Research Article

Survival Exponents for Some Gaussian Processes

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The problem is a power-law asymptotics of the probability that a self-similar process does not exceed a fixed level during long time. The exponent in such asymptotics is estimated for some Gaussian processes, including the fractional Brownian motion (FBM) in $(-T_-, T)$, $T \geq T_- \gg 1$ and the integrated FBM in $(0, T)$, $T \gg 1$.

1. The Problem

Let $x(t)$, $x(0) = 0$ be a real-valued stochastic process with the following asymptotics:

$$P(x(t) < 1, t \in \Delta_T) = T^{-\theta_x + o(1)}, \quad T \rightarrow \infty, \quad (1.1)$$

where θ_x is the so-called *survival* exponent of $x(t)$. Below we focus on estimating θ_x for some self-similar Gaussian processes in extended intervals $\Delta_T = (0, T)$ and $(-T_-, T)$, $T \geq T_- \gg 1$. Usually the estimation of the survival exponents is based on Slepian's lemma. The estimation requires reference processes with explicit or almost explicit values of θ . Unfortunately, the list of such processes is very short. This includes the fractional Brownian motion (FBM), $w_H(t)$, of order $0 < H < 1$ both with one- and multidimensional time. According to Molchan ([1])

$$\theta_{w_H} = 1 - H \quad \text{for } \Delta_T = (0, T), \quad \theta_{w_H} = d \quad \text{for } \Delta_T = (-T, T)^d. \quad (1.2)$$

Another important example is the integrated Brownian motion $I(t) = \int_0^t w(s)ds$ with the exponent

$$\theta_I = \frac{1}{4}, \quad \Delta_T = (0, T), \quad (1.3)$$

(Sinai [2]).

The nature of this result is best understood in terms of a series of generalizations where the integrand is a random walk with discrete or continuous time (see, e.g., Isozaki and Watanabe [3]; Isozaki and Kotani [4]; Simon [5]; Vysotsky [6, 7]; Aurzada and Dereich [8]; Dembo et al. [9]; Denisov and Wachtel [10]). The extension of (1.3) to include the case of the integrated fractional Brownian motion, $I_H(t) = \int_0^t w_H(s)ds$, remains an important; but as yet unsolved problem.

Below we consider the survival exponents for the following Gaussian processes: $I_H(t)$, $t \in (0, T)$; $\chi_H(t) = \text{sign}(t)w_H(t)$, $t \in (-T, T)$; FBM in $\Delta_T = (-T^\alpha, T)$, $0 \leq \alpha \leq 1$; the Laplace transform of white noise with $\Delta_T = (0, T)$; the fractional Slepian's stationary process whose correlation function is $B_{S_H}(t) = (1 - |t|^{2H})_+$, $0 < H \leq 1/2$.

Our approach to the estimation of θ is more or less traditional. Namely, any self-similar process $x(t)$ in $\Delta_T = (0, T)$ generates a *dual stationary process* $\tilde{x}(s) = e^{-hs}x(e^s)$, $s < \ln T := \tilde{T}$, where h is the self-similarity index of $x(t)$. For a large class of Gaussian processes, relation (1.1) induces the dual asymptotics

$$P(\tilde{x}(s) \leq 0, 0 < s < \tilde{T}) = \exp(-\tilde{\theta}_x \tilde{T}(1 + o(1))), \quad \tilde{T} \rightarrow \infty, \quad (1.4)$$

with the same exponent $\tilde{\theta}_x = \theta_x$, [1, 11]. More generally, the dual exponent is defined by the asymptotics

$$P(x(t) \leq 0, t \in \Delta_T \setminus (-1, 1)) = \exp(-\tilde{\theta}_x \tilde{T}(1 + o(1))). \quad (1.5)$$

To formulate the simplest condition for the exponents to be equal, we define one more exponent $\check{\theta}_x$ by means of the asymptotics

$$P(|t_T^*| \leq 1) = T^{-\check{\theta}_x + o(1)}, \quad (1.6)$$

where t_T^* is the position of the maximum of $x(t)$ in Δ_T , that is, $x(t_T^*) = \sup(x(t), t \in \Delta_T)$.

Lemma 1.1 (see [1, 11]). *Let $x(t)$, $x(0) = 0$ be a self-similar continuous Gaussian process in $\Delta_T = (-T_-, T)$, $T_- \leq T$ and $(H_x(\Delta_T), \|\cdot\|_T)$ be the reproducing kernel Hilbert space associated with $x(t)$. Suppose that there exists such an element φ of $H_x(\Delta_T)$ that $\varphi(t) \geq 1$, $|t| > 1$ and $\|\varphi\|_T^2 = o(\ln T)$. Then θ_x , $\tilde{\theta}_x$, and $\check{\theta}_x$ can exist simultaneously only; moreover, the exponents are equal to each other.*

The equality $\theta = \tilde{\theta}$ reduces the original problem to the estimation of $\tilde{\theta}$. Nonnegativity of the correlation function of $\tilde{x}(s)$ guarantees the existence of the exponent $\tilde{\theta}$, [12]. In turn, the inequality of two correlation functions, $B_1(s) \leq B_2(s)$, $B_i(0) = 1$, implies, by Slepian's lemma, the inverted inequality for the corresponding exponents: $\tilde{\theta}_1 \geq \tilde{\theta}_2$.

An essentially different approach is required to find the explicit value of θ for FBM in $\Delta_T = (-T^\alpha, T)$ and to estimate $\tilde{\theta}$ in (1.4) for the fractional Slepian process with a small parameter H .

Figure 1: The survival exponents $\tilde{\theta}_{I_H}$ for the integrated fractional Brownian motion in $\Delta_T = (-T, T)$: hypothetical values (parabolic line), empirical estimates (small circles, squares), and theoretical bounds (shaded zone given by Proposition 2.1(b, c)). The empirical exponents are based on the model ((2.9), $\alpha(H) = 0$) in three time intervals of $\tilde{T} = \ln T : \ln(1/\varepsilon) \leq \tilde{T}(1-H)H \leq \ln(10/\varepsilon)$, where $\varepsilon = 0.01, 0.003$, and 0.001 (see more in [13]).

2. Examples

2.1. Integrated Fractional Brownian Motion

Consider the process

$$I_H(t) = \int_0^t w_H(s) ds, \quad (2.1)$$

where $w_H(t)$ is the fractional Brownian motion, that is, a Gaussian random process with the stationary increments: $E|w_H(t) - w_H(s)|^2 = |t - s|^{2H}$, $w_H(0) = 0$. Molchan and Khokhlov [13, 14] analyzed the exponent θ_{I_H} theoretically and numerically and formulated the following Hypothesis: $\theta_{I_H} = H(1-H)$ for $\Delta_T = (0, T)$ and $\theta_{I_H} = 1-H$ for $\Delta_T = (-T, T)$.

The unexpected symmetry $\theta_{I_H} = \theta_{I_{1-H}}$ for $\Delta_T = (0, T)$ caused some doubt as to the numerical results. To support the hypothesis, Molchan [11] derived the following estimates of θ for $I_H(t)$:

$$\rho H(1-H) \leq \theta_{I_H}^+ \leq \theta_{I_H}^{-/+} \leq (1-H), \quad (2.2)$$

where ρ is a small constant and (+) and (-/+) are indicators of the intervals $\Delta_T = (0, T)$ and $\Delta_T = (-T, T)$, respectively. Note that, in the case of $H < 1/2$ and $\Delta_T = (-T, T)$, it is unknown whether the exponent exists. In such cases we have to operate with upper $\bar{\theta}$ and lower $\underline{\theta}$ exponents. Therefore, $\theta_{I_H}^{-/+}$ in (2.2) for $H < 1/2$ is any number from the interval $(\underline{\theta}, \bar{\theta})$. The relation (2.2) can be improved as follows.

