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*Research Article*

# Graph Polynomials

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One of the most important and applied concepts in graph theory is to find the edge cover, vertex cover, and dominating sets with minimum cardinal also to find independence and matching sets with maximum cardinal and their polynomials. Although there exist some algorithms for finding some of them (Kuhn and Wattenhofer, 2003; and Mihelic and Robic, 2005), but in this paper we want to study all of these concepts from viewpoint linear and binary programming and we compute the coefficients of the polynomials by solving a system of linear equations with $\{0, 1\}$ variables.

## 1. Introduction

All graphs in this note are simple, connected, finite, and undirected, though it is probable that some of the obtained results are extendable to general or directed graphs.

Let $G = (V, E)$ be a simple and connected graph with $|V| = n$ and $|E| = m$; then the edge cover and edge dominating polynomials are of degree $m$, and the vertex cover and dominating polynomials are of degree $n$, in which coefficient of $x^k$ is the number of edge cover, edge dominating, vertex cover, and dominating sets with $k$ elements, respectively. Also the independence and matching polynomials are at most of degree $n$ such that coefficient of $x^k$ is the number of independence and matching sets with $k$ elements, respectively, for some positive integer $k$.

For some notation being not defined here we refer the reader to [1].

A set $L \subseteq E$ is an *edge cover* if every vertex $v \in V$ is incident to some edge of $L$.

A set $Q \subseteq V$ is a *vertex cover* if every edge $e \in E$ has at least one endpoint in $Q$.

A set $S \subseteq V$ is an *independent set* if non of two vertex in $S$ are not adjacent.

The maximum size of an independent set is named *independence number*.

A *matching* in graph $G$ is a set $M \subseteq E$ with no shared endpoints.

In graph $G$ a set $D \subseteq V$ is a *dominating set* if every vertex number in $D$ has a neighbor in $D$, and finally a set $W \subseteq E$ is an *edge dominating set* if every edge number in $W$ has a neighbor in $W$.

We set

$$\begin{aligned} \operatorname{Min}|L| &= \beta', \\ \operatorname{Max}|S| &= \alpha, \\ \operatorname{Min}|Q| &= \beta, \\ \operatorname{Max}|M| &= \alpha', \\ \operatorname{Min}|D| &= \gamma, \\ \operatorname{Min}|W| &= \gamma'. \end{aligned} \tag{1.1}$$

By [1],

(i) $\alpha + \beta = n$,

(ii) $\alpha' + \beta' = n$.

Also in every bipartite graph

(iii) $\alpha = \beta'$,

(iv) $\alpha' = \beta$.

We denote the *Adjacency matrix* by $A$ and *Incidence matrix* by $R$, in which $A = [a_{ij}]_{n\times n}$ such that

$$a_{ij} = \text{the numbers of edges with endpoints } v_i \text{ and } v_j, \tag{1.2}$$

and also $R = [r_{ij}]_{n\times m}$ in which

$$r_{ij} = \begin{cases} 1, & v_i \text{ is an endpoint of } e_j, \\ 0, & \text{otherwise}. \end{cases} \tag{1.3}$$

We also define an *Edge adjacency matrix* $B = [b_{ij}]_{m\times m}$ as follows:

$$b_{ij} = \begin{cases} 1, & e_i \text{ is adjacent to } e_j, \\ 0, & \text{otherwise}, \end{cases} \tag{1.4}$$

and $b_{ii} = 0$.

From now on we set

$$\begin{aligned} V &= (v_1, v_2, \ldots, v_n)^t, \\ E &= (e_1, e_2, \ldots, e_m)^t, \\ 1_n &= (1, 1, \ldots, 1)^t_{1\times n}. \end{aligned} \tag{1.5}$$

## 2. Edge Cover Set and Edge Cover Polynomial

As previous notations we have the following theorem for obtaining the minimum size of edge cover set.

