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Research Article

On the Survival Time of a Duplex System: A Sokhotski-Plemelj Problem

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We analyze the survival time of a renewable duplex system characterized by warm standby and subjected to a priority rule. In order to obtain the Laplace transform of the survival function, we employ a stochastic process endowed with time-dependent transition measures satisfying coupled partial differential equations. The solution procedure is based on the theory of sectionally holomorphic functions combined with the notion of dual transforms. Finally, we introduce a security interval related to a prescribed security level and a suitable risk criterion based on the survival function of the system. As an example, we consider the particular case of deterministic repair. A computer-plotted graph displays the survival function together with the security interval corresponding to a security level of 90%.

1. Introduction

Standby provides a powerful tool to enhance the reliability, availability, quality, and safety of operational plants (see, e.g., [1–4]). Standby systems are often subjected to priority rules. For instance, the *external* power supply station of a technical plant has usually overall (break-in) priority in operation with regard to an *internal* (local) power generator kept in cold or warm standby; that is, the local generator is only deployed if the external unit is down. The notion of “cold” standby signifies that the local generator has a zero failure rate in standby, whereas the notion of “warm” standby means that the failure-free time of the local generator is stochastically larger [5] in standby than in the operative state. Note that the warm standby mode of a unit is often indispensable to perform an instantaneous switch from standby into the operative state, allowing continuous operation of an operational system upon failure of the online unit.

Cold or warm standby systems, subjected to priority rules, have received considerable attention in previous literature (see, e.g., [6–20]). As a variant, we introduce a duplex system consisting of a priority unit (the **p**-unit) with a back-up nonpriority unit (the **n**-unit) in warm standby and attended by a repair facility. The **p**-unit has overall (break-in) priority in operation with regard to the **n**-unit; that is, the **n**-unit is only deployed if and only if the **p**-unit is down. In order to avoid undesirable delays in repairing failed units, we assume that the twin system is attended by two heterogeneous repairmen. Each repairman has his own particular task. Repairman $\mathcal{N}$ is skilled at repairing the failed **n**-unit, whereas repairman $\mathcal{P}$ is supposed to be an expert in repairing the failed **p**-unit. Both repairmen are jointly busy if both units (the **p**-unit and **n**-unit) are down. Otherwise, at least one repairman is idle. Any repair is assumed to be perfect. The entire system (henceforth called the **T**-system) is up if at least one unit is up. Otherwise, the **T**-system is down.

In order to determine the survival function of the **T**-system, we introduce a stochastic process endowed with time-dependent transition measures satisfying coupled partial differential equations. The solution procedure is based on a refined application of the theory of sectionally holomorphic functions (see, e.g., [21]) combined with the notion of dual transforms. Furthermore, we introduce a security interval $[0,\tau)$ related to a security level $0 < \delta < 1$ and a risk criterion based on the survival function of the **T**-system. The security interval ensures a survival of the **T**-system up to time τ with a probability larger than δ. Finally, we consider the particular case of deterministic repair (replacement). A computer-plotted graph displays the survival function together with the security interval corresponding to a security level of 90%.

2. Formulation

Consider the **T**-system subjected to the following conditions.

(i) The **p**-unit has a general failure-free time distribution $F(\cdot)$ with finite mean and a general repair time distribution $R(\cdot),\ R(0) = 0$. The failure-free time and the repair time are denoted by f and r. We assume that $F(\cdot)$ is Lebesgue absolutely continuous with a density function (in the Radon-Nikodym sense) of bounded variation on $[0,\infty)$.

(ii) The **n**-unit has a constant failure rate $\lambda > 0$ in the operative state and a constant failure rate $0 < \lambda_s < \lambda$ in standby. Note that the inequality $\lambda_s < \lambda$ is consistent with the notion of warm standby. The failure-free time of the **n**-unit in warm standby (resp., in operation) is denoted by f_s (resp., f_o). The *common* repair time of any **n**-failure is denoted by r_s with (common) repair time distribution $R_s(\cdot),\ R_s(0) = 0$. In addition, we assume that r_s has finite mean and variance.

(iii) The random variables f, r, f_s, f_o, and r_s are assumed to be statistically independent and any repair is perfect.

