International Scholarly Research Network
ISRN Algebra
Volume 2011, Article ID 208478, 4 pages
doi:10.5402/2011/208478

Research Article

A Characterization of Uniform Matroids

Brahim Chaourar

Sciences College, Al Imam Muhammad Ibn Saud Islamic University, P.O. Box 286574, Riyadh 11323, Saudi Arabia

Correspondence should be addressed to Brahim Chaourar, bchaourar@hotmail.com

Received 18 September 2011; Accepted 23 October 2011

Academic Editor: A. Jaballah

This paper gives a characterization of uniform matroids by means of locked subsets. Locked subsets are 2-connected subsets, their complements are 2-connected in the dual, and the minimum rank of both is 2. Locked subsets give the nontrivial facets of the bases polytope.

1. Introduction

Sets and their characteristic vectors will not be distinguished. We refer to Oxley [1] for the terminology about matroids and to Schrijver [2] for the terminology about polyhedra.

Let E be a finite set, and let M be a matroid defined on E. If M is 2-connected, then we will say that a proper subset L of E; that is, $\emptyset \neq L \neq E$, is *locked* if L is nonseparable or 2-connected in M, $E \setminus L$ is nonseparable or 2-connected in M^* and $r(L) \geq \max\{2, 2 + r(E) - |E \setminus L|\}$ or $\min\{r(L), r^*(E \setminus L)\} \geq 2$. Observe that L is locked in a matroid M if and only if $M \mid L$ and M/L are both connected and $\min\{r(L), r^*(E \setminus L)\} \geq 2$. Locked subsets give the nontrivial facets of the bases polytope. We will denote the class of these subsets by $\Lambda(M)$ or Λ. If M is not 2-connected, then $M = M_1 \oplus M_2 \oplus \cdots \oplus M_k$ where each M_j is 2-connected, $j = 1, 2, \ldots, k$, and the class of locked subsets $\Lambda(M)$ is the union of such classes in each 2-connected component involved in the direct sums. Locked subsets were introduced by Chaourar ([3–5]) to describe some facets of the cone and the polytope generated by the matroid bases. We will denote the 2-sum of two matroids M and N using the basepoint $\{e\}$ by $M \oplus_e N$ or $M \oplus_2 N$ if there is no confusion. Since 2-sums with $U_{1,2}$ and $U_{1,1}$ are, respectively, identity and deletion, then we will consider only proper 2-sums without $U_{1,2}$ nor $U_{1,1}$.

Let M be the class of matroids obtained by means of 1-sums (or direct sums) and (proper) 2-sums of uniform matroids together with all minors of such matroids. Since $(M \oplus N)^* = M^* \oplus N^*$ and $(M \oplus_2 N)^* = M^* \oplus_2 N^*$ (see [1]), then M is closed under the taking of duals. It is also clear that M is closed under the taking of minors.

If M is not a 3-connected matroid, then, using a theorem of Oxley (see [1]), M can be constructed from 3-connected minors of it by a sequence of the operations of 1-sum and 2-sum.

The purpose of this paper is to characterize uniform matroids by means of locked subsets. There are exactly five 3-connected matroids of rank 3 on a 6-element set. These matroids can be obtained from $M(K_4)$ by relaxing zero, one, two, three, or four circuit-hyperplanes. The matroids are, respectively, $M(K_4)$, the rank-3 whirl W^3, Q_6, P_6, and the uniform matroid $U_{3,6}$ (see [1]).

The remaining of the paper is organized as follows: in Section 2, we will give a characterization of uniform matroids by means of locked subsets, two consequences are given in Section 3, and the conclusion is given is Section 4.

2. The Characterization

We will need to three lemmas in this section.

Lemma 2.1. $|\Lambda(M^*)| = |\Lambda(M)|$.

Proof. Direct from the definition of a locked subset. □

Lemma 2.2. *Let N be a 3-connected minor of a 2-connected matroid M. If $\Lambda(N) \neq \emptyset$, then $\Lambda(M) \neq \emptyset$.*

Proof. Using duality and Lemma 2.1, it suffices to prove that, if $\Lambda(M \setminus e) \neq \emptyset$, then $\Lambda(M) \neq \emptyset$.

Suppose that L is locked in $M \setminus e$. We establish that $L \cup \{e\}$, when $e \in$ closure (L), or L, when $e \notin$ closure (L), is locked in M. If L spans e in M, then $M \mid (L \cup \{e\})$ is connected because e is not a loop of M. As

$$M/(L \cup \{e\}) = (M/L)/e = (M/L) \setminus e = (M \setminus e)/L \tag{2.1}$$

is connected, it follows that $L \cup \{e\}$ is locked in M. If L does not span e in M, then e is not a loop of M/L. Therefore, M/L is connected because $(M/L) \setminus e = (M \setminus e)/L$ is connected (remember that L is locked in $M \setminus e$). Thus, L is locked in M. □

The following last lemma of this section was proved by Walton [6] and we give here a new proof based on locked subsets.