**Theorem 2.1.** *One has*

$$\begin{gathered} \beta' = \min \sum_{i=1}^{m} e_i \\ \textit{subject to} \quad RE \geqq 1_n, \\ e_i \in \{0, 1\}, \quad \textit{where } i = 1, 2, \ldots, m. \end{gathered} \tag{2.1}$$

*Proof.* Since an edge cover set of $G$ is a set $L$ of edges such that every vertex of $G$ is incident to some edge of $L$ and we want to obtain the optimal size of the sets in covering problems, so we will have a minimize problem; that is, the object function is $\beta' = \min \sum_{i=1}^{m} e_i$; on the other hand for each $v_i \in V$ at least one edge with endpoint $v_i$ must belong to $L$; in other words from every row of matrix $R$ at least one entry $(e_i)$ must be equal to 1. Therefore

$$\begin{gathered} r_{11}e_1 + r_{12}e_2 + \cdots + r_{1m}e_m \geq 1, \\ r_{21}e_1 + r_{22}e_2 + \cdots + r_{2m}e_m \geq 1, \\ \vdots \\ r_{n1}e_1 + r_{n2}e_2 + \cdots + r_{nm}e_m \geq 1, \\ e_i \in \{0, 1\}, \quad \text{where } i = 1, 2, \ldots, m. \end{gathered} \tag{2.2}$$

□

*Definition 2.2.* An *edge cover polynomial* is as follows:

$$L(x) = a_0 x^{\beta'} + a_1 x^{\beta'+1} + \cdots + a_{m-\beta'} x^m, \tag{2.3}$$

where $\beta'$ is the same as in (2.1) and $a_i$'s are the number of edge cover sets with $\beta' + i$ elements.

**Theorem 2.3.** *The coefficients* $a_0, a_1, \ldots, a_{m-\beta'}$ *in edge cover polynomial are all of solutions of the following system for* $i = 0, i = 1, \ldots, i = m - \beta'$*, respectively,*

$$RE \geq 1_n, \tag{$*$}$$

$$\begin{gathered} e_1 + e_2 + \cdots + e_m = \beta' + i, \\ e_j \in \{0,1\}, \quad \text{where } j = 1, 2, \ldots, m. \end{gathered} \tag{$**$}$$

*Proof.* The first inequality $(*)$ is the condition for a set to be an edge cover set and $(**)$ for each $i$ causes that we have the edge cover sets with cardinality $\beta', \beta' + 1, \ldots, m$, respectively, and with this process we can compute $a_0, a_1, \ldots, a_{m-\beta'}$. It is trivial that $a_{m-\beta'} = 1$ and this completes the proof. □

*Algorithm 2.4* (For computation $a_i$). One has the following.

*Step 1.* Solve

$$\begin{gathered} \beta' = \min \sum_{i=1}^{m} e_i, \\ RE \geq 1_n, \\ e_j \in \{0,1\}, \quad \text{where } j = 1, 2, \ldots, m, \end{gathered} \tag{2.4}$$

and obtain $\beta'$.

*Step 2.* For $i = 0$ to $m - \beta' - 1$, compute all of solutions:

$$\begin{gathered} RE \geq 1_n, \\ e_1 + e_2 + \cdots + e_m = \beta' + i, \\ e_j \in \{0,1\}, \quad \text{where } j = 1, 2, \ldots, m. \end{gathered} \tag{2.5}$$

*Step 3.* Set $a_i$ to be equal to all solutions of Step 2.

## 3. Independence Set and Independence Polynomial

In an independence set $S$ from every two adjacent vertices at most one of them belongs to $S$; this means that for all $e_i \in E$ with end points $v_i$ and $v_j$ at most $v_i$ or $v_j$ belongs to $S$. Therefore we have the following.

**Theorem 3.1.** *One has*

$$\begin{aligned} \alpha &= \max \sum_{i=1}^{n} v_i \\ &\text{subject to} \quad R^t V \le 1_m, \\ v_i &\in \{0,1\}, \quad \text{where } i = 1,2,\ldots,n. \end{aligned} \tag{3.1}$$

*Definition 3.2.* An *independence polynomial* is as follows:

$$S(x) = b_0 x^{\alpha} + b_1 x^{\alpha-1} + \cdots + b_{\alpha-1} x, \tag{3.2}$$

where $\alpha$ is the same as in (3.1) and $b_i$'s are the numbers of independence sets with $\alpha - i$ elements.