(iv) Characteristic functions are formulated in terms of a complex transform variable. For instance,

$$\mathbf{E}e^{i\omega r} = \int_0^\infty e^{i\omega x} dR(x), \quad \operatorname{Im}\omega \geq 0. \tag{2.1}$$

Note that

$$\mathbf{E}e^{-i\omega r} = \int_{-\infty}^0 e^{i\omega x} d\{1 - R((-x) -)\}, \quad \operatorname{Im}\omega \leq 0. \tag{2.2}$$

The corresponding Fourier-Stieltjes transforms are called *dual* transforms. Without loss of generality (see Remark 7.4), we may assume that R and R_s have density functions of bounded variation on $[0,\infty)$. Note that the bounded variation property implies that, for instance,

$$\left|\mathbf{E}e^{i\tau f}\right| = O\left(\frac{1}{|\tau|}\right), \quad |\tau| \longrightarrow \infty. \tag{2.3}$$

(v) In order to derive the survival function of the $\mathbf{T}$-system, we employ a stochastic process $\{N_t,\ t \geq 0\}$ with discrete state space $\{A,B,C,D\} \subset [0,\infty)$ and absorbing state D characterized by the following exhaustive set of mutually exclusive events.

$\{N_t = A\}$: the **p**-unit is operative and the **n**-unit is in warm standby at time t.

$\{N_t = B\}$: the **n**-unit is operative and the **p**-unit is under progressive repair at time t.

$\{N_t = C\}$: the **p**-unit is operative and the **n**-unit is under progressive repair at time t.

$\{N_t = D\}$: the $\mathbf{T}$-system is down at time t.

Note that the absorbing state D implies that a transition of the process $\{N_t\}$ into state A is only possible via states B or C, whereas a transition from state B or C into state D terminates the lifetime of the system. Therefore, the inclusion of the absorbing state D into the state space of the process $\{N_t,\ t \geq 0\}$ triggers the introduction of a so-called *stopping time*. Consequently, we first define the non-Markovian process $\{N_t,\ t \geq 0\}$ on a filtered probability space $\{\Omega, \mathscr{A}, \mathbf{P}, \mathcal{F}\}$ where the *history* $\mathcal{F} := \{\mathcal{F}_t,\ t \geq 0\}$ satisfies the Dellacherie conditions:

(i) $\mathcal{F}_0$ contains the $\mathbf{P}$-null sets of $\mathscr{A}$;

(ii) for all $t \geq 0$, $\mathcal{F}_t = \bigcap_{u>t}\mathcal{F}_u$; that is, the family $\mathcal{F}$ is right-continuous.

Consider the $\mathcal{F}$-stopping time

$$\theta := \inf\left\{t : N_t = D \mid N_0 = A,\ V_0 = 0\right\}, \tag{2.4}$$

where V_t is the past failure-free time of the **p**-unit being operative at time t. We assume that the $\mathbf{T}$-system starts functioning at some time origin $t = 0$ in state A; that is, let $N_0 = A,\ V_0 = 0$, $\mathbf{P}$-a.s. Thus, from $t = 0$ onwards, θ is the *survival* time (lifetime) of the $\mathbf{T}$-system. The corresponding survival function is denoted by $\mathfrak{R}(\cdot)$. Clearly, $\mathfrak{R}(t) = \mathbf{P}\{\theta > t\},\ t \geq 0$. A (vector) Markov characterization of the non-Markovian process $\{N_t,\ t \geq 0\}$ with absorbing state D is piecewise and conditionally defined by

(1) $\{(N_t, U_t)\}$ if $N_t = A$ (i.e., if the event $\{N_t = A\}$ occurs), where U_t denotes the remaining failure-free time of the **p**-unit being up at time t;

(2) $\{(N_t, X_t)\}$ if $N_t = B$, where X_t denotes the remaining repair time of the **p**-unit being under progressive repair at time t;

(3) $\{(N_t, U_t, Y_t)\}$ if $N_t = C$, where Y_t denotes the remaining repair time of the **n**-unit being under progressive repair at time t;

(4) $\{N_t\}$ if $N_t = D$ (the absorbing state).

The state space of the underlying Markov process, with absorbing state D, is given by

$$\{(A,u)\}\bigcup\{(B,x)\}\bigcup\{(C,u,y)\}\bigcup\{D\}, \quad u \geq 0,\ x \geq 0,\ y \geq 0. \tag{2.5}$$

For $K = A,\ B,\ C,\ D$, let $p_K(t) := \mathbf{P}\{N_t = K\}$, $t \geq 0$.

