ON FUZZY $\varphi\psi$ -CONTINUOUS MULTIFUNCTION

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Here, we would study and characterize fuzzy $\varphi\psi$ -continuity for fuzzy multifunctions which extend fuzzy $\varphi\psi$ -continuity of fuzzy functions. Moreover, we obtain some results in fuzzy multifunctions.

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1. Introduction and preliminaries

In the last three decades, the theory of multifunctions has advanced in a variety of ways and applications of this theory can be found, specially in functional analysis and fixed point theory. Recently many authors, for example, Albrycht and Matłoka [1] and Beg [3] have studied fuzzy multifunctions and have characterized some property of fuzzy multifunctions defined on a fuzzy topological space. Several authors have studied some type of fuzzy continuity for fuzzy functions and fuzzy multifunctions [1–5], [8–12]. In [3] fuzzy $\varphi\psi$ -continuous functions have been studied. But this brand of fuzzy continuity has not considered for fuzzy multifunctions which we attempt to study and characterize.

The *fuzzy set* in (on) a universe X is a function with domain X and values in $I = [0, 1]$. The class of all fuzzy sets on X will be denoted by I^X and symbols $A, B, \dots$ are used for fuzzy sets on X . 01_X is called *empty fuzzy set*, where 1_X is the characteristic function on X . For any fuzzy set A in X , the function value $A(x)$ is called the *grade of membership* of x in A . We write $x \in A$ if $A(x) > 0$. For any fuzzy set A , the fuzzy set $1 - A(x)$ is called the *complement* of A which is denoted by A^c . Let A and B be fuzzy sets in X , we write $A \leq B$ if $A(x) \leq B(x)$ for all x in X . For any family $\{A_\alpha\}_{\alpha \in \mathcal{A}}$ of fuzzy sets in X , $\bigvee_{\alpha \in \mathcal{A}} A_\alpha$ and $\bigwedge_{\alpha \in \mathcal{A}} A_\alpha$ are defined by $\sup_{\alpha} A_\alpha(x)$ and $\inf_{\alpha} A_\alpha(x)$, respectively. A family τ of fuzzy sets in X is called a *fuzzy topology* for X if (i) $\alpha 1_X \in \tau$ for each $\alpha \in I$; (ii) $A \wedge B \in \tau$ where $A, B \in \tau$ and (iii) $\bigvee_{\alpha \in \mathcal{A}} A_\alpha \in \tau$ whenever $A_\alpha \in \tau$ for all α in $\mathcal{A}$. The pair (X, τ) is called a *fuzzy topological space* [6]. Every member of τ is called *fuzzy open set* and its complements

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are called *fuzzy closed sets* [6]. In a fuzzy topological space X the *interior* and the *closure* of a fuzzy set A (simply $\text{int}(A)$ and $\text{cl}(B)$, resp.) are defined by

$$\begin{aligned}\text{int}(A) &= \bigvee \{U : A \leq U, U \text{ is a fuzzy open set}\}, \\ \text{cl}(A) &= \bigwedge \{F : A \leq F, F \text{ is a fuzzy closed set}\}.\end{aligned}\tag{1.1}$$

A *neighborhood* of a fuzzy set A in a fuzzy topological space X is any fuzzy set B for which there is a fuzzy open set V satisfying $A \leq V \leq B$. Any fuzzy open set V that satisfies $A \leq V$ is called a *fuzzy open neighborhood* of A [10]. A fuzzy set A is called a *fuzzy point* if it takes the value 0 for all $y \in X$ except one, say $x \in X$. If its value at x is ϵ ($0 \leq \epsilon \leq 1$), we denote this fuzzy point by x_ϵ [11]. For any fuzzy point x_ϵ and any fuzzy set A we write $x_\epsilon \in A$ if and only if $\epsilon \leq A(x)$. Let f be a function from X to Y . A fuzzy function $f : X \rightarrow Y$ is defined by