Proposition 2.1. For $\Delta_T = (0, T)$, one has

- (a) $\theta_{I_H} \geq \theta_{I_{1-H}}$, $0 < H \leq 0.5$,
- (b) $0.5(H \wedge \bar{H}) \leq \theta_{I_H} \leq \bar{H}$, $\bar{H} = 1-H$,

$$(c) \theta_{I_H} \leq \sqrt{(1 - (H \wedge \overline{H})^2)/12}.$$

Proof. The identity of the dual exponents for $I_H(t)$ follows from [14]; the dual survival exponent exists because the dual correlation function,

$$\tilde{B}_{I_H}(s) = (2 + 4H)^{-1} \left[(2 + 2H) \left(e^{Hs} + e^{-Hs} \right) - e^{(1+H)s} - e^{-(1+H)s} + \left(e^{s/2} - e^{-s/2} \right)^{2H+2} \right] \quad (2.3)$$

is positive. The inequality (a) is a consequence of the relation

$$\tilde{B}_{I_H}(t) \leq \tilde{B}_{I_{1-H}}(t), \quad 0 < H \leq \frac{1}{2}. \quad (2.4)$$

To prove (b, c), we use the correlation function of the process $\tilde{I}_{1/2}(ps)$, that is,

$$\tilde{B}_{I_{1/2}}(ps) = \frac{1}{2} \left(3 \exp\left(-\frac{p|s|}{2}\right) - \exp\left(-\frac{3p|s|}{2}\right) \right), \quad (2.5)$$

and the respective exponent $\tilde{\theta} = p/4$ (see (1.3)). The relation

$$\tilde{B}_{I_H}(t) \leq \tilde{B}_{I_{1/2}}(pt), \quad H \geq \frac{1}{2}, \quad p = 2(1 - H), \quad (2.6)$$

implies $\theta_{I_H} \geq (1 - H)/2$ for $H \geq 1/2$. Using (a) in addition, we come to the lower bound in (b) because $\theta_{I_H} \geq \theta_{I_{1-H}} \geq H/2$ for $H \leq 1/2$.

Similarly, the relation

$$\tilde{B}_{I_H}(t) \geq \tilde{B}_{I_{1/2}}(pt), \quad H \leq \frac{1}{2}, \quad p = 2\sqrt{\frac{1 - H^2}{3}}, \quad (2.7)$$

implies (c) for all H . A test of the purely analytical facts (2.4), (2.6), and (2.7) is given in the appendix. $\square$

Remark 2.2. Proposition 2.1(a) follows from the more informative relation:

$$P\left(\tilde{I}_H(s) \leq 0, s \in (0, \tilde{T})\right) \leq P\left(\tilde{I}_{1-H}(s) \leq 0, s \in (0, \tilde{T})\right). \quad (2.8)$$

This inequality is important for understanding the numerical result by Molchan and Khokhlov [13] represented in the form of empirical estimates of $\tilde{\theta}_{I_H}$ in Figure 1. We can see that the empirical estimates show small but one-sided deviations from the hypothetical curve $\theta = H(1 - H)$ before and after $H = 1/2$. The signs of these deviations are consistent with (2.8), while the amplitudes are compatible with the model

$$P\left(\tilde{I}_H(s) \leq 0, s \in (0, \tilde{T})\right) \approx C \tilde{T}^{\alpha(H)} \exp\left(-H(1 - H)\tilde{T}\right), \quad \tilde{T} \gg 1, \quad \text{sgn } \alpha(H) = \text{sign}(H - 0.5), \quad (2.9)$$

and $\alpha(H) = H - 0.5$ (more can be found in [13]).

2.2. The Laplace Transform of White Noise

Consider the process $L(t) = t \int_0^\infty e^{-tu} dw(u)$, where $w(u)$ is Brownian motion. The dual stationary process $\tilde{L}(s)$ has the correlation function $\tilde{B}_L(s) = 1/\cosh(s/2)$. Using (2.5) as a majorant of $\tilde{B}_L(s)$, we improve the lower bound of $\tilde{\theta}_L$ as follows.

Proposition 2.3. $3^{-1/2} \leq 4\tilde{\theta}_L \leq 1$.

Proof. That the exponents for the dual processes L and $\tilde{L}$ are equal follows from Lemma 1.1 with $\varphi(t) = t(1 + \varepsilon_T)/(t + \varepsilon_T)$, where $\varepsilon_T = 1/\sqrt{\ln T}$. For indeed, $\varphi(t) = EL(t)\eta$, where $\eta = (1 + \varepsilon_T^{-1})L(\varepsilon_T)$. By definition of the Hilbert space $H_L(\Delta_T)$, we have the desired estimate:

$$\|\varphi\|_T^2 = E\eta^2 = \frac{\varepsilon_T^{-1}(\varepsilon_T + 1)^2}{2} = O(\sqrt{\ln T}). \quad (2.10)$$

By (1.3) and Slepian's lemma, the relation

$$\tilde{B}_{L_{1/2}}(t) \leq \tilde{B}_L(pt), \quad p \leq 1 \quad (2.11)$$

has as a consequence the estimate $4p\tilde{\theta}_L \leq 1$. The opposite inequality

$$\tilde{B}_{L_{1/2}}(t) \geq \tilde{B}_L(pt), \quad p^2 \geq 3, \quad (2.12)$$

implies $4p\tilde{\theta}_L \geq 1$. The test of ((2.11), $p = 1$) and ((2.12), $p = 2$) is very simple and yields the Li and Shao [12, 15] estimates: $0.5 < 4\tilde{\theta}_L < 1$. The appendix contains a proof of (2.11), (2.12) for all interesting values of p : 1, 2, and $\sqrt{3}$. $\square$

Remark 2.4. The *dual survival* exponent of $L(t)$ is of interest as a parameter of the following asymptotic relation:

$$P\left(\sum_0^{2n} \xi_i x^i \neq 0, x \in R^1\right) = (2n)^{-4\tilde{\theta}_L + o(1)}, \quad n \longrightarrow \infty, \quad (2.13)$$

for random polynomials with the standard Gaussian independent coefficients [16]. A continuous analogue of the polynomial on any of four intervals $0 < \pm x^{\pm 1} \leq 1$ is the Laplace transform of white noise, which partially explains the appearance of $\tilde{\theta}_L$ in the asymptotic relation (2.13). Simulations suggest $4\tilde{\theta}_L = 0.76 \pm 0.03$, [16] and $4\tilde{\theta}_L \approx 0.75$, [17].