(vi) Finally, we introduce the transition measures:

$$p_A(t,u)du := \mathbf{P}\{N_t = A,\ U_t \in du\},$$
$$p_B(t,x)dx := \mathbf{P}\{N_t = B,\ X_t \in dx\}, \tag{2.6}$$
$$p_C(t,u,y)du\,dy := \mathbf{P}\{N_t = C,\ U_t \in du,\ Y_t \in dy\}.$$

Note that, for instance,

$$\begin{aligned} p_C(t) &= \int_0^\infty \int_0^\infty d_u\, d_y \mathbf{P}\{N_t = C,\ U_t \le u,\ Y_t \le y\} \\ &= \int_0^\infty \int_0^\infty p_C(t,u,y)du\,dy. \end{aligned} \tag{2.7}$$

3. Notations

(i) The indicator (function) of an event $\{N_t = K\}$ is denoted by $\mathbf{1}\{N_t = K\}$.

(ii) The complex plane and the real line are, respectively, denoted by $\mathbf{C}$ and $\mathbf{R}$ with obvious superscript notations such as $\mathbf{C}^+$ and $\mathbf{C}^-$. For instance, $\mathbf{C}^+ := \{\omega \in \mathbf{C} : \operatorname{Im}\,\omega > 0\}$.

(iii) We frequently use the characteristic function:

$$\gamma_s^+(\tau) := \begin{cases} \dfrac{\mathbf{E}e^{i\tau r_s} - 1}{i\tau \mathbf{E}r_s} & \text{if } \tau \neq 0, \\ 1 & \text{if } \tau = 0. \end{cases} \tag{3.1}$$

Note that

$$\gamma_s^+(\omega) = \frac{1}{\mathbf{E}r_s}\int_0^\infty e^{i\omega x}(1 - R_s(x))dx, \quad \operatorname{Im}\,\omega \ge 0. \tag{3.2}$$

Property 3.1 (see [22, Appendix]). The function $1 + \lambda_s \mathbf{E}r_s \gamma_s^+(\omega)$, $\operatorname{Im}\ \omega \ge 0$, has no zeros in $\mathbf{C}^+ \cup \mathbf{R}$.

(iv) The Heaviside unit step function, with the unit step at $t = t_0$, is denoted by $H_{t_0}(\cdot)$, that is,

$$H_{t_0}(t) := \begin{cases} 1 & \text{if } t \ge t_0 > 0, \\ 0 & \text{if } t < t_0. \end{cases} \tag{3.3}$$

(v) The greatest integer function is denoted by $[\cdot]$.

(vi) The Laplace transform of any locally integrable and bounded function on $[0,\infty)$ is denoted by the corresponding character marked with an asterisk. For instance,

$$p_A^*(z) := \int_0^\infty e^{-zt} p_A(t)dt, \quad \operatorname{Re}\ z > 0. \tag{3.4}$$

Observe that

$$\mathfrak{R}^*(z) = \frac{1 - \mathbf{E}e^{-z\theta}}{z}, \quad \operatorname{Re} z > 0. \tag{3.5}$$

Moreover, by the product rule for Lebesgue-Stieltjes integrals (see, e.g., [23, Appendix])

$$zp_D^*(z) = \int_{0-}^{\infty} e^{-zt}\, dp_D(t) = \mathbf{E}e^{-z\theta}, \quad \operatorname{Re} z > 0. \tag{3.6}$$

(vii) Let $\varphi(\tau),\ \tau \in \mathbf{R}$, be a bounded and continuous function. φ is called Γ-integrable if

$$\lim_{\substack{T\to\infty \\ \epsilon\downarrow 0}} \int_{\Gamma_{T,\epsilon}} \varphi(\tau)\frac{d\tau}{\tau - u}, \quad u \in \mathbf{R}, \tag{3.7}$$

exists, where $\Gamma_{T,\epsilon} := (-T, u-\epsilon] \cup [u+\epsilon, T)$. The corresponding integral, denoted by

$$\frac{1}{2\pi i}\int_{\Gamma} \varphi(\tau)\frac{d\tau}{\tau - u}, \tag{3.8}$$

is called a Cauchy principal value in double sense.

(viii) A function $\varphi(\tau),\ \tau \in \mathbf{R}$, is called Hölder-continuous on $\mathbf{R}$ if for all $\tau_1, \tau_2 \in \mathbf{R}$, there exists $(\beta, A),\ 0 < \beta \leq 1,\ A > 0:$

$$\left|\varphi(\tau_2) - \varphi(\tau_1)\right| \leq A\left|\tau_2 - \tau_1\right|^{\beta}. \tag{3.9}$$

The function $\varphi(\tau),\ \tau \in \mathbf{R}$, is called Hölder-continuous at infinity if there exists $\gamma > 0$:

$$\left|\varphi(\tau)\right| = O\left(\frac{1}{|\tau|^{\gamma}}\right), \quad |\tau| \longrightarrow \infty. \tag{3.10}$$

Hölder-continuous functions with exponent $\beta = \gamma = 1$ are called Lipschitz-continuous.

