MEAN CONVERGENCE THEOREM FOR MULTIDIMENSIONAL ARRAYS OF RANDOM ELEMENTS IN BANACH SPACES

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For a d -dimensional array of random elements $\{V_n, n \in \mathbb{Z}_+^d\}$ in a real separable stable type p ($1 \leq p < 2$) Banach space, a mean convergence theorem is established. Moreover, the conditions for the convergence in mean of order p are shown to completely characterize stable-type p Banach spaces.

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1. Introduction

Let $\mathbb{Z}_+^d$, where d is a positive integer, denote the positive integer d -dimensional lattice points. The notation $m < n$, where $m = (m_1, m_2, \dots, m_d)$ and $n = (n_1, n_2, \dots, n_d) \in \mathbb{Z}_+^d$, means that $m_i \leq n_i$, $1 \leq i \leq d$, $|n|$ is used for $\prod_{i=1}^d n_i$.

Gut [5] proved that if $\{X, X_n, n \in \mathbb{Z}_+^d\}$ is a d -dimensional array of i.i.d. random variables with $E|X|^p < \infty$ ($0 < p < 2$) and $EX = 0$ if $1 \leq p < 2$, then

$$\frac{\sum_{j < n} X_j}{|n|^{1/p}} \longrightarrow 0 \quad \text{in } L^p \text{ as } \min_{1 \leq i \leq d} n_i \longrightarrow \infty, \quad (1.1)$$

where $(n_1, n_2, \dots, n_d) = n \in \mathbb{Z}_+^d$.

Recently, Thanh [11] proved (1.1) under condition of uniform integrability of $\{|X_n|^p, n \in \mathbb{Z}_+^d\}$.

Mean convergence theorems for sums of random elements Banach-valued are studied by many authors. The reader may refer to Wei and Taylor [12], Adler et al. [2], Rosalsky and Sreehari [9], or more recently, Rosalsky et al. [10], Cabrera and Volodin [3]. However, we are unaware of any literature of investigation on the mean convergence for multidimensional arrays of random elements in Banach spaces.

Consider a d -dimensional array $\{V_n, n \in \mathbb{Z}_+^d\}$ of independent random elements defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a real separable Banach space

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$\mathcal{X}$ with norm $\|\cdot\|$. In the current work, we establish the convergence in mean of order p ($1 \leq p < 2$) of the sums $\sum_{j \leq n} V_j / |n|^{1/p}$, $n \in \mathbb{Z}_+^d$, under the condition that $\{\|V_n\|^p, n \in \mathbb{Z}_+^d\}$ is uniformly integrable. The main results of this paper are Theorems 2.1 and 2.2. Theorem 2.1 is a stable-type p Banach space version of the main result of Thanh [11]. While the proof of Theorem 2.1 and the proof of the main result in Thanh [11] are similar, we will show in Theorem 2.2 that the implication in Theorem 2.1 indeed completely characterizes stable-type p Banach spaces.

Let $0 < p \leq 2$ and let $\{\theta_n, n \geq 1\}$ be independent and identically distributed stable random variables each with characteristic function $\phi(t) = \exp\{-|t|^p\}$. The real separable Banach space $\mathcal{X}$ is said to be of *stable-type p* if $\sum_{n=1}^\infty \theta_n v_n$ converges a.s. whenever $v_n \in \mathcal{X}$, $n \geq 1$ with $\sum_{n=1}^\infty \|v_n\|^p < \infty$. Equivalent characterizations of a Banach space being of stable-type p , properties of stable-type p Banach spaces, as well as various relationships between the conditions Rademacher-type p , and stable-type p may be found by Woyczyński in [13], by Marcus and Woyczyński in [7], and by Pisier in [8], see also the discussion by Adler et al. in [1]. We now mention explicitly some characterizations of this concept. The first theorem was obtained by Mandrekar and Zinn [6] and by Marcus and Woyczyński [7].

THEOREM 1.1. *Let $1 \leq p < 2$ and let $\mathcal{X}$ be a real separable Banach space. Then the following statements are equivalent.*

- (i) $\mathcal{X}$ is of stable-type p .
- (ii) For every symmetric random elements V , the condition $n^p P(\|V\| > n) \rightarrow 0$ as $n \rightarrow \infty$ implies that

$$\frac{\sum_{j=1}^n V_j}{n^{1/p}} \rightarrow 0 \quad \text{in probability,} \quad (1.2)$$

where $\{V_j, j \geq 1\}$ are independent copies of V .