2.3. Fractional Slepian's Process

We reserve this term for a Gaussian stationary process $S_H(t)$ with correlation function

$$B_{S_H}(t) = \left(1 - |t|^{2H}\right)_+, \quad 0 < H \leq \frac{1}{2}, \quad (2.14)$$

because $S_{1/2}(t)$ is known as the Slepian process and $S_H(t) - S_H(0)$, $0 < t \leq 1$, is equal in distribution to the fractional Brownian motion on the interval $(0,1)$. By the Polya criterion, the fractional Slepian process exists because $B_{S_H}(t)$ is a nonincreasing and a convex function on the semiaxis. The fact of the correlation function being nonnegative guarantees the existence of $\tilde{\theta}_{S_H}$ in (1.4). $S_H(t)$ can be useful as a reference process in estimation of the survival exponents. Therefore it is important to have accurate estimates of the exponent for $S_H(t)$. The case of small H is the most interesting because it describes a transition of $S_H(t)$ to white noise. Our estimates of $\tilde{\theta}_{S_H}$ are based on two lemmas, where we use the following notation:

$$\tilde{\theta}(f, \Delta) = -|\Delta|^{-1} \log P(x(t) \leq f(t), t \in \Delta). \quad (2.15)$$

Lemma 2.5 (see [12]). *Let $x(t)$ be a centered Gaussian stationary process with a finite nonnegative correlation function, that is, $B_x(t) \geq 0$ and $B_x(t) = 0$ for $|t| \geq T_0$. Then the limit*

$$\tilde{\theta}(a) = \lim_{T \rightarrow \infty} \tilde{\theta}(a, (0, T)), \quad (2.16)$$

exists for every $a \in \mathbb{R}^1$. Moreover,

$$\left(1 + \frac{1}{k}\right)^{-1} \tilde{\theta}(a, k\Delta_0) \leq \tilde{\theta}(a) \leq \tilde{\theta}(a, k\Delta_0), \quad \Delta_0 = (0, T_0). \quad (2.17)$$

Remark 2.6. Lemma 1.1 was derived by Li and Shao [12] for the Slepian process, $S_{1/2}(t)$, but the proof remains valid for the general case. There is an explicit but very complicated formula for $\tilde{\theta}_{S_H}(0, \Delta)$ with $H = 1/2$ [18]. In case of $\Delta = (0, 2)$, this result reduces to

$$P(S_{1/2}(t) \leq 0, t \in (0, 2)) = \frac{1}{6} - \frac{2 + \sqrt{3}}{8\pi} \quad (2.18)$$

and gives $1.336 < \tilde{\theta}_{S_{1/2}} < 2.004$.

Lemma 2.7 (see [8]). *Let $x(t)$ be a centered Gaussian process in an interval Δ with a correlation function $B(t, s)$ and $(H_x(\Delta), \|\cdot\|_\Delta)$ be the Hilbert space with the reproducing kernel $B(t, s)$ on $\Delta \times \Delta$. If $0 < \tilde{\theta}(a, \Delta) < \infty$, then*

$$\left| \sqrt{\tilde{\theta}(a + f, \Delta)} - \sqrt{\tilde{\theta}(a, \Delta)} \right| \leq \frac{\|f\|_\Delta}{\sqrt{2|\Delta|}}. \quad (2.19)$$

Remark 2.8. Lemma 2.7 is a version of Proposition 1.6 from the paper by Aurzada and Dereich [8]; relation (2.19) successfully supplements the original Lemma 1.1.

Proposition 2.9. *The persistence exponent of process $S_H(t)$ has the following estimates:*

$$-(1-H)H^{-1} \ln(2H) \leq \tilde{\theta}_{S_H} \leq 49H^{-2}, \quad (2.20)$$

where the left inequality holds for $0 < H \leq e^{-2}/2$.

Corollary 2.10. *If $w_H^-(t) = (w_H(t) - w_H(-t))/2$ is the odd component of the fractional Brownian motion, then*

$$\tilde{\theta}_{w_H^-} \leq \frac{(7/H)^2}{2}, \quad 0 < H < 0.5. \quad (2.21)$$

Proof. The dual stationary process $\tilde{w}_H^-$ has the correlation function

$$\tilde{B}_{w_H^-}(t) = \left(\cosh \frac{t}{2} \right)^{2H} - \left(\sinh \frac{t}{2} \right)^{2H}, \quad (2.22)$$

which is positive. Hence the exponent $\tilde{\theta}_{w_H^-}$ exists. The inequality

$$\tilde{B}_{w_H^-}(2t) = (\cosh t)^{2H} (1 - (\tanh t)^{2H}) \geq (\cosh t)^{2H} (1 - |t|^{2H})_+ \geq \tilde{B}_{S_H}(t), \quad (2.23)$$

and Proposition 2.9 immediately imply the corollary. $\square$

Remark 2.11. The following estimates of $\tilde{\theta}_{w_H^-}$ are due to Krug et al. [19]:

$$\begin{aligned} \tilde{\theta}_{w_H^-} &\geq \min \left(\frac{(1-H)^2}{H}, (1-H)2^{1/(2H)-1} \right), \quad 0 < H < 0.5, \\ \tilde{\theta}_{w_H^-} &\leq \frac{(1-H)^2}{H}, \quad 0.1549 < H < 0.5. \end{aligned} \quad (2.24)$$

For small H these estimates are one-sided only.

Remark 2.12. A considerable difference in the behavior of $\tilde{\theta}_{w_H^-}$ and $\tilde{\theta}_{w_H} = 1 - H$ for small H is expected. Heuristically this can be explained as follows. As $H \rightarrow 0$, the discrete processes $\tilde{w}_H^-(k\Delta)$ and $\tilde{w}_H(k\Delta)$ have different weak limits: $\{\xi_k\}$ and $\{\xi_k - \eta/\sqrt{2}\}$, respectively, where $\{\xi_k\}$ and η are independent standard Gaussian variables. The probability (1.4) for the limiting processes is quite different:

$$P(\xi_k < 0, k = 1 \div N) = 2^{-N}, \quad P(\xi_k - \eta \leq 0, k = 1 \div N) = (N+1)^{-1}. \quad (2.25)$$

Unfortunately, this argument fails to predict the behavior of $\tilde{\theta}_{S_H}$ for small H , because the step Δ cannot be arbitrary and is a function of H .

2.4. Khanin's Problem

The survival exponent for fractional Brownian motion in the intervals $\Delta_T = (-T, T)$ is independent of the parameter H : $\theta_{w_H} = 1$. This interesting fact follows from both self-similarity of w_H and the stationarity of its increments [1].

In the case $H < 0.5$, the variables $w_H(t)$ and $w_H(-t)$ are positive correlated. Therefore, a possible power-law asymptotics

$$P(w_H(t) < 1, -w_H(-t) < 1, t \in (0, T)) = T^{-\theta+o(1)}, \quad (2.26)$$

where we change sign before $w_H(t)$ for negative t only, may have a radically different exponent compared with $\theta_{w_H} = 1$. The question of finding bounds on the exponent θ_{χ_H} for the process

$$\chi_H(t) = \text{sign}(t)w_H(t), \quad \Delta_T = (-T, T), \quad (2.27)$$

was asked by K. Khanin. The next proposition contains a partial answer to this question.

Proposition 2.13. (1) *In the case $0.5 \leq H < 1$, the exponent θ_{χ_H} for $\Delta_T = (-T, T)$ exists and admits of the following estimates:*

$$1 < \theta_{\chi_H}(1-H)^{-1} \leq 2, \quad 0.5 \leq H < 1, \quad (2.28)$$

in addition, $\theta_{\chi_{1/2}} = 1$.

(2) *Let $\underline{\theta}_{\chi_H}$ be the lower exponent in (2.26), then*

$$\begin{aligned} \underline{\theta}_{\chi_H}(1-H)^{-1} &\geq (H^{-1} - 1) \wedge 2^{1/2H-1}, \quad 0 < H < 0.25, \\ \underline{\theta}_{\chi_H}(1-H)^{-1} &\geq 2 \quad 0.25 < H \leq 0.5. \end{aligned} \quad (2.29)$$

Remark 2.14. To clarify why $\theta_{\chi_H}/\theta_{w_H}$ is unbounded for small H in the case $\Delta_T = (-T, T)$, we consider again the limiting sequence for $w_H(k\Delta)$ as $H \rightarrow 0$. This is $\{(\xi_k - \xi_0)/\sqrt{2}\}$, where the $\{\xi_k\}$ are independent standard Gaussian variables. The probability (1.1) for the limit sequence is

$$P\{\xi_k < \xi_0 + \sqrt{2}, |k| \leq N\} = (2N+1)^{-1}l(N), \quad (2.30)$$

where $l(N)$ is a slowly varying function, whereas for the limit sequence of $\chi_H(k\Delta)$ we have

$$P\{\xi_{-k} - \sqrt{2} < \xi_0 < \xi_k + \sqrt{2}, 0 < k \leq N\} \approx \sqrt{\pi}eN^{-1/2}\Phi(\sqrt{2})^{2N}, \quad (2.31)$$

where $\Phi(x)$ is the Gaussian distribution function. As in Remark 2.12, we have nontrivial exponential asymptotics where the threshold for $\{\xi_k\}$ is constant or bounded. Indeed, if the event in (2.31) is true, then

$$|\xi_0| < \sqrt{2} + \frac{\max\left(\left|\sum_1^N \xi_{-k}\right|, \left|\sum_1^N \xi_k\right|\right)}{N} = \sqrt{2} + \frac{O_p(1)}{\sqrt{N}}. \quad (2.32)$$

2.5. An Explicit Value of θ_x

We have two explicit but isolated results for the fractional Brownian motion: $\theta_{w_H} = (1 - H)$ for $\Delta_T = (0, T)$ and $\theta_{w_H} = 1$ for $\Delta_T = (-T, T)$. These results can be combined as follows.