(ix) Note that the Hölder continuity of $\varphi(\cdot)$ on $\mathbf{R}$ and at infinity is sufficient for the existence of the Cauchy-type integral:

$$\frac{1}{2\pi i}\int_{\Gamma} \varphi(\tau)\frac{d\tau}{\tau - \omega}, \quad \omega \in \mathbf{C}. \tag{3.11}$$

4. Differential equations

In order to derive a system of differential equations, we observe the random behavior of the $\mathbf{T}$-system in some time interval $(t, t+\Delta)$, $\Delta \downarrow 0$. Grouping terms of $o(\Delta)$ and taking the absorbing state D into account reveal that

$$\begin{aligned}
p_A(t+\Delta, u-\Delta) &= p_A(t,u)(1-\lambda_s\Delta) + p_B(t,0)\frac{\mathrm{d}F}{\mathrm{d}u}(u)\Delta + p_C(t,u,0)\Delta + o(\Delta),\\
p_B(t+\Delta, x-\Delta) &= p_B(t,x)(1-\lambda\Delta) + p_A(t,0)\frac{\mathrm{d}R}{\mathrm{d}x}(x)\Delta + o(\Delta),\\
p_C(t+\Delta, u-\Delta, y-\Delta) &= p_C(t,u,y) + \lambda_s p_A(t,u)\frac{\mathrm{d}R_s}{\mathrm{d}y}(y)\Delta + o(\Delta),\\
p_D(t+\Delta) &= p_D(t) + \lambda\int_0^\infty p_B(t,x)dx\Delta + \int_0^\infty p_C(t,0,y)dy\Delta + o(\Delta).
\end{aligned} \tag{4.1}$$

Taking the definition of *directional* derivative into account, for instance,

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u} - \frac{\partial}{\partial y}\right)p_C(t,u,y) := \lim_{\Delta\downarrow 0}\frac{p_C(t+\Delta, u-\Delta, y-\Delta) - p_C(t,u,y)}{\Delta}, \tag{4.2}$$

entails that for $t>0$, $u>0$, $x>0$, and $y>0$,

$$\left(\lambda_s + \frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right)p_A(t,u) = p_B(t,0)\frac{\mathrm{d}F}{\mathrm{d}u}(u) + p_C(t,u,0), \tag{4.3}$$

$$\left(\lambda + \frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)p_B(t,x) = p_A(t,0)\frac{\mathrm{d}R}{\mathrm{d}x}(x), \tag{4.4}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u} - \frac{\partial}{\partial y}\right)p_C(t,u,y) = \lambda_s p_A(t,u)\frac{\mathrm{d}R_s}{\mathrm{d}y}(y), \tag{4.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}p_D(t) = \lambda p_B(t) + \int_0^\infty p_C(t,0,y)dy. \tag{4.6}$$

Note that the initial condition $N_0 = A$, $V_0 = 0$, $\mathbf{P}$-a.s. implies that

$$p_A(0,u) = \frac{\mathrm{d}F}{\mathrm{d}u}(u), \quad u>0. \tag{4.7}$$

Moreover, $\mathbf{P}\{\theta \le t\} = p_D(t)$. Finally, observe that (4.3)–(4.6) are consistent with the probability law $\sum_K p_K(t) = 1$ and that $p_A(0) = 1$.

5. Functional equation

First, we remark that our system of differential equations is well adapted to a Laplace-Fourier transformation. As a matter of fact, the transition functions are bounded on their appropriate regions and locally integrable with respect to t. Consequently, each Laplace transform exists for $\operatorname{Re} z > 0$. Moreover, the integrability of the density functions and the transition functions with regard to u, x, and y also implies the integrability of the corresponding partial derivatives.