**Theorem 3.3.** *The coefficients $b_0, b_1, \ldots, b_{\alpha-1}$ are all of solutions of the following system $i = 0$, $i = 1, \ldots, i = \alpha - 1$, respectively,*

$$\begin{gathered} R^t V \le 1_m, \\ v_1 + v_2 + \cdots + v_n = \alpha - i, \\ v_j \in \{0,1\}, \quad \text{where } j = 1,2,\ldots,n. \end{gathered} \tag{3.3}$$

*Algorithm 3.4* (For computation $b_i$). One has the following.

*Step 1.* Solve

$$\begin{gathered} \alpha = \max \sum_{i=1}^{n} v_i, \\ R^t V \le 1_m, \\ v_j \in \{0,1\}, \quad \text{where } j = 1,2,\ldots,n, \end{gathered} \tag{3.4}$$

and obtain $\alpha$.

*Step 2.* For $i = 0$ to $\alpha - 2$, compute all of solutions:

$$\begin{gathered} R^t V \le 1_m, \\ v_1 + v_2 + \cdots + v_n = \alpha - i, \\ v_j \in \{0,1\}, \quad \text{where } j = 1,2,\ldots,n. \end{gathered} \tag{3.5}$$

*Step 3.* Set $b_i$ to be equal to all solutions of Step 2 of course $b_{\alpha-1} = n$.

## 4. Vertex Cover Set and Vertex Cover Polynomial

We have the following theorem for vertex cover set.

**Theorem 4.1.** *One has*

$$\begin{aligned} \beta &= \min \sum_{i=1}^{n} v_i \\ &\quad \textit{subject to} \quad R^t V \geqq 1_m, \\ v_i &\in \{0,1\}, \quad \textit{where } i = 1,2,\ldots,n. \end{aligned} \tag{4.1}$$

*Proof.* Since a vertex cover set of $G$ is a set $Q$ of vertices such that every edge of $G$ is incident to some vertex of $Q$ and we want to obtain the optimal size of the sets in covering problems, so we will have a minimize problem; that is, the object function is $\beta = \min \sum_{i=1}^{n} v_i$; on the other hand for each $e_i \in E$ with endpoint $v_i$ and $v_j$ at least one of them must belong to $Q$; in other words from every row of matrix $R^t$ at least one entry $(v_i)$ must be equal to 1. Therefore

$$\begin{aligned} r_{11}v_1 + r_{21}v_2 + \cdots + r_{n1}v_n &\geq 1, \\ r_{12}v_1 + r_{22}v_2 + \cdots + r_{n2}v_n &\geq 1, \\ &\vdots \\ r_{1n}v_1 + r_{2n}v_2 + \cdots + r_{mn}v_n &\geq 1, \\ v_i \in \{0,1\}, \quad \textit{where } i &= 1,2,\ldots,n. \end{aligned} \tag{4.2}$$

□

*Definition 4.2.* A *vertex cover polynomial* is as follows:

$$Q(x) = c_0 x^{\beta} + c_1 x^{\beta+1} + \cdots + c_{n-\beta} x^n, \tag{4.3}$$

where $\beta$ is the same as in (4.1) and $c_i$'s are the number of vertex cover sets with $\beta + i$ elements.

**Theorem 4.3.** *The coefficients $c_0, c_1, \ldots, c_{n-\beta}$ are all of solutions of the following system for $i = 0$, $i = 1, \ldots, i = n - \beta$, respectively,*

$$R^t V \geq 1, \tag{$\imath$}$$

$$\begin{aligned} v_1 + v_2 + \cdots + v_n &= \beta + i, \\ v_j \in \{0,1\}, \quad \textit{where } j &= 1,2,\ldots,n. \end{aligned} \tag{$\imath\imath$}$$

*Proof.* The first inequality $(\imath)$ is the condition for a set to be a vertex cover set and $(\imath\imath)$ for each $i$ causes that we have the vertex cover sets with cardinality $\beta, \beta + 1, \ldots, n$, respectively, and with this process we can compute $c_0, c_1, \ldots, c_{n-\beta}$. It is trivial that $c_{n-\beta} = 1$ and this completes the proof. □

*Algorithm 4.4* (For computation $c_i$). One has the following.