Proposition 2.15. *If $\Delta_T = (-T^\alpha, T)$, $0 \leq \alpha \leq 1$, then $\theta_{w_H} = \alpha H + (1 - H)$.*

Remark 2.16. The result is based on the following properties of the position t_Δ^* of the maximum of $w_H(t)$ in $\Delta = [0, 1]$: t_Δ^* has a continuous probability density $f_\Delta^*(t)$ in $(0, 1)$ and $f_\Delta^*(t) \approx O(t^{-H})$ as $t \rightarrow 0$. In the case of multidimensional time, the behavior of $f_\Delta^*(t)$, $\Delta = (0, 1)^d$ near $t = 0$ is a key to the survival exponent of $w_H(t)$ for $\Delta_T = (-T^\alpha, T)^d$, $0 < \alpha < 1$ and $H < 1$. By (1.2), $\theta_{w_H} = d$ in the case $\alpha = 1$, and $\theta_{w_H} = \alpha d$ in the degenerate case: $H = 1$.

3. Proofs

Proof of Proposition 2.9

Lower Bound. Let $\tilde{w}_H(t)$ be a dual fractional Brownian motion with the parameter H , that is, a Gaussian stationary process with correlation function $\tilde{B}_{w_H}(t) = \cosh(Ht) - 0.5(2 \sinh(t/2))^{2H}$. We prove in the appendix that for $0 < H \leq e^{-2}/2$,

$$\tilde{B}_{w_H}(pt) \geq B_{S_H}(t), \quad p = -H^{-1} \ln(2H). \quad (3.1)$$

Applying Slepian's lemma, one has $\tilde{\theta}_{S_H} \geq p(1 - H)$ because $\tilde{\theta}_{w_H} = (1 - H)$.

Upper Bound. The random variable $\eta = \int_0^1 S_H(t) dt$ corresponds to an element $f_\eta(t)$ of the Hilbert space, $H_S(\Delta)$, $\Delta = (0, 1)$, with the reproducing kernel $B(t, s) = 1 - |t - s|^{2H}$. By definition of $H_S(\Delta)$, we have

$$\begin{aligned} f_\eta(t) &= ES_H(t)\eta = 1 - \frac{t^{1+2H} + (1-t)^{1+2H}}{1+2H}, \\ \|f_\eta\|_\Delta^2 &= E\eta^2 = H(3+2H)(1+H)^{-1}(1+2H)^{-1}. \end{aligned} \quad (3.2)$$

It is easy to see that $f_\eta(0) \leq f_\eta(t) \leq f_\eta(1/2)$. Therefore,

$$H < f_\eta(t) < 2H \ln(2e), \quad \frac{4H}{3} < \|f_\eta\|_\Delta^2 < 3H. \quad (3.3)$$

Let m_H be the median of the random variable $M = \max\{S_H(t), t \in \Delta\}$, where $\Delta = (0, 1)$. Then

$$0.5 = P(S_H(t) < m_H, t \in \Delta) < P(S_H(t) < m_H H^{-1} f_\eta(t), t \in \Delta), \quad (3.4)$$

because $H^{-1} f_\eta(t) > 1$. Setting $x(t) = S_H(t)$ in Lemma 2.5 and using notation (2.15), one has

$$\begin{aligned} \tilde{\theta}(m_H H^{-1} f_\eta(t), \Delta) &< \ln 2, \\ \sqrt{\tilde{\theta}(0, \Delta)} &< \sqrt{\ln 2} + \frac{m_H H^{-1} \|f_\eta\|_\Delta}{\sqrt{2}}. \end{aligned} \quad (3.5)$$

Using Lemma 1.1 and the inequality $\|f_\eta\|_\Delta < \sqrt{3H}$, we have

$$\tilde{\theta}_{S_H} < \tilde{\theta}(0, \Delta) < \left(\sqrt{\ln 2} + m_H \sqrt{\frac{1.5}{H}} \right)^2. \quad (3.6)$$

It is well known (see, e.g., [20]) that $m_H < 4\sqrt{2}D(\Delta, \sigma/2)$, where $\sigma^2 = \max_\Delta ES_H(t)$ and D is the Dudley entropy integral related to the semimetrics on Δ : $\rho^2(t, s) = E(S_H(t) - S_H(s))^2$.

In our case $\rho(t, s) = \sqrt{2}|t - s|^H$, $\sigma = 1$ and therefore

$$m_H < \frac{c_H}{\sqrt{H}}, \quad (3.7)$$

where

$$c_H = 4\sqrt{(1-H)\ln 2} + 2^{3-H}\sqrt{\pi}\Phi(-\sqrt{1-H}\ln 4) < 5.36, \quad H < \frac{1}{2}, \quad (3.8)$$

and $\Phi(x)$ is the standard Gaussian distribution. Hence,

$$\tilde{\theta}_{S_H} < \left(\sqrt{\ln 2} + \frac{5.36\sqrt{1.5}}{H} \right)^2 < \left(\frac{7}{H} \right)^2. \quad (3.9)$$

Proof of Proposition 2.13

Part (1). In the case of $H \geq 0.5$, the process $\chi_H(t) = \text{sign}(t)w_H(t)$ has nonnegative correlations on R^1 . In the standard manner, this implies the existence of θ_{χ_H} for $\Delta_T = (-T, T)$. More precisely, starting from a self-similar 2D process $x(t) = (w_H(t), -w_H(-t))$ on R_+^1 , we consider the dual 2D stationary process $\tilde{x}(t) = x(e^t) \exp(-Ht)$ whose correlation matrix has positive elements. By [12], we conclude that the exponent $\tilde{\theta}_{\chi_H}$ for $\tilde{x}(t)$ exists.

The equality $\tilde{\theta}_{\chi_H} = \theta_{\chi_H}$ for $\Delta_T = (-T, T)$. We will use Lemma 1.1. By the relation $\chi_H(t) = \text{sign}(t)w_H(t)$, the map $\varphi(t) \mapsto \text{sign}(t)\varphi(t)$ is an isometry between the Hilbert spaces

$H_{\chi_H}(\Delta_T)$ and $H_{w_H}(\Delta_T)$ associated with $\chi_H(t)$ and $w_H(t)$ on $\Delta_T = (-T, T)$, respectively. To prove the equality of the dual exponents, it is enough to find $\varphi(t) \in (H_{w_H}(R^1), \|\cdot\|_R)$ such that $\text{sgn}(t)\varphi(t) \geq 1$ for $|t| \geq 1$. We can use

$$\varphi(t) = \text{sgn}(t) \min(|t|, 1) = \int \left(e^{it\lambda} - 1 \right) \frac{\sin \lambda}{i\pi\lambda^2} d\lambda, \quad (3.10)$$

because

$$\|\varphi\|_R^2 = k_H \int \frac{(\sin \lambda)^2}{(\pi\lambda^2)^2} |\lambda|^{1+2H} d\lambda < \infty, \quad (3.11)$$

(see [14]).