Applying a Laplace-Fourier transform technique to (4.3)–(4.6) and taking the initial condition into account reveal that for $\operatorname{Re} z > 0$, $\operatorname{Im} \omega \geq 0$, $\operatorname{Im} \eta \geq 0$, and $\operatorname{Im} \zeta \geq 0$,

$$\begin{aligned}(\lambda_s + z + i\zeta)\int_0^\infty e^{-zt}\mathbf{E}\left(e^{i\zeta U_t}\mathbf{1}\{N_t = A\}\right)dt + p_A^*(z,0)\\ = p_B^*(z,0)\mathbf{E}e^{i\zeta f} + \int_0^\infty e^{i\zeta u}p_C^*(z,u,0)du + \mathbf{E}e^{i\zeta f},\end{aligned} \tag{5.1}$$

$$(\lambda + z + i\omega)\int_0^\infty e^{-zt}\mathbf{E}\left(e^{i\omega X_t}\mathbf{1}\{N_t = B\}\right)dt + p_B^*(z,0) = p_A^*(z,0)\mathbf{E}e^{i\omega r}, \tag{5.2}$$

$$\begin{aligned}(z + i\zeta + i\eta)\int_0^\infty e^{-zt}\mathbf{E}\left(e^{i\zeta U_t}e^{i\eta Y_t}\mathbf{1}\{N_t = C\}\right)dt + \int_0^\infty e^{i\zeta u}p_C^*(z,u,0)du + \int_0^\infty e^{i\eta y}p_C^*(z,0,y)dy\\ = \lambda_s\int_0^\infty e^{-zt}\mathbf{E}\left(e^{i\zeta U_t}\mathbf{1}\{N_t = A\}\right)dt\, \mathbf{E}e^{i\eta r_s},\end{aligned} \tag{5.3}$$

$$zp_D^*(z) = \lambda p_B^*(z) + \left.\int_0^\infty e^{i\eta y}p_C^*(z,0,y)dy\right|_{\eta=0}. \tag{5.4}$$

Adding (5.1) and (5.3) yields the functional equation

$$\begin{aligned}\left(\lambda_s\left(1 - \mathbf{E}e^{i\eta r_s}\right) + z + i\zeta\right)\int_0^\infty e^{-zt}\mathbf{E}\left(e^{i\zeta U_t}\mathbf{1}\{N_t = A\}\right)dt\\ + p_A^*(z,0) - p_B^*(z,0)\mathbf{E}e^{i\zeta f} + \int_0^\infty e^{i\eta y}p_C^*(z,0,y)dy\\ + (z + i\zeta + i\eta)\int_0^\infty e^{-zt}\mathbf{E}\left(e^{i\zeta U_t}e^{i\eta Y_t}\mathbf{1}\{N_t = C\}\right)dt = \mathbf{E}e^{i\zeta f}.\end{aligned} \tag{5.5}$$

6. Survival function

In order to obtain the Laplace transform of the survival function, we first remark that by (5.4) and (3.6),

$$\mathbf{E}e^{-z\theta} = \lambda p_B^*(z) + \left.\int_0^\infty e^{i\eta y}p_C^*(z,0,y)dy\right|_{\eta=0}. \tag{6.1}$$

Inserting $\omega = i(\lambda + z)$ (resp., $\omega = 0$) into (5.2) entails that

$$p_B^*(z,0) = p_A^*(z,0)\mathbf{E}e^{-(\lambda+z)r}, \tag{6.2}$$

$$(z+\lambda)p_B^*(z) + p_B^*(z,0) = p_A^*(z,0). \tag{6.3}$$

Finally, inserting $\zeta = iz,\ \eta = 0$ into the functional equation (5.5) reveals that

$$\mathbf{E}e^{-zf} = p_A^*(z,0) - p_B^*(z,0)\mathbf{E}e^{-zf} + \int_0^\infty e^{i\eta y} p_C^*(z,0,y)dy\bigg|_{\eta=0}. \tag{6.4}$$

Invoking the relation

$$\mathfrak{R}^*(z) = \frac{1-\mathbf{E}e^{-z\theta}}{z}, \quad \operatorname{Re} z > 0, \tag{6.5}$$

yields by (6.1)–(6.4) that

$$\mathfrak{R}^*(z) = \frac{1-\mathbf{E}e^{-zf}}{z}\left(1 + p_A^*(z,0)\mathbf{E}e^{-(\lambda+z)r}\right) + p_A^*(z,0)\frac{1-\mathbf{E}e^{-(z+\lambda)r}}{z+\lambda}. \tag{6.6}$$

Hence, we only have to determine $p_A^*(z,0)$.

