}} ds$.

THEOREM 3.5 (see [5]). *$A(t, B_t)$ has the decomposition*

$$A(t, B_t) = \int_0^t L_s^{B_t} dB_s + X_t, \quad (3.9)$$

where

$$X_t = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \{L_s^{B_t} - L_s^{B_t - \varepsilon}\} ds = t + \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \{L_s^{B_t + \varepsilon} - L_s^{B_t}\} ds. \quad (3.10)$$

The limits exist in probability, uniformly for t in compact sets.

Our example follows by recalling the following property:

$$\int_0^t \int_{\mathbb{R}} A_\varepsilon(s, x) dL_s^x = -\frac{1}{\varepsilon} \int_0^t \{L_s^{B_t} - L_s^{B_t - \varepsilon}\} ds. \quad (3.11)$$

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