Estimation of θ_{χ_H} , $H > 1/2$. Since $E\chi_H(t)\chi_H(s) \geq 0$ for any t, s , we have, by Slepian's lemma,

$$p_T := P(w_H(t) < 1, -w_H(-t) < 1, t \in (0, T)) \geq [P(w_H(t) < 1, t \in (0, T))]^2. \quad (3.12)$$

Using (1.2), one has $\theta_{\chi_H} \leq 2(1 - H)$.

Obviously, $p_T \leq P(w_H(t) < 1, t \in (0, T))$. Therefore, $\theta_{\chi_H} \geq (1 - H)$ for any H .

Part (2). Let $0 < H \leq 1/2$, then $Ew_H(t)(-w_H(-s)) \leq 0$ for $t, s > 0$. Hence,

$$p_T \leq [P(w_H(t) < 1, t \in (0, T))]^2, \quad \theta_{\chi_H} \geq 2(1 - H). \quad (3.13)$$

Finally,

$$p_T \leq P(w_H(t) - w_H(-t) < 2, t \in (0, T)) = P(w_H^-(t) < 1, t \in (0, T)). \quad (3.14)$$

But then, $\theta_{w_H^-} \geq \theta_{w_H^-}$ for all H . If $\theta_{w_H^-} = \tilde{\theta}_{w_H^-}$, then we get a lower bound of $\tilde{\theta}_{\chi_H}$ for $0 < H \leq 1/4$.

The equality $\theta_{w_H^-} = \tilde{\theta}_{w_H^-}$. Let $H_{w_H^-}(\Delta)$ and $H_{w_H}(\Delta)$ be the reproducing Kernel Hilbert spaces associated with $w_H^-(t)$ and $w_H(t)$, respectively. By the definition of $w_H^-(t)$, the map $(\varphi(t), t > 0) \mapsto (\text{sign}(t)\varphi(|t|), |t| < \infty)$ is an isometric embedding of $H_{w_H^-}(R_+^1)$ in $H_{w_H}(R^1)$. To prove that the exponents are equal, it is enough to find $\varphi(t)$, $t \geq 0$ such that $\text{sign}(t)\varphi(|t|) \in (H_{w_H}(R^1), \|\cdot\|_R)$, $\varphi(t) \geq 1$ for $t \geq 1$, and $\|\varphi\|_R < \infty$. As we showed above, this can be $\varphi(t) = \min(t, 1)$, $t > 0$.

Proof of Proposition 2.15

Consider the fractional Brownian motion in $\Delta_T = (-T^\alpha, T)$, $0 \leq \alpha \leq 1$. By Lemma 1.1, we can focus on the exponent related to the position of the maximum of $w_H(t)$ in Δ_T , $t_{\Delta_T}^*$.

Distribution of t_{Δ}^ .* We remind the main properties of the distribution function, $F^*(x)$, of t_{Δ}^* related to the normalized interval $\Delta = (0, 1)$ (see [1, 14]):

- (i) $F^*(x)$ has a continuous density $f_{\Delta}^*(x) > 0$, $0 < x < 1$ such that $(1 - x)f_{\Delta}^*(x)$ decreases and $xf_{\Delta}^*(x)$ increases on Δ ;

(ii) $F^*(x)$ have the following estimates:

$$x^{1-H}l^{-1}(x) \leq F^*(x) \leq x^{1-H}l(x), \quad (3.15)$$

where $l(x) = \exp(c\sqrt{-\ln x})$, $c > 0$.

Due to monotonicity of $(1-x)f_\Delta^*(x)$ and $xf_\Delta^*(x)$, one has

$$(1-x)f_\Delta^*(x) \leq x^{-1} \int_0^x (1-u)f_\Delta^*(u)du \leq x^{-1}F^*(x), \quad (3.16)$$

$$xf_\Delta^*(x) \geq x^{-1} \int_{xq}^x uf_\Delta^*(u)du \geq q(F^*(x) - F^*(xq)), \quad 0 < q < 1. \quad (3.17)$$

By (3.15), (3.16),

$$f_\Delta^*(x) \leq x^{-H}l(x)(1-x)^{-1}. \quad (3.18)$$

Using (3.15), (3.17), one has

$$f_\Delta^*(x) \geq qx^{-H}l^{-1}(x)(1-l(x)l(xq)q^{1-H}). \quad (3.19)$$

If we set $q^{1-H} = l^{-2}(x)/2$, then

$$f_\Delta^*(x) \geq \frac{qx^{-H}l^{-1}(x)}{2} = c_H x^{-H}l^{-\nu_H}(x), \quad (3.20)$$

where $\nu_H = (3-H)/(1-H)$, $c_H = 2^{-(2-H)/(1-H)}$.

Distribution of $t_{\Delta_T}^$.* Let $T_1 = T_- + T$, where $T_- = T^\alpha$, then the processes $w_H(T_1\tau - T_-) - w_H(-T_-)$ and $w_H(\tau)T_1^H$ on $\Delta = (0, 1)$ are equal in distribution. Hence, $t_{\Delta_T}^*$ and $T_1 t_\Delta^* - T_-$ have the same distribution as well. Therefore,

$$p_T := P\left(\left|t_{\Delta_T}^*\right| \leq 0.5\right) = P\left(\left|t_\Delta^* - \frac{T_-}{T_1}\right| \leq \frac{0.5}{T_1}\right) = T_1^{-1}f_\Delta^*\left(\frac{T_- + \varepsilon}{T_1}\right), \quad (3.21)$$

where $|\varepsilon| \leq 0.5$. We have used here the existence and continuity of $f_\Delta^*(x)$.

Exponent $\check{\theta}_{w_H}$. Set $\alpha = 1$. Then (3.21) implies $\lim_{T \rightarrow \infty} T p_T = 0.5 f_\Delta^*(0.5)$.

Let $\alpha < 1$, then $(T_- + \varepsilon)/T_1 = o(1)$ as $T \rightarrow \infty$, and (3.20), (3.21) give a lower bound on p_T :

$$T_1 p_T \geq c_H (a_T^+)^{-H} l^{-\nu_H}(a_T^+). \quad (3.22)$$

Here and below $a_T^\pm = (T_- \pm 0.5)/T_1$.

Using (3.18), (3.21), we get an upper bound on p_T :

$$T_1 p_T = f_{\Delta}^* \left(\frac{T^{\alpha} + \varepsilon}{T_1} \right) \leq \frac{(a_T^-)^{-H} l(a_T^-) T_1}{T + 1} \leq 2(a_T^-)^{-H} l(a_T^-). \quad (3.23)$$

By substituting $T_- = T^{\alpha}$, we have

$$\ln a_T^{\pm} = -(1 - \alpha) \ln T + O(T^{-\beta}), \quad \beta = \alpha \wedge (1 - \alpha), \quad \ln l(a_T^{\pm}) = O(\sqrt{\ln T}). \quad (3.24)$$

Hence,

$$-\ln p_T = (1 - (1 - \alpha)H) \ln T + O(\sqrt{\ln T}), \quad (3.25)$$

that is, $\check{\theta}_{w_H} = \alpha H + (1 - H)$.