7. Methodology

In order to derive the unknown $p_A^*(z,0)$, we first eliminate the function

$$\int_0^\infty e^{-zt}\mathbf{E}\left(e^{i\zeta U_t}e^{i\eta Y_t}\mathbf{1}\{N_t = C\}\right)dt, \tag{7.1}$$

by the substitution of $\eta = \tau,\ \zeta = -\tau + iz,\ \tau \in \mathbf{R},\ \operatorname{Re} z > 0$. Noting that $z + i\zeta + i\eta = 0$ reveals that by (5.5),

$$\begin{aligned} &p_A^*(z,0)\left(1 - \mathbf{E}e^{-(z+\lambda)r}\mathbf{E}e^{-i(\tau-iz)f}\right) + \int_0^\infty e^{i\tau y}p_C^*(z,0,y))dy \\ &\quad - \left(1 + \lambda_s \mathbf{E}r_s\gamma_s^+(\tau)\right)i\tau\int_0^\infty e^{-zt}\mathbf{E}\left(e^{-i(\tau-iz)U_t}\mathbf{1}\{N_t = A\}\right)dt = \mathbf{E}e^{-i(\tau-iz)f}. \end{aligned} \tag{7.2}$$

Dividing (7.2) by the factor $1 + \lambda_s\mathbf{E}r_s\gamma_s^+(\tau)$, taking Property 3.1 into account, yields the boundary value equation

$$\psi^+(z,\tau) - \psi^-(z,\tau) = \left(1 + p_A^*(z,0)\mathbf{E}e^{-(\lambda+z)r}\right)\varphi(z,\tau), \tag{7.3}$$

where

$$
\begin{aligned}
\psi^{+}(z,\omega) &:= \frac{\phi^{+}(z,\omega)}{1+\lambda_s \mathbf{E} r_s \gamma_s^{+}(\omega)}, \\
\phi^{+}(z,\omega) &:= \int_0^{\infty} e^{i\omega y} p_C^{*}(z,0,y)\,dy - \lambda_s \mathbf{E} r_s \gamma_s^{+}(\omega) p_A^{*}(z,0), \quad \operatorname{Im}\omega \geq 0, \\
\psi^{-}(z,\omega) &:= i\omega \int_0^{\infty} e^{-zt} \mathbf{E}\left(e^{-i(\omega-iz)U_t}\mathbf{1}\{N_t = A\}\right)dt - p_A^{*}(z,0), \quad \operatorname{Im}\omega \leq 0, \\
\varphi(z,\tau) &:= \frac{\mathbf{E} e^{-i(\tau-iz)f}}{1+\lambda_s \mathbf{E} r_s \gamma_s^{+}(\tau)}.
\end{aligned}
\tag{7.4}
$$

Equation (7.3) constitutes a z-dependent Sokhotski-Plemelj problem on $\mathbf{R}$, solvable by the theory of sectionally holomorphic functions (see, e.g., [21]). First, we need the following property.

Lemma 7.1. *The function $\varphi(z,\tau)$, $\operatorname{Re} z \geq 0$, is Lipschitz-continuous on $\mathbf{R}$ and at infinity.*

Proof. Note that Property 3.1 implies that $\sup_{\tau\in\mathbf{R}} |1+\lambda_s \mathbf{E} r_s \gamma_s^{+}(\tau)|^{-1} < \infty$. Hence, the existence of $\mathbf{E}f$, $\mathbf{E}r_s$, and $\mathbf{E}r_s^2$ entails that

$$
\sup_{\tau\in\mathbf{R}} \left| \frac{\partial}{\partial\tau} \varphi(z,\tau) \right| < \infty. \tag{7.5}
$$

Consequently, by the mean value theorem (see, e.g., [24]), there exists a constant K such that for all $\tau_1, \tau_2 \in \mathbf{R}$,

$$
|\varphi(z,\tau_1) - \varphi(z,\tau_2)| \leq K|\tau_1 - \tau_2|. \tag{7.6}
$$

Hence, $\varphi(z,\tau)$ is Lipschitz-continuous on $\mathbf{R}$.

Finally, note that the Lipschitz continuity of $\varphi(z,\tau)$ at infinity follows from the boundedness of $|1+\lambda_s \mathbf{E} r_s \gamma_s^{+}(\tau)|^{-1}$ and (2.3). □

Corollary 7.2. *The function*

$$
\frac{1}{2\pi i} \int_{\Gamma} \varphi(z,\tau) \frac{d\tau}{\tau-\omega}, \quad \omega \in \mathbf{C}, \tag{7.7}
$$

is sectionally holomorphic and regular.