*Step 1.* Solve

$$\begin{gathered}\beta = \min \sum_{i=1}^{n} v_i, \\ R^t V \geq 1_m, \\ v_i \in \{0,1\}, \quad \text{where } i = 1,2,\ldots,n,\end{gathered} \tag{4.4}$$

and obtain $\beta$.

*Step 2.* For $i = 0$ to $n - \beta - 1$, compute all of solutions:

$$\begin{gathered}R^t V \geq 1_m, \\ v_1 + v_2 + \cdots + v_n = \beta + i, \\ v_j \in \{0,1\}, \quad \text{where } j = 1,2,\ldots,n.\end{gathered} \tag{4.5}$$

*Step 3.* Set $c_i$ to be equal to all solutions of Step 2.

## 5. Matching Set and Matching Polynomial

In a matching set $(M)$ from every two adjacent edges at most one of them belongs to $M$ and this means that for all $e_i, e_j \in E$ with common endpoint $v_i$ at most $e_i$ or $e_j$ belongs to $M$. Therefore we have the following.

**Theorem 5.1.** *One has*

$$\begin{gathered}\alpha' = \max \sum_{i=1}^{m} e_i \\ \textit{subject to} \quad RE \leq 1_n, \\ e_j \in \{0,1\}, \quad \textit{where } j = 1,2,\ldots,m.\end{gathered} \tag{5.1}$$

*Definition 5.2.* A *matching polynomial* is as follows:

$$M(x) = d_0 x^{\alpha'} + d_1 x^{\alpha'-1} + \cdots + d_{\alpha'-1} x, \tag{5.2}$$

where $\alpha'$ is the same as in (5.1) and $d_i$'s are the number of matching sets with $\alpha' - i$ elements.

**Theorem 5.3.** *The coefficients $d_0, d_1, \ldots, d_{\alpha'-1}$ are all of solutions of the following system, respectively, $i = 0$, $i = 1, \ldots, i = \alpha' - 1$,*

$$\begin{gathered} RE \leq 1_n, \\ e_1 + e_2 + \cdots + e_m = \alpha' - i, \\ e_j \in \{0,1\}, \quad \text{where } j = 1,2,\ldots,m. \end{gathered} \tag{5.3}$$

*Algorithm 5.4* (For computation $d_i$). One has the following.

*Step 1.* Solve

$$\begin{gathered} \alpha' = \max \sum_{i=1}^{m} e_i, \\ RE \leq 1_n, \\ e_j \in \{0,1\}, \quad \text{where } j = 1,2,\ldots,m, \end{gathered} \tag{5.4}$$

and obtain $\alpha'$.

*Step 2.* For $i = 0$ to $\alpha' - 2$, compute all of solutions:

$$\begin{gathered} RE \leq 1_n, \\ e_1 + e_2 + \cdots + e_m = \alpha' - i, \\ e_j \in \{0,1\}, \quad \text{where } j = 1,2,\ldots,m. \end{gathered} \tag{5.5}$$

*Step 3.* Set $d_i$ to be equal to all solutions of Step 2, $i = 0, 1, \ldots, \alpha' - 2$, of course $d_{\alpha'-1} = 1$.

## 6. Dominating Set and Dominating Polynomial

With the same argument in the previous sections we have the following theorem.

**Theorem 6.1.** *One has*

$$\begin{gathered} \gamma \;= \min \sum_{i=1}^{n} v_i, \\ \text{subject to} \quad (A + I_n)V \geqq 1_n, \\ v_i \in \{0,1\}, \quad \text{where } i = 1,2,\ldots,n. \end{gathered} \tag{6.1}$$

*Definition 6.2.* A *dominating polynomial* is as follows:

$$D(x) = f_0 x^{\gamma} + f_1 x^{\gamma+1} + \cdots + f_{n-\gamma} x^n, \tag{6.2}$$

where $\gamma$ is the same as in (6.1) and $f_i$'s are the number of dominating sets with $\gamma + i$ elements.