The equality $\check{\theta}_{w_H} = \theta_{w_H}$. Consider the Hilbert space $(H_{w_H}(R^1), \|\cdot\|_R)$ related to FBM and a function

$$\varphi(t) = \min(|t|, 1) = \int (e^{it\lambda} - 1) \left(\frac{\sin \lambda/2}{\sqrt{2\pi} \lambda/2} \right)^2 d\lambda. \quad (3.26)$$

The standard spectral representation of the kernel $E w_H(t) w_H(s)$ and the representation (3.26) yield

$$\|\varphi\|_R^2 = k_H \int \left(\frac{\sin \lambda/2}{\sqrt{2\pi} \lambda/2} \right)^4 |\lambda|^{1+2H} d\lambda < \infty, \quad (3.27)$$

where $k_H = \int |e^{i\lambda} - 1|^2 |\lambda|^{-1-2H}$. Setting $\varphi_T := \{\varphi(t), t \in \Delta_T\}$, the desired statement follows from Lemma 1.1 because $\varphi_T \in (H_{w_H}(\Delta_T), \|\cdot\|_T)$ and $\|\varphi_T\|_T \leq \|\varphi\|_R$.

Appendix

Relation (2.4). $(\tilde{B}_{I_H}(t) \leq \tilde{B}_{I_{1-H}}(t))$.

By (2.3), one has for small and large t

$$\begin{aligned} \tilde{B}_{I_H}(t) &= 1 - \frac{(1 - H^2)t^2}{2} + (2 + 4H)^{-1} t^{2+2H} (1 + o(1)), \quad t \rightarrow 0, \\ \tilde{B}_{I_H}(t) &= (1 + H)(1 + 2H)^{-1} e^{-Ht} (1 - e^{-t}) \\ &\quad + 0.5(1 + H) e^{-\bar{H}t} (1 + O(e^{-t})), \quad t \rightarrow \infty, \end{aligned} \quad (A.1)$$

where $\overline{H} = 1 - H$. Therefore, we have the following asymptotics for $\Delta(t) = \tilde{B}_{I_H}(t) - \tilde{B}_{I_{\overline{H}}}(t)$:

$$\begin{aligned}\Delta(t) &= -\frac{(1-2H)t^2}{2} + O(t^{2+2H}), \quad t \rightarrow 0, \\ \Delta(t) &= -(1-2H)H(2+4H)^{-1}e^{-Ht} - (1-2\overline{H})\overline{H}(2+4\overline{H})^{-1}e^{-\overline{H}t} + O(e^{-t}), \quad t \rightarrow \infty.\end{aligned}\tag{A.2}$$

These relations support (2.4) both for small and large enough t . To verify (2.4) in the general case, we consider the following test function: $(2+4H)(2+4\overline{H})\Delta(t)\exp(-1.5t)$. Using new variables: $x = \exp(-t)$, $\alpha = 1 - 2H$, the test function is transformed to a function ψ on the square $S = (0, 1) \times (0, 1)$. Namely, $\psi = U(x, \alpha) - U(x, -\alpha)$, where

$$U(x, \alpha) = (4 - \alpha^2)x^{\alpha/2}(3 - \alpha) \int_0^x \left[(x-u) \left((1-u)^{1-\alpha} - u^{1-\alpha} \right) + u^{1-\alpha} \right] du. \tag{A.3}$$

We have to show that $\psi \leq 0$. It is easy to see that $\psi = 0$ at the boundary of S . By (A.1), $\psi \leq 0$ in vicinities of two sides of S : $x = 0$ and $x = 1$. The same is true for the other sides: $\alpha = 0$ and $\alpha = 1$ because

$$\begin{aligned}\frac{\partial \psi}{\partial \alpha}(x, 0) &= -4(1-x^2) \int_{1-x}^1 \ln\left(\frac{1}{u}\right) du < 0, \\ \frac{\partial \psi}{\partial \alpha}(x, 1) &= (1-x)x^{-1/2}f(x) > 0.\end{aligned}\tag{A.4}$$

Here

$$f(x) = -x(1-x) + x^3 \ln \frac{1}{x} + (1-x^3) \ln \frac{1}{(1-x)}. \tag{A.5}$$

To verify $f(x) > 0$, $0 < x < 1$, note that $f'(x) = 3x^2(1+v+\ln v)$, where $v = (1-x)/x$. Obviously, f' has a single zero in $(0, 1)$, that is, f has a single extreme point. But $f(0) = 0 = f(1)$ and $f(x) > 0$ for small x . Therefore, $f(x) \geq 0$, $0 < x < 1$.

Numerical testing supports the desired inequality $\psi < 0$ for interior points of S .

Comment 1. Our preliminary numerical test was concerned with points on a grid with a step of 0.005. The first derivatives of ψ are uniformly bounded from above on S . This fact helps us to find the final grid step to prove $\psi < 0$ for all interior points of S . The relevant analysis is cumbersome and so has been omitted.

Relation (2.6). $(\tilde{B}_{I_H}(t) \leq \tilde{B}_{I_{1/2}}(pt), H \geq 1/2, p = 2(1 - H))$.

To verify the inequality $\Delta(t) = \tilde{B}_{I_H}(t) - \tilde{B}_{I_{1/2}}(2(1 - H)t) \leq 0$, we consider the following test function: $(2 + 4H)\Delta(t) \exp(-(1 + H)t)$. Using (2.3), (2.5), and new variables $(x = \exp(-t), \alpha = 2H - 1) \in S = (0, 1) \times (0, 1)$, we will have the following representation for the test function:

$$\begin{aligned} \psi(x, \alpha) = & (3 + \alpha) \left(x + x^{\alpha+2} \right) - 1 - x^{\alpha+3} + (1 - x)^{\alpha+3} \\ & - 3(\alpha + 2)x^2 + (\alpha + 2)x^{3-\alpha}. \end{aligned} \quad (\text{A.6})$$

One has $\psi(x, \alpha) \leq 0$ in vicinities of two sides of S : $x = 0$ and $x = 1$, because

$$\begin{aligned} \psi(x, \alpha) = & -\frac{(\alpha + 2)(3 - \alpha)x^2}{2} + O\left(x^{(\alpha+2) \wedge (3-\alpha)}\right) < 0, \quad x \rightarrow 0, \\ \psi(x, \alpha) = & -\frac{2\alpha(1 - \alpha)(3 - \alpha)(1 - x)^2}{2} + O\left((1 - x)^3\right) \leq 0, \quad x \rightarrow 1. \end{aligned} \quad (\text{A.7})$$

The same is true for the other sides: $\alpha = 0$ and $\alpha = 1$.

Side $\alpha = 0$

One has $\psi(x, 0) = 0$ and

$$\frac{\partial \psi}{\partial \alpha}(x, 0) = (1 - x) \left[x(1 - x) + 3x^2 \ln x + (1 - x)^2 \ln(1 - x) \right] := (1 - x)\varphi_3(x) \leq 0 \quad (\text{A.8})$$

because

$$\varphi_a(x) = x(1 - x) + ax^2 \ln x + (1 - x)^2 \ln(1 - x) \leq 0, \quad a > 1. \quad (\text{A.9})$$

To prove (A.9), note that $\varphi_a(0) = \varphi_a(1) = 0$ and $\varphi_a(x) = ax^2 \ln x + O(x^2) \leq 0$ as $x \rightarrow 0$. Hence, (A.9) holds if $\varphi_a(x)$ has a single extremum in $(0, 1)$. By

$$\varphi_a^{(4)}(x) = -2ax^{-2} - 2(1 - x)^{-2} \leq 0, \quad (\text{A.10})$$

we conclude that

$$\varphi_a''(x) = (3a + 1) + 2a \ln x + 2 \ln(1 - x), \quad (\text{A.11})$$

is a concave function with two zeroes in $(0, 1)$, because $\varphi_a''(1/2) > 0$ and $\varphi_a''(x) \rightarrow -\infty$ as $x \rightarrow 0$ or 1 .