Moreover, by (7.3),

$$
\psi^{-}(z,\omega) = \left(1 + p_A^{*}(z,0) \mathbf{E} e^{-(z+\lambda)r}\right) \frac{1}{2\pi i} \int_{\Gamma} \varphi(z,\tau) \frac{d\tau}{\tau-\omega}, \quad \omega \in \mathbf{C}^{-}. \tag{7.8}
$$

Note that (7.8) is only valid for $\omega \in \mathbf{C}^-$. However, by the Sokhotski-Plemelj formula (see, e.g., [21, page 36]),

$$\lim_{\substack{\omega \to 0 \\ \omega \in \mathbf{C}^-}} \psi^-(z,\omega) = \left(1 + p_A^*(z,0)\mathbf{E}e^{-(z+\lambda)r}\right)\alpha(z), \tag{7.9}$$

where

$$\alpha(z) := \lim_{\substack{\omega \to 0 \\ \omega \in \mathbf{C}^-}} \frac{1}{2\pi i}\int_\Gamma \varphi(z,\tau)\frac{d\tau}{\tau-\omega} = -\frac{1}{2}\varphi(z,0) + \frac{1}{2\pi i}\int_\Gamma \varphi(z,\tau)\frac{d\tau}{\tau}. \tag{7.10}$$

On the other hand, we have by continuity $\lim_{\substack{\omega \to 0 \\ \omega \in \mathbf{C}^-}} \psi^-(z,\omega) = \psi^-(z,0) = -p_A^*(z,0)$. Hence,

$$p_A^*(z,0) = -\frac{\alpha(z)}{1+\alpha(z)\mathbf{E}e^{-(z+\lambda)r}}. \tag{7.11}$$

The function $\mathfrak{R}^*(z)$ is now completely determined by (7.11) and (6.6). We summarize the following result.

Property 7.3. The Laplace transform of the survival function is given by

$$\mathfrak{R}^*(z) = \frac{1}{1+\alpha(z)\mathbf{E}e^{-(z+\lambda)r}}\left\{\frac{1-\mathbf{E}e^{-zf}}{z} - \frac{1-\mathbf{E}e^{-(z+\lambda)r}}{z+\lambda}\alpha(z)\right\}, \tag{7.12}$$

where

$$\begin{gathered}\alpha(z) = -\frac{1}{2}\frac{\mathbf{E}e^{-zf}}{1+\lambda_s\mathbf{E}r_s} + \frac{1}{2\pi i}\int_\Gamma \frac{\mathbf{E}e^{-i(\tau-iz)f}}{1+\lambda_s\mathbf{E}r_s\gamma_s^+(\tau)}\frac{d\tau}{\tau}, \\ \gamma_s^+(\tau) := \begin{cases} \dfrac{\mathbf{E}e^{i\tau r_s}-1}{i\tau\mathbf{E}r_s} & \text{if } \tau \neq 0, \\ 1 & \text{if } \tau = 0. \end{cases}\end{gathered} \tag{7.13}$$

Remark 7.4. It should be noted that Property 3.1 also holds for an *arbitrary* R_s with finite mean. Moreover, the existence of moments does not depend on the canonical structure (Lebesgue decomposition) of the underlying distribution. For instance, the inequality

$$\left|\frac{\partial}{\partial\tau}\gamma_s^+(\tau)\right| \leq \frac{1}{\mathbf{E}r_s}\int_0^\infty x\left(1-R_s(x)\right)dx = \frac{1}{2}\frac{\mathbf{E}r_s^2}{\mathbf{E}r_s} < \infty \tag{7.14}$$

also holds for an arbitrary R_s with finite mean and variance. Therefore, Lemma 7.1 remains valid for arbitrary R_s. The requirement of a finite variance $\sigma_{r_s}^2$ is extremely mild. In fact,

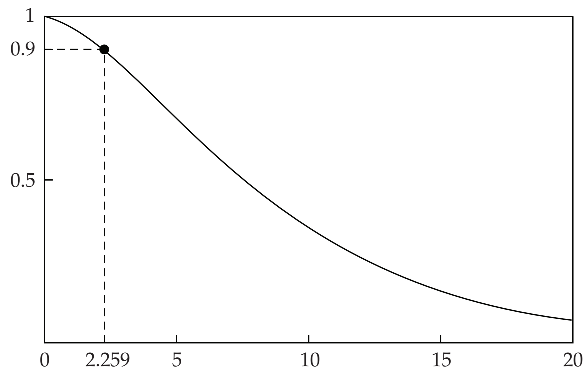

Figure 1: Graph of $\mathfrak{R}(t)$, $0 \leq t \leq 20$, $\lambda = 0.5$, and $\lambda_s = 0.25$ with security interval $[0, \tau]$, $\tau = 2.259$, and security level $\delta = 0.9$.

the current probability distributions of interest to statistical reliability engineering even have moments of any order. Finally, the functional

$$\mathbf{E}e^{-(z+\lambda)r} = \int_0^\infty e^{-(z+\lambda)x}\, \mathrm{d}R(x) \tag{7.15}$$

exists for an arbitrary R as a Lebesgue-Stieltjes integral on $[0, \infty)$ and has no impact on the existence of the integral

$$\frac{1}{2\pi i}\int_\Gamma \varphi(z, \tau)\frac{d\tau}{\tau - \omega}, \quad \omega \in C. \tag{7.16}$$

Consequently, Property 7.3 holds for arbitrary repair time distributions.