THEOREM 1.2 (see [13, Theorem V.9.3]). *Let $1 \leq p < 2$ and let $\mathcal{X}$ be a real separable Banach space. Then the following statements are equivalent.*

- (i) $\mathcal{X}$ is of stable-type p .
- (ii) For each bounded sequence $\{x_n, n \geq 1\}$ of elements of $\mathcal{X}$,

$$\frac{\sum_{j=1}^n x_k \epsilon_k}{n^{1/p}} \rightarrow 0 \quad \text{a.s.,} \quad (1.3)$$

where $\{\epsilon_n, n \geq 1\}$ is a Rademacher sequence.

The symbol C denotes throughout a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

2. Main results

We can now present the main results. Theorem 2.1 is a stable-type p Banach space version of the main result of Thanh [11].

THEOREM 2.1. Let $\{V_n, n \in \mathbb{Z}_+^d\}$ be a d -dimensional array of independent mean-zero random elements in a real separable stable-type p ($1 \leq p < 2$) Banach space $\mathcal{X}$. If

$$\{\|V_n\|^p, n \in \mathbb{Z}_+^d\} \text{ is uniformly integrable,} \quad (2.1)$$

then

$$\frac{\sum_{j < n} V_j}{|n|^{1/p}} \rightarrow 0 \quad \text{in } L^p \text{ as } |n| \rightarrow \infty. \quad (2.2)$$

Proof. For arbitrary $\epsilon > 0$, there exists $M > 0$ such that

$$E(\|V_n\|^p I(\|V_n\| > M)) < \epsilon, \quad \forall n \in \mathbb{Z}_+^d. \quad (2.3)$$

Set

$$V'_n = V_n I(\|V_n\| \leq M), \quad n \in \mathbb{Z}_+^d, \quad V''_n = V_n I(\|V_n\| > M), \quad n \in \mathbb{Z}_+^d. \quad (2.4)$$

Since $\mathcal{X}$ is of stable-type p and $p < 2$, it is of Rademacher-type q for some $p < q < 2$. Thus

$$\begin{aligned} E \left\| \sum_{j < n} V_j \right\|^p &\leq 2^{p-1} \left[E \left\| \sum_{j < n} (V'_j - EV'_j) \right\|^p + E \left\| \sum_{j < n} (V''_j - EV''_j) \right\|^p \right] \\ &\leq 2^{p-1} E \left\| \sum_{j < n} (V'_j - EV'_j) \right\|^p + C \sum_{j < n} E \|V''_j - EV''_j\|^p \\ &\leq 2^{p-1} \left(E \left\| \sum_{j < n} (V'_j - EV'_j) \right\|^q \right)^{p/q} + C \sum_{j < n} E \|V''_j - EV''_j\|^p \\ &\quad \text{(by the Jensen inequality)} \\ &\leq C \left(\sum_{j < n} E \|V'_j - EV'_j\|^q \right)^{p/q} + C \sum_{j < n} E \|V''_j - EV''_j\|^p \\ &\leq C(|n|M^q)^{p/q} + C|n|\epsilon \\ &= o(|n|), \quad \text{as } |n| \rightarrow \infty. \end{aligned} \quad (2.5)$$

□

While the proof of Theorem 2.1 and the proof of the main result in Thanh [11] are similar, we now show in Theorem 2.2 that the implication $((2.1) \Rightarrow (2.2))$ in Theorem 2.1 indeed completely characterizes stable-type p Banach spaces.

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THEOREM 2.2. *Let $1 \leq p < 2$ and let $\mathcal{X}$ be a real separable Banach space. Then the following statements are equivalent.*

- (i) $\mathcal{X}$ is of stable-type p .
- (ii) For every d -dimensional array $\{V_n, n \in \mathbb{Z}_+^d\}$ of independent mean-zero random elements in $\mathcal{X}$, the condition (2.1) implies (2.2).
- (iii) For every d -dimensional array $\{V, V_n, n \in \mathbb{Z}_+^d\}$ of independent mean-zero random elements in $\mathcal{X}$, the conditions

$$E\|V\|^p < \infty, \quad \sup_{n \in \mathbb{Z}_+^d} P\{\|V_n\| > t\} \leq CP\{\|V\| > t\}, \quad \forall t > 0, \quad (2.6)$$

imply (2.2).