This means that

$$\varphi_a'(x) = (a - 1)x + 2ax \ln x - 2(1 - x) \ln(1 - x), \quad (\text{A.12})$$

has two extremums in $(0,1)$ only. But $\varphi'_a(0) = 0$, $\varphi'_a(1) = a - 1 > 0$, and $\varphi'_a(x) \leq 0$ for small x because $\varphi''_a(x) \rightarrow -\infty$ as $x \rightarrow 0$. Hence $\varphi'_a(x)$ has a single zero in $(0,1)$ and $\varphi_a(x)$ has a single extremum.

We have proved that $\varphi(x, \alpha) \leq 0$ for small α .

Side $\alpha = 1$

Here $\varphi(x, 1) = 0$ and

$$\frac{\partial \varphi}{\partial \alpha}(x, 1) = (1-x)(3-x)x^2 \ln\left(\frac{1}{x}\right) + (1-x)^2 \left[x + (1-x)^2 \ln(1-x) \right] \geq 0, \quad (\text{A.13})$$

because $[x + (1-x)^2 \ln(1-x)] \geq x + (1-x) \ln(1-x) = -\int_0^x \ln(1-u) du \geq 0$.

Hence, $\varphi(x, \alpha) = \varphi'_\alpha(x, 1) (\alpha - 1) (1 + o(1 - \alpha)) \leq 0$, $\alpha \rightarrow 1$.

As a result $\varphi(x, \alpha) \leq 0$ near the boundary of $S = (0, 1) \times (0, 1)$. Numerical testing supports the desired inequality $\varphi < 0$ for the interior of S (see more in the Comment 1 from the appendix section "Relation (2.4)").

Relation (2.7). $(\tilde{B}_{I_H}(t) \geq \tilde{B}_{I_{1/2}}(pt), H \leq 1/2, p = 2\sqrt{(1-H^2)/3})$.

Let $\varphi = (2 + 4H)(\tilde{B}_{I_H}(t) - \tilde{B}_{I_{1/2}}(pt))e^{-(1+H)t}$. By change of variables: $x = \exp(-t)$ and $\alpha = 2H$, we get a test function

$$\begin{aligned} \varphi(x, \alpha) = & (2 + \alpha) \left(x + x^{\alpha+1} \right) - 1 - x^{\alpha+2} + (1-x)^{\alpha+2} \\ & - 3(\alpha + 1)x^{1+(\alpha+p)/2} + (\alpha + 1)x^{1+(\alpha+3p)/2}, \end{aligned} \quad (\text{A.14})$$

on $S = (0, 1) \times (0, 1)$ and the relation between p and α is

$$3\left(\frac{p}{2}\right)^2 + \left(\frac{\alpha}{2}\right)^2 = 1. \quad (\text{A.15})$$

One has

$$\begin{aligned} \varphi(x, \alpha) = & (2 + \alpha)x^{1+\alpha} - 3(1 + \alpha)x^{1+(p+\alpha)/2} + O(x^2) \geq 0, \quad x \rightarrow 0, \\ \varphi(x, \alpha) = & (1-x)^{2+\alpha} + O((1-x)^3) \geq 0, \quad x \rightarrow 1. \end{aligned} \quad (\text{A.16})$$

In addition,

$$\varphi(x, 0) = x \left(2 - 3x^{3^{-1/2}} + x^{3^{1/2}} \right) \geq 0. \quad (\text{A.17})$$

Finally, $\varphi(x, 1) = 0$ and

$$\frac{\partial \varphi}{\partial \alpha}(x, 1) = \bar{x} \left(x\bar{x} + 2x^2 \ln x + \bar{x}^2 \ln \bar{x} \right) = \bar{x}\varphi_2(x), \quad (\text{A.18})$$

where $\bar{x} = 1 - x$. By (A.9), $\varphi_2(x) \leq 0$.

Therefore $\varphi(x, \alpha) \leq 0$ near the boundary of $S = (0, 1) \times (0, 1)$. The numerical testing supports this conclusion for the interior of S (see more in the Comment 1 from the appendix section "Relation (2.4)").

Relations (2.11), (2.12)

Consider $\Delta(t) = \tilde{B}_{I_{1/2}}(t) - \tilde{B}_L(pt)$, where $\tilde{B}_L(t) = 1/\cosh(t/2)$ and $\tilde{B}_{I_{1/2}}(t)$ is given in (2.5). By the change of variables $x = e^{-t/2}$, we transform the test function $2(1 + e^{-pt})\Delta(t)$ to a function φ on $(0, 1)$ such that

$$\varphi(x) = (3x - x^3)(1 + x^{2p}) - 4x^p. \quad (\text{A.19})$$

Taking into account the asymptotics of φ near 0, we come to a necessary condition for φ to be negative, namely, $p \leq 1$. Let $p = 1$, then $\varphi = -(1 - x^2)^2 x \leq 0$, that is, $4\theta_L \leq 1$.

The Case $p > 1$. Here, $\varphi \geq 0$ as $x \rightarrow 0$. An additional condition on $p > 1$ we can get from the relation $\varphi \geq 0$ as $x \rightarrow 1$. One has $\varphi = xQ(x)$, where

$$Q(x) = (3 - x^2)(1 + x^{2p}) - 4x^{p-1}. \quad (\text{A.20})$$

By $Q(0) = 3$, $Q(1) = Q'(1) = 0$, we have $Q(x) = (1 - x)^2 P(x)$ and $P(1) = 0.5Q''(1) = 2(p^2 - 3)$. Thus $Q(1) \geq 0$ if $p^2 \geq 3$.

The Case $p = 2$. Here, $P(x)$ is a polynomial, $P(x) = 3 + 2x - 2x^3 - x^4$, and $P''(x) = -12x(1 + x) \leq 0$, that is, $P(x)$ is a concave function with $P(0) = 3$, $P(1) = 2$. Therefore, $P(x) \geq 0$ and as a result, $4\theta_L \geq 1/p = 0.5$.

Consider $p = \sqrt{3}$. One has $Q(x) = 8(1 - x)^3(1 + o(1))$, $x \rightarrow 1$ and $Q(0) = 3 > 0$. Therefore, $Q(x) \geq 0$, if $Q(x)$ is convex, that is, $Q''(x) \geq 0$. To verify this property, note that

$$\begin{aligned} 0.5x^2 Q''(x) &= 2(3p - 5)x^{p-1} + 3(6 - p)x^{2p} - x^2 - (7 + 3p)x^{2+2p} \\ &= (7 + 3p)x^{2p}(1 - x^2) + \rho x^{p-1} + (1 - \rho)x^{2p} - x^2 := \varphi(x), \end{aligned} \quad (\text{A.21})$$

where $\rho = 6p - 10$.

Obviously, $\varphi(x) \geq 0$ if $\rho x^{p-1} - x^2 \geq 0$. This holds for $0 < x < x_0 = 0.478$.

For $x > x_0$,

$$\rho x^{p-1} + (1 - \rho)x^{2p} - x^2 \geq (\rho + (1 - \rho)x_0^{p+1})x^{p-1} - x^2. \quad (\text{A.22})$$

The right part here is positive for $x < 0.55$, that is, $\varphi(x) \geq 0$ for $x \leq 0.5$.

Let $x > 0.5$. Then

$$\begin{aligned} \varphi(x) &\geq (7 + 3p)2^{-2p}(1 - x^2) + \rho x^{p-1} + (1 - \rho)x^{2p} - x^2 \\ &= C - (C + 1)x^2 + \rho x^{p-1} + (1 - \rho)x^{2p} := u(x), \end{aligned} \quad (\text{A.23})$$

where $C = (7 + 3p)2^{-2p}$. We have $u(0) = C$, $u(1) = 0$ and

$$\begin{aligned} u'(x) &= -2(C+1)x + \rho(p-1)x^{p-2} + 2(1-\rho)px^{2p-1} \\ &= -\left(C+1-2(1-\rho)px^{2p-2}\right)x - \left((C+1)x^{3-p} - \rho(p-1)\right)x^{p-2}. \end{aligned} \quad (\text{A.24})$$

It is easy to see that both terms in parentheses are positive on $(0.5, 1)$.