8. Risk criterion

Along with the survival function of the **T**-system, we now introduce a security interval $[0, \tau)$, where

$$\tau := \sup\left\{t \geq 0 : \mathfrak{R}(t-) > \delta\right\} \tag{8.1}$$

for some $0 < \delta < 1$, which is called the security level. In practice, δ is usually large. For instance, $\delta = 0.9$. Therefore, we require that the **T**-system satisfy the risk criterion $\lim_{t\uparrow\tau}\mathfrak{R}(t) > \delta \gg 0$. Note that the security interval, corresponding to the security level δ, ensures a continuous operation (survival) of the **T**-system up to time τ with probability larger than δ. See the forthcoming example.

9. Deterministic repair

As an example, we consider the particular case of deterministic repair (replacement); that is, let $R(\cdot) = R_s(\cdot) = H_{t_0}(\cdot)$, where $t_0 = 1$ is taken as time unit. Clearly, $\mathbf{E}e^{-zr} = \mathbf{E}e^{-zr_s} = e^{-z}$.

Furthermore, let $F(u) = 1 - e^{-\lambda u},\ u \geq 0$. Note that

$$\mathbf{E}e^{-i(\tau - iz)f} = \frac{-i\lambda}{\tau - i(\lambda + z)}. \tag{9.1}$$

By Property 7.3, we have

$$\Re^*(z) = \frac{1}{\lambda + z}\left(1 - \frac{\alpha(z)}{1 + \alpha(z)e^{-(\lambda+z)}}\right). \tag{9.2}$$

We recall that

$$\alpha(z) = \lim_{\substack{\omega \to 0 \\ \omega \in \mathbf{C}^-}} \frac{1}{2\pi i}\int_\Gamma \varphi(z,\tau)\frac{d\tau}{\tau - \omega} = \lim_{\substack{\omega \to 0 \\ \omega \in \mathbf{C}^-}} \frac{1}{2\pi i}\int_\Gamma \frac{-i\lambda(\tau-\omega)^{-1}\left(\tau - i(\lambda+z)\right)^{-1}}{1 + \lambda_s \mathbf{E}r_s\gamma_s^+(\tau)} d\tau. \tag{9.3}$$

For $\omega \in \mathbf{C}^-,\ \operatorname{Re} z \geq 0$, the integrand represents a *meromorphic* function in $\mathbf{C}^+$ with single pole $i(\lambda + z)$. Moreover, the function vanishes at infinity in $\mathbf{C}^+ \cup \mathbf{R}$. An application of the residue theorem entails that

$$\alpha(z) = \lim_{\substack{\omega \to 0 \\ \omega \in \mathbf{C}^-}} \frac{i\lambda(\lambda + z)}{\left(\omega - i(\lambda+z)\right)\left(\lambda + z + \lambda_s\left(1 - e^{-(\lambda+z)}\right)\right)} = -\frac{\lambda}{\lambda + z + \lambda_s\left(1 - e^{-(\lambda+z)}\right)}. \tag{9.4}$$

Hence, by (9.2),

$$\Re^*(z) = \frac{1}{z + \lambda}\left(1 + \frac{\lambda}{z + \lambda_s + \lambda - (\lambda_s + \lambda)e^{-(z+\lambda)}}\right). \tag{9.5}$$

Applying the inversion technology presented in [25] yields the *exact* survival function

$$\Re(t) = e^{-\lambda t}\left(1 + \frac{\lambda}{\lambda_s}\sum_{k=0}^{[t]}\left(\frac{\lambda}{\lambda + \lambda_s}\right)^k\left(1 - e^{-\lambda_s(t-k)}\sum_{j=0}^{k}\frac{(\lambda_s(t-k))^j}{j!}\right)\right). \tag{9.6}$$

Figure 1 displays the graph of $\Re(t),\ 0 \leq t \leq 20,\ \lambda = 0.5$, and $\lambda_s = 0.25$ with the security interval $[0, \tau]$, $\tau = 2.259$. The interval ensures a continuous operation of the **T**-system up to time $\tau = 2.259$ with a probability of at least 90%.

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