$$f(A)(y) = \begin{cases} \bigvee_{x \in f^{-1}(\{y\})} A(x) & f^{-1}(\{y\}) \neq \emptyset, \\ 0 & f^{-1}(\{y\}) = \emptyset, \end{cases}\tag{1.2}$$

for all y in Y , where A is an arbitrary fuzzy set in X [12]. A fuzzy function $f : X \rightarrow Y$ is called *fuzzy continuous* if for each $x_\epsilon \in X$ and each fuzzy neighborhood B of $f(x_\epsilon)$ there exists a fuzzy neighborhood A of x_ϵ such that $f(A) \leq B$ [11]. A *fuzzy multifunction* $f : X \rightarrow \rightarrow Y$ assigns to each x in X a fuzzy set $f(x)$ of Y [2]. If A is a fuzzy set in X , then the fuzzy set $f(A)$ in Y is defined by

$$f(A)(y) = \bigvee_{x \in X} (f(x)(y) \wedge A(x)).\tag{1.3}$$

For more details about fuzzy multifunctions and their properties, the reader is referred to [1, 2, 10]. Throughout this paper, (X, τ) and (Y, ν) are fuzzy topological spaces. The symbol $f : X \rightarrow \rightarrow Y$ is used for a fuzzy multifunction from X to Y , while $f : X \rightarrow Y$ for a fuzzy function from X to Y .

Main results

Definition 1.1. (i) A fuzzy function φ on X is called a *fuzzy operation* on X , if $\varphi(01_X) = 01_X$ and $\text{int}(A) \leq \varphi(A)$, where A is any nonempty fuzzy set in X . φ is called a *monotonous fuzzy operation*, if $\varphi(A) \leq \varphi(B)$, whenever $A, B \in I^X$ and $A \leq B$ [5].

(ii) $f : X \rightarrow Y$ is called a *fuzzy $\varphi\psi$ -continuous function* at $x_\epsilon \in X$ if for each fuzzy open neighborhood B of $f(x_\epsilon)$, there is a fuzzy open neighborhood A of x_ϵ such that $f(\varphi(A)) \leq \psi(B)$, where φ and ψ are fuzzy operation on X and Y , respectively. $f : X \rightarrow Y$ is said to be a *fuzzy $\varphi\psi$ -continuous function* if it is a fuzzy $\varphi\psi$ -continuous function at each $x_\epsilon \in X$.

Definition 1.2. (i) $f : X \rightarrow \rightarrow Y$ is called a *fuzzy $\varphi\psi$ -continuous multifunction* at $x_\epsilon \in X$ if for each fuzzy open neighborhood B of $f(x_\epsilon)$, there is a fuzzy open neighborhood A of x_ϵ such that $f(\varphi(A)) \leq \psi(B)$, where φ and ψ are fuzzy operation on X and Y , respectively. $f : X \rightarrow \rightarrow Y$ is said to be a *fuzzy $\varphi\psi$ -continuous multifunction* if it is a fuzzy $\varphi\psi$ -continuous multifunction at each $x_\epsilon \in X$.

(ii) $f : X \rightarrow \rightarrow Y$ is called a *single valued fuzzy multifunction* if f at each x is a fuzzy point $1_{\{y_x\}}$, where $y_x \in Y$. In this case it would induce a fuzzy function $\tilde{f} : X \rightarrow Y$ by $\tilde{f}(x) = y_x$. Therefore, $f(x) = 1_{\{\tilde{f}(x)\}}$.

PROPOSITION 1.3. *Suppose $f : X \rightarrow \rightarrow Y$ be a single valued fuzzy multifunction. Then for any fuzzy set A in X ; $f(A) = \tilde{f}(A)$. Therefore, f is a $\varphi\psi$ -continuous multifunction if and only if $\tilde{f}$ is a fuzzy $\varphi\psi$ -continuous function.*

Proof. The equivalence for any fuzzy set A of X can be derived from the following fact:

$$f(\varphi(A))(y) = \bigvee_{z \in X} (f(z)(y) \wedge \varphi(A)(z)) = \bigvee_{z \in X} (1_{\{\tilde{f}(z)\}}(y) \wedge \varphi(A)(z)). \quad (1.4)$$