Thus, $u(x)$ decreases to $u(1) = 0$. This means that $Q''(x) \geq 0$.

Relation (3.1). $(\tilde{B}_{w_H}(pt) \geq B_{S_H}(t), pH = -\ln(2H), 0 < H < e^{-2}/2)$.

The difference between the correlation functions is

$$\Delta(t) = \left(\cosh(Hpt) - 0.5 \left(2 \sinh\left(\frac{pt}{2}\right) \right)^{2H} \right) - \left(1 - |t|^{2H} \right)_+. \quad (\text{A.25})$$

Let $t > 1$, then $\Delta(t) = \tilde{B}_{w_H}(pt) \geq 0$.

Let $2H < t < 1$. It is enough to show that the first term, φ , in the following representation:

$$\Delta(t) = \left[0.5e^{-Hpt} - 1 + t^{2H} \right] + 0.5e^{Hpt} \left(1 - (1 - e^{-pt})^{2H} \right) := \varphi + R, \quad (\text{A.26})$$

is nonnegative. Setting $Hp = -\ln(2H)$, $\alpha = 2H$ one has

$$\varphi(t) = 0.5\alpha^t + t^\alpha - 1. \quad (\text{A.27})$$

Let us show that φ is decreasing. In this case φ is positive because $\varphi(1) = \alpha/2$.

We have

$$\varphi'(t) = \alpha^t \left(-0.5 \ln\left(\frac{1}{\alpha}\right) + \varphi(t) \right), \quad (\text{A.28})$$

where $\varphi(t) = \alpha^{1-t}/t^{1-\alpha}$. The function $\varphi(t)$ has a single extreme point in the interval: $t^* = (1 - \alpha)/\ln(1/\alpha)$. But $\varphi(t^*) = \min$, because $\varphi(t)$ decreases near $t = \alpha$:

$$\varphi(\alpha) = 1, \quad \varphi'(\alpha) = \frac{(\alpha \ln(e/\alpha) - 1)}{\alpha} \leq 0 \quad \text{for } 0 < \alpha < 1. \quad (\text{A.29})$$

Hence, $\varphi(t) \leq \max(\varphi(\alpha), \varphi(1)) = 1$. As a result,

$$\varphi'(t) \leq \alpha^t \left(-0.5 \ln\left(\frac{1}{\alpha}\right) + 1 \right) \leq 0. \quad (\text{A.30})$$

The last inequality holds for $0 < \alpha < e^{-2}$, so we have

$$\Delta(t) \geq 0, \quad 2H < t < 1 \quad \text{for } 0 < \alpha < e^{-2}. \quad (\text{A.31})$$

Let $0 < t < 2H$. Use

$$\Delta(t) = \cosh(Hpt) - 1 + t^{2H} \left[1 - 0.5 \left(2t^{-1} \sinh\left(\frac{pt}{2}\right) \right)^{2H} \right], \quad (\text{A.32})$$

then $\Delta(t) \geq 0$ if

$$2^{1/(2H)} \geq \max_{(0,2H)} \left(2t^{-1} \sinh\left(\frac{pt}{2}\right) \right) = H^{-1} \sinh(pH) = (2H)^{-2} - 1. \quad (\text{A.33})$$

This inequality holds for $0 < 2H < 1/4$.

Combining the above inequalities, we get (3.1) for $2H \leq e^{-2} \wedge 1/4$.

References

- [1] G. M. Molchan, "Maximum of a fractional Brownian motion: probabilities of small values," *Communications in Mathematical Physics*, vol. 205, no. 1, pp. 97–111, 1999.
- [2] Ya. G. Sinai, "Distribution of some functionals of the integral of a random walk," *Rossiiskaya Akademiya Nauk. Teoreticheskaya i Matematicheskaya Fizika*, vol. 90, no. 3, pp. 323–353, 1992.
- [3] Y. Isozaki and S. Watanabe, "An asymptotic formula for the Kolmogorov diffusion and a refinement of Sinai's estimates for the integral of Brownian motion," *Proceedings of the Japan Academy, Series A*, vol. 70, no. 9, pp. 271–276, 1994.
- [4] Y. Isozaki and S. Kotani, "Asymptotic estimates for the first hitting time of fluctuating additive functionals of Brownian motion," in *Séminaire de Probabilités, 34*, vol. 1729 of *Lecture Notes in Mathematics*, pp. 374–387, Springer, Berlin, Germany, 2000.
- [5] T. Simon, "The lower tail problem for homogeneous functionals of stable processes with no negative jumps," *ALEA. Latin American Journal of Probability and Mathematical Statistics*, vol. 3, pp. 165–179, 2007.
- [6] V. Vysotsky, "On the probability that integrated random walks stay positive," *Stochastic Processes and Their Applications*, vol. 120, no. 7, pp. 1178–1193, 2010.
- [7] V. Vysotsky, "Positivity of integrated random walks," <http://arxiv.org/abs/1107>.
- [8] F. Aurzada and S. Dereich, "Universality of the asymptotics of the one-sided exit problem for integrated processes," *Annales de l'Institut Henri Poincaré (B)*, <http://arxiv.org/abs/1008.0485>.
- [9] A. Dembo, J. Ding, and F. Gao, "Persistence of iterated partial sums," <http://arxiv.org/abs/1205.5596>.
- [10] D. Denisov and V. Wachtel, "Exit times for integrated random walks," <http://arxiv.org/pdf/1207.2270v1>.
- [11] G. Molchan, "Unilateral small deviations of processes related to the fractional Brownian motion," *Stochastic Processes and Their Applications*, vol. 118, no. 11, pp. 2085–2097, 2008.
- [12] W. V. Li and Q.-M. Shao, "Lower tail probabilities for Gaussian processes," *The Annals of Probability*, vol. 32, no. 1, pp. 216–242, 2004.
- [13] G. Molchan and A. Khokhlov, "Unilateral small deviations for the integral of fractional Brownian motion," <http://arxiv.org/abs/math/0310413>.
- [14] G. Molchan and A. Khokhlov, "Small values of the maximum for the integral of fractional Brownian motion," *Journal of Statistical Physics*, vol. 114, no. 3–4, pp. 923–946, 2004.
- [15] W. V. Li and Q.-M. Shao, "A normal comparison inequality and its applications," *Probability Theory and Related Fields*, vol. 122, no. 4, pp. 494–508, 2002.
- [16] A. Dembo, B. Poonen, Q.-M. Shao, and O. Zeitouni, "Random polynomials having few or no real zeros," *Journal of the American Mathematical Society*, vol. 15, no. 4, pp. 857–892, 2002.

- [17] T. Newman and W. Loinaz, "Critical dimensions of the diffusion equation," *Physical Review Letters*, vol. 86, no. 13, pp. 2712–2715, 2001.
- [18] L. A. Shepp, "First passage time for a particular Gaussian process," *Annals of Mathematical Statistics*, vol. 42, pp. 946–951, 1971.
- [19] J. Krug, H. Kallabis, S. N. Majumdar, S. J. Cornell, A. J. Bray, and C. Sire, "Persistence exponents for fluctuating interfaces," *Physical Review E*, vol. 56, no. 3, pp. 2702–2712, 1997.
- [20] M. A. Lifshits, *Gaussian Random Functions*, vol. 322 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1995.