**Theorem 6.3.** *The coefficients* $f_0, f_1, \ldots, f_{n-\gamma}$ *are all of solutions of the following system, respectively,* $i = 0, i = 1, \ldots, i = n - \gamma$,

$$(A + I_n)V \geq 1_n, \tag{†}$$

$$\begin{aligned} v_1 + v_2 + \cdots + v_n &= \gamma + i, \\ v_j \in \{0,1\}, &\quad \textit{where } j = 1, 2, \ldots, n. \end{aligned} \tag{††}$$

*Proof.* The first inequality (†) is the condition for a set to be a dominating set and (††) for each $i$ causes that we have the dominating sets with cardinality $\gamma, \gamma+1, \ldots, n$, respectively, and with this process we can compute $f_0, f_1, \ldots, f_{n-\gamma}$. It is trivial that $f_{n-\gamma} = 1$ and this completes the proof. □

*Algorithm 6.4* (For computation $f_i$). One has the following.

*Step 1.* Solve

$$\begin{gathered} \gamma = \min \sum_{i=1}^{n} v_i, \\ (A + I_n)V \geq 1_n, \\ v_i \in \{0,1\}, \quad \text{where } i = 1, 2, \ldots, n, \end{gathered} \tag{6.3}$$

and obtain $\gamma$.

*Step 2.* For $i = 0$ to $n - \gamma - 1$, compute all of solutions:

$$\begin{gathered} (A + I_n)V \geq 1_n, \\ v_1 + v_2 + \cdots + v_n = \gamma + i, \\ v_j \in \{0,1\}, \quad \text{where } j = 1, 2, \ldots, n. \end{gathered} \tag{6.4}$$

*Step 3.* Set $f_i$ to be equal to all solutions of Step 2.

## 7. Edge Dominating Set and Edge Dominating Polynomial

With the same argument in previous sections we have Theorems 7.1 and 7.3.

**Theorem 7.1.** *One has*

$$\begin{aligned} \gamma' &= \min \sum_{i=1}^{m} e_i \\ &\quad \textit{subject to} \quad (B + I_m)E \geqq 1_m, \\ e_i \in \{0,1\}, &\quad \textit{where } i = 1, 2, \ldots, m. \end{aligned} \tag{7.1}$$

*Definition 7.2.* An *edge dominating polynomial* is a polynomial such as

$$W(x) = f_0' x^{\gamma'} + f_1' x^{\gamma'+1} + \cdots + f_{m-\gamma'}' x^m, \tag{7.2}$$

where $\gamma'$ is the same as in (7.1) and $f_i'$'s are the number of edge dominating sets with $\gamma' + i$ elements.

**Theorem 7.3.** *The coefficients $f_0', f_1', \ldots, f_{m-\gamma'}'$ are all of solutions of the following system, respectively, $i = 0$, $i = 1, \ldots, i = m - \gamma'$,*

$$\begin{gathered}(B + I_m)E \geq 1_m, \\ e_1 + e_2 + \cdots + e_m = \gamma' + i, \\ e_j \in \{0, 1\}, \quad \textit{where } j = 1, 2, \ldots, m.\end{gathered} \tag{7.3}$$

*Algorithm 7.4* (For computation $f_i$). One has the following.

*Step 1.* Solve

$$\begin{gathered}\gamma' = \min \sum_{i=1}^{m} e_i, \\ (B + I_m)E \geq 1_m, \\ e_i \in \{0, 1\}, \quad \text{where } i = 1, 2, \ldots, m,\end{gathered} \tag{7.4}$$

and obtain $\gamma'$.

*Step 2.* For $i = 0$ to $m - \gamma' - 1$, compute all of solutions:

$$\begin{gathered}(B + I_m)E \geq 1_m, \\ e_1 + e_2 + \cdots + e_m = \gamma' + i, \\ e_j \in \{0, 1\}, \quad \text{where } j = 1, 2, \ldots, m.\end{gathered} \tag{7.5}$$

*Step 3.* Set $f_i'$ to be equal to all solutions of Step 2.

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