Lemma 2.3. *Let M be a 3-connected matroid having no isomorphic minor to any of $M(K_4), W^3, Q_6$, and P_6. Then M is uniform.*

Proof. Suppose by contradiction that M is not uniform. It follows that there exists a subset F of M such that $|F| = r(M)$ and F contains a circuit C. Without loss of generality, we can suppose that $|C| = 3$ because, if it is not, we can contract some elements of C keeping C as a circuit and decreasing its cardinality. Now we delete all elements of $F - C$. Let N be the obtained matroid.

Case 1. If $r_N(E - C) = r(N)$, then let B be a base of N included into $E - C$. If $|B| > 3$, then contract some elements of B keeping B as a base and C as a circuit with $|B| = |C| = 3$. C is a locked subset of the matroid $N \mid (B \cup C)$ because C is 2-connected, B is a cocircuit, and $r(C) = r^*(B) = 2$. Thus, $N \mid (B \cup C)$ is one of the excluded minors, a contradiction.

Case 2. If $r_N(E - C) < r(N)$, then N is a series extension of a uniform matroid. By induction on $|E(M)|$, the matroid U, obtained by contracting one element in the series closure S, is uniform. But S intersect C so there are two parallel elements e and f in U. Since $r(\{e, f\}) = 1$, then $r(U) = 1$, a contradiction. □

Here we give our main result.

Theorem 2.4. *If M is a 3-connected matroid, then the following assertions are equivalent:*

(i) *M is a uniform matroid,*

(ii) $\Lambda(M) = \emptyset$.

Proof. (i)⇒(ii) Using the fact that there is a unique closed and 2-connected subset which is E.

(ii)⇒(i) Using Lemma 2.2, any minor N of M verifies $\Lambda(N) = \emptyset$. So M has no isomorphic minor to any of $M(K_4), W^3, Q_6$, and P_6, because any of these excluded minors has at least one locked subset (circuit of rank 3). By Lemma 2.3, M is uniform. □

Note that (i) implies (ii), in Theorem 2.4, even if M is not 3-connected.

3. Some Consequences

We will give two corollaries of our characterization.

The first one is a characterization by excluded minors and is almost a restatement of Lemma 2.3, and Walton should be credited for this result:

Corollary 3.1. *The following assertions are equivalent for a matroid M:*

(i) *M is a minor of 1-sums and 2-sums of uniform matroids,*

(ii) *M has no isomorphic to any of $M(K_4), W^3, Q_6$ and P_6.*

Proof. (i)⇒(ii) By contradiction, suppose that M has one isomorphic to any of the excluded minors. Since all the excluded minors are 3-connected, then at least one of the 3-connected components used to construct M by means of 1-sums and 2-sums has one such excluded minor. Let N be this excluded minor. Since the number of locked subsets for any excluded minor is at least 1, then, using Lemmas 2.2 and 2.3 and Theorem 2.4, $\Lambda(N) \neq \emptyset$ and N is not uniform.

(ii)⇒(i) If M is 3-connected, then, by Lemma 2.3, M is uniform. If M is not 3-connected, then M can be construct using 3-connected matroids by means of 1-sum and 2-sum. It follows that no one of these matroids has an isomorphic to any of the excluded minors and, by Lemma 2.3, all these matroids are uniform. □

We will need the following result of Chaourar [5] to deduce the second corollary.

Theorem 3.2. *If M is a 2-connected matroid, then its bases' polytope is given by the following constraints:*

$$x(E) = r(E), \tag{3.1}$$

$$x(S) \geq |S| - 1 \quad \text{for any series closure } S \text{ of } M, \tag{3.2}$$

$$x(P) \leq 1 \quad \text{for any parallel closure } P \text{ of } M, \tag{3.3}$$

$$x(H) \leq r(H) \quad \text{for any locked subset } H \text{ of } M. \tag{3.4}$$

Corollary 3.3. *If M is a 2-connected and uniform matroid, then its bases' polytope is given by constraints* (3.1)–(3.3).

Proof. Direct from Theorems 2.4 and 3.2. □

4. Conclusion

We have given a characterization of uniform matroids by means of locked subsets and two consequences of this characterization.

Acknowledgment

Many thanks are due to Al Imam Muhammad Bin Saud University for financially supporting this research under the Grant no. 301216.

References

[1] J. G. Oxley, *Matroid Theory*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, NY, USA, 1992.
[2] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons, Chichester, UK, 1986.
[3] B. Chaourar, *Hilbert bases and combinatorial applications*, Ph.D. thesis, Joseph Fourier University, Grenoble, France, 1993.
[4] B. Chaourar, "On greedy bases packing in matroids," *European Journal of Combinatorics*, vol. 23, no. 7, pp. 769–776, 2002.
[5] B. Chaourar, "On the Kth best base of a matroid," *Operations Research Letters*, vol. 36, no. 2, pp. 239–242, 2008.
[6] P. N. Walton, *Some topics in combinatorial theory*, Ph.D. thesis, University of Oxford, 1981.