On the other hand,

$$\tilde{f}(\varphi(A))(y) = \begin{cases} \bigvee_{h \in \tilde{f}^{-1}(\{y\})} \varphi(A)(h) & \tilde{f}^{-1}(\{y\}) \neq \emptyset, \\ 0 & \tilde{f}^{-1}(\{y\}) = \emptyset. \end{cases} \quad (1.5)$$

Therefore, $f(\varphi(A)) = \tilde{f}(\varphi(A))$. Now replacing identity function as a fuzzy operation on X instead of φ completes the proof. $\square$

From the above result this brand of continuity for fuzzy multifunctions is in fact a generalization of $\varphi\psi$ -continuity introduced in [3]. Next, we would like to present a result showing the relation between fuzzy $\varphi\psi$ -continuity and fuzzy $\varphi\psi$ -continuity in respect of nets. First we note to the following results.

PROPOSITION 1.4. *Suppose $f : X \rightarrow \rightarrow Y$ be a fuzzy multifunction and $A, B \in I^X$ such that $A \leq B$. Then $f(A) \leq f(B)$.*

Proof.

$$f(A)(y) = \bigvee_{z \in X} (f(z)(y) \wedge A(z)) \leq \bigvee_{z \in X} (f(z)(y) \wedge B(z)) = f(B)(y). \quad (1.6)$$

$\square$

LEMMA 1.5. *Let $f : X \rightarrow \rightarrow Y$ be a fuzzy multifunction and let x_ϵ be a fuzzy point in X . Then $f(x_\epsilon) = f(x) \wedge \epsilon$.*

Proof. It is straightforward. $\square$

We say that a net $(x_{\epsilon_\alpha}^\alpha)_{\alpha \in \mathcal{A}}$ of fuzzy points in a fuzzy topological space X is φ -convergent to a fuzzy point x_ϵ (we will denote it by $x_{\epsilon_\alpha}^\alpha \xrightarrow{\varphi} x_\epsilon$) if for any neighborhood set A of x_ϵ , there is an $\alpha_0 \in \mathcal{A}$ in which $x_{\epsilon_\alpha}^\alpha \in \varphi(A)$ for all $\alpha \geq \alpha_0$.

LEMMA 1.6. *Consider a fuzzy set A and a convergent net (A_α) of fuzzy sets which $A_\alpha \rightarrow A$ in X . Then $A_\alpha \xrightarrow{\varphi} A$.*

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Proof. From the assumption given any fuzzy open neighborhood B of A , there is $\alpha_0 \in \mathcal{A}$ such that for all $\alpha \geq \alpha_0$,

$$A_\alpha \leq B = \text{int}(B) \leq \phi(B). \quad (1.7)$$

□

A fuzzy multifunction $f : X \rightarrow Y$ is called *net-fuzzy $\phi\psi$ -continuous* if for each net of fuzzy points $x_{\epsilon_\alpha}^\alpha$ and x_ϵ in X , $f(x_{\epsilon_\alpha}^\alpha) \xrightarrow{\psi} f(x_\epsilon)$, where $(x_{\epsilon_\alpha}^\alpha) \xrightarrow{\phi} x_\epsilon$.

THEOREM 1.7. *Let X be a fuzzy topological space. For any fuzzy multifunction $f : X \rightarrow Y$ the following are equivalent:*

- (i) *f is a fuzzy $\phi\psi$ -continuous;*
- (ii) *f is a net-fuzzy $\phi\psi$ -continuous.*

Proof. (i) $\Rightarrow$ (ii).

For any fuzzy open neighborhood B of $f(x_\epsilon)$, there is a fuzzy open neighborhood A of x_ϵ such that

$$f(\phi(A)) \leq \psi(B). \quad (1.8)$$

From the assumption, there is $\alpha_0 \in A$ for which

$$x_{\epsilon_\alpha}^\alpha \leq \phi(A) \quad (\forall \alpha \geq \alpha_0). \quad (1.9)$$

According to Proposition 1.4, $f(x_{\epsilon_\alpha}^\alpha) \leq f(\phi(A)) \leq \psi(B)$.