Proof. The implication ((i) $\Rightarrow$ (ii)) is precisely Theorem 2.1, whereas the implication ((ii) $\Rightarrow$ (iii)) is immediate. It remains to verify the implication ((iii) $\Rightarrow$ (i)). For reasons of clarity, we collect some of the steps in the following lemmas. The first lemma is a slight modification of de Acosta [4, Theorem 3.1] which holds for sequences of independent identically distributed random elements. The proof of the following modification can be obtained from de Acosta [4, Theorem 3.1] line by line, and so will be omitted. $\square$

LEMMA 2.3. *Let $\mathcal{X}$ be a real separable Banach space, $1 \leq p < 2$. Let $\{V, W_k, k \geq 1\}$ be sequence of independent random elements such that $E\|V\|^p < \infty$ and $\sup_{k \geq 1} P\{\|W_k\| > t\} \leq CP\{\|V\| > t\}$ for all $t > 0$. Then*

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n W_k}{n^{1/p}} = 0 \quad \text{in probability} \quad (2.7)$$

if and only if

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n W_k}{n^{1/p}} = 0 \quad \text{a.s.} \quad (2.8)$$

LEMMA 2.4. *Let $1 \leq p < 2$ and let $\mathcal{X}$ be a real separable Banach space. Suppose that for every sequence $\{V, W_k, k \geq 1\}$ of independent mean-zero random elements in $\mathcal{X}$, the conditions*

$$E\|V\|^p < \infty, \quad \sup_{k \geq 1} P\{\|W_k\| > t\} \leq CP\{\|V\| > t\}, \quad \forall t > 0, \quad (2.9)$$

imply that

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n W_k}{n^{1/p}} = 0 \quad \text{in probability.} \quad (2.10)$$

Then $\mathcal{X}$ is of stable-type p .

Proof of Lemma 2.4. Let $\{\varepsilon_k, k \geq 1\}$ be a Rademacher sequence and let $\{x_k, k \geq 1\}$ be a sequence of elements in $\mathcal{X}$ such that

$$\sup_{k \geq 1} \|x_k\| < \infty. \quad (2.11)$$

Then by the hypothesis of the lemma,

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \varepsilon_k x_k}{n^{1/p}} = 0 \quad \text{in probability.} \quad (2.12)$$

By Lemma 2.3,

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \varepsilon_k x_k}{n^{1/p}} = 0 \quad \text{a.s.} \quad (2.13)$$

Hence, by Theorem 1.2, $\mathcal{X}$ is of stable-type p . The proof of Lemma 2.4 is completed. $\square$

We now prove the implication ((iii) $\Rightarrow$ (i)). If $d = 1$, then the conclusion follows directly from Lemma 2.4. So, we can assume that $d \geq 2$. Let $\{V, W_k, k \geq 1\}$ be a sequence of independent mean-zero random elements in $\mathcal{X}$ such that $E\|V\|^p < \infty$ and $\sup_{k \geq 1} P\{\|W_k\| > t\} \leq CP\{\|V\| > t\}$ for all $t > 0$. For $n = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$, set

$$\begin{aligned} V_{(n_1, \dots, n_d)} &= W_{n_1}, \quad \text{if } n_2 = \dots = n_d = 1, \\ V_{(n_1, \dots, n_d)} &= 0, \quad \text{if } \max\{n_2, \dots, n_d\} \geq 2. \end{aligned} \quad (2.14)$$

Then $\{V_n, n \in \mathbb{Z}_+^d\}$ is an array of independent mean-zero random elements, and

$$\sup_{n \in \mathbb{Z}_+^d} P\{\|V_n\| > t\} \leq CP\{\|V\| > t\}, \quad \forall t > 0. \quad (2.15)$$

By (iii),

$$\frac{1}{|n|^{1/p}} \sum_{j < n} V_j \rightarrow 0 \quad \text{in } L^p \text{ as } |n| \rightarrow \infty. \quad (2.16)$$

This implies by taking $n_2 = \dots = n_d = 1$ and letting $n_1 \rightarrow \infty$ that

$$\frac{1}{n_1^{1/p}} \sum_{k=1}^{n_1} W_k \rightarrow 0 \quad \text{in } L^p, \text{ so in probability as } n_1 \rightarrow \infty. \quad (2.17)$$

By Lemma 2.4, $\mathcal{X}$ is of stable-type p .

Remark 2.5. In Theorem 2.1, if $0 < p < 1$, then the independence hypothesis and the hypothesis that the $\{V_n, n \in \mathbb{Z}_+^d\}$ have mean-zero are not needed for the theorem to hold.

Indeed, for arbitrary $\epsilon > 0$, define V'_n and $V''_n, n \in \mathbb{Z}_+^d$ as in the proof of Theorem 2.1. If $0 < p < 1$, then

$$\begin{aligned} E \left\| \sum_{j < n} V_j \right\|^p &\leq E \left\| \sum_{j < n} V'_j \right\|^p + E \left\| \sum_{j < n} V''_j \right\|^p \\ &\leq E \left\| \sum_{j < n} V'_j \right\|^p + \sum_{j < n} E \|V''_j\|^p \\ &\leq (|n|M)^p + |n|\epsilon \\ &= o(|n|), \quad \text{as } |n| \rightarrow \infty. \end{aligned} \quad (2.18)$$

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