(ii) $\Rightarrow$ (i).

On the contrary, there is a fuzzy point x_ϵ in X , a fuzzy open neighborhood B of $f(x_\epsilon)$ such that, there is not a fuzzy neighborhood A of x_ϵ satisfying in $f(\phi(A)) \leq \psi(B)$. This means that there is $z_A \in Y$ with the following property:

$$f(\phi(A))(z_A) > \psi(B)(z_A). \quad (1.10)$$

Therefore,

$$\bigvee_{x \in X} (f(x)(z_A) \wedge \phi(A)(x)) > \psi(B)(z_A). \quad (1.11)$$

Then $f(x_A)(z_A) \wedge \phi(A)(z_A) > \psi(B)(z_A)$ for a suitable x_A of X . We conclude that

$$f(x_A)(z_A) > \psi(B)(z_A). \quad (1.12)$$

Consider $\{A_\alpha : \alpha \in \mathcal{A}\}$ as a system of fuzzy neighborhoods at x_ϵ . The following order makes $\mathcal{A}$ as a directed set and so it makes $\{A_\alpha : \alpha \in \mathcal{A}\}$ as a net:

$$\alpha \leq \beta \iff A_\beta \leq A_\alpha. \quad (1.13)$$

Applying (1.12) for A_α instead of A , there is $x_{\epsilon_\alpha}^\alpha$ in A_α for which $f(x_{\epsilon_\alpha}^\alpha) > \psi(B)$. From the choice of $x_{\epsilon_\alpha}^\alpha$ in A_α , $x_{\epsilon_\alpha}^\alpha \rightarrow x_\epsilon$. Lemma 1.6 implies that $x_{\epsilon_\alpha}^\alpha \xrightarrow{\phi} x_\epsilon$. Since $f(x_{\epsilon_\alpha}^\alpha)(z_A) = f(x_\alpha)(z_\alpha) \wedge \epsilon_\alpha \leq f(x_\alpha)(z_\alpha)$ so from (1.12), $f(x_{\epsilon_\alpha}^\alpha)$ is not ψ -convergent to $f(x_\epsilon)$, which completes the proof. □

In the following result we show continuity of the composition of two fuzzy multifunction. Suppose $f : X \rightarrow \rightarrow Y$ and $g : Y \rightarrow \rightarrow Z$. We define *composition* $gof : X \rightarrow \rightarrow Z$ by $(gof)(x) = g(f(x)) = \bigcup_{t \in f(x)} g(t)$.

COROLLARY 1.8. *Suppose $f : X \rightarrow \rightarrow Y$ be a fuzzy $\phi\psi$ -continuous single valued multifunction and suppose $g : Y \rightarrow \rightarrow Z$ be $\psi\eta$ -fuzzy continuous multifunction. Then, $gof : X \rightarrow \rightarrow Z$ is $\phi\eta$ -fuzzy continuous multifunction.*

Proof. Assume that $x_{\epsilon_\alpha}^{\alpha} \xrightarrow{\phi} x_\epsilon$ in X . Since f is fuzzy $\phi\psi$ -continuous multifunction, so

$$f(x_{\epsilon_\alpha}^{\alpha}) \xrightarrow{\psi} f(x_\epsilon). \quad (1.14)$$

Assume that g is $\psi\eta$ -fuzzy continuous multifunction and f is fuzzy $\phi\psi$ -continuous single valued multifunction, so

$$g(f(x_{\epsilon_\alpha}^{\alpha})) \xrightarrow{\eta} g(f(x_\epsilon)). \quad (1.15)$$

Therefore,

$$(gof)(x_{\epsilon_\alpha}^{\alpha}) \xrightarrow{\eta} (gof)(x_\epsilon). \quad (1.16)$$

Theorem 1.7 completes the proof. $\square$

Definition 1.9. Let X_0 be a subset of X , let $i : X_0 \rightarrow X$ be the inclusion map, and let $f : X \rightarrow \rightarrow Y$ be a fuzzy multifunction. Say that f_{oi} is the *restriction* of f to X_0 .

LEMMA 1.10. *Assuming ϕ is a fuzzy operation on X and $X_0 \subseteq X$. Then $\tilde{\phi}(A) = \phi(\tilde{A})$ defines a fuzzy operation on X_0 , where $\tilde{A}$ is the extension of A by zero to X .*

Proof. It is easy to see that $\tilde{\phi}$ is a well-defined map and $\tilde{\phi}(01_X) = 01_X$. ϕ is a fuzzy operation, so $\text{int}(\tilde{A}) \leq \phi(\tilde{A})$. But,

$$\begin{aligned} \text{int}(A) &= \bigvee \{ \dot{U} : \dot{U} \leq A, \dot{U} \text{ is a fuzzy open set} \} \\ &= \bigvee \{ Uoi : U \leq A, U \text{ is a fuzzy open set} \}. \end{aligned} \quad (1.17)$$

For $x_0 \in X_0$, $\text{int}(A)(x_0) = \text{int}(\tilde{A})(x_0)$. This shows that

$$\text{int}(A) \leq \phi(\tilde{A}) \leq \tilde{\phi}(A). \quad (1.18)$$

$\square$

The following result shows the fuzzy continuity of the restriction of fuzzy multifunction.

THEOREM 1.11. *Suppose $f : X \rightarrow \rightarrow Y$ be a fuzzy $\phi\psi$ -continuous multifunction and $X_0 \subseteq X$. Then f_{oi} is a $\tilde{\phi}\psi$ -fuzzy continuous multifunction, where ϕ is a monotonous fuzzy operation.*

Proof. For any fuzzy point x_ϵ in X_0 , $j(x_\epsilon)$ is a fuzzy point in X . It shows that for any fuzzy open neighborhood B of $f(i(x_\epsilon))$, there is a fuzzy open neighborhood A of $i(x_\epsilon)$

for which $f(\varphi(A)) \leq \psi(B)$. But Aoi is a fuzzy open neighborhood of x_ϵ in X_0 , only we must show that $f oi(\widetilde{\varphi}(Aoi)) \leq \psi(B)$. To see this,

$$\begin{aligned}
 f oi(\widetilde{\varphi}(Aoi))(y) &= \bigvee_{z \in X_0} ((f oi)(z)(y) \wedge \widetilde{\varphi}(Aoi)(z)) \\
 &= \bigvee_{z \in X_0} (f(z)(y) \wedge \widetilde{\varphi(Aoi)oi}(z)) \\
 &\leq \bigvee_{z \in X_0} (f(z)(y) \wedge \varphi(A)(z)) \\
 &= f(\varphi(A))(y) \\
 &\leq \psi(B)(y).
 \end{aligned} \tag{1.19}$$

□

PROPOSITION 1.12. *Suppose (X, τ) and (Y, η) be fuzzy topological spaces, φ and ψ are fuzzy operations on X and Y , respectively, where φ is a monotonous fuzzy operation. Let $f : X \rightarrow Y$ be any fuzzy multifunction and let $\mathcal{B}$ be a base for η . Then f is fuzzy $\varphi\psi$ -continuous multifunction if and only if f is fuzzy $\varphi\psi$ -continuous multifunction with respect to $\mathcal{B}$.*

Proof. $(\Rightarrow)$ is straightforward.

For $(\Leftarrow)$ consider any fuzzy point x_ϵ in X and any fuzzy open neighborhood B of $f(x_\epsilon)$. $C \in \mathcal{B}$ exists such that $f(x_\epsilon) \leq C$ and $C \leq B$. From the assumption there is a fuzzy open neighborhood A of x_ϵ such that $f(\varphi(A)) \leq \psi(C)$. But ψ is monotonous so $f(\varphi(A)) \leq \psi(B)$. □

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