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Research Article

Certain Transformation Formulae for Polybasic Hypergeometric Series

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Making use of Bailey's transformation and certain known summations of truncated series, an attempt has been made to establish transformation formulae involving polybasic hypergeometric series.

1. Introduction

The remarkable contribution in the field of hypergeometric and basic hypergeometric series mainly due to Bailey [1] has appeared in Proceeding of London Mathematical society in 1947. The key result of the paper later on recognized as Bailley's transformation is as follows:

$$\begin{aligned} \text{if } \beta_n &= \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}, \\ \gamma_n &= \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{n+r}, \\ \text{then } \sum_{n=0}^{\infty} \alpha_n \gamma_n &= \sum_{n=0}^{\infty} \beta_n \delta_n, \end{aligned} \tag{1.1}$$

where $\alpha_r, \delta_r, u_r, v_r$ are functions of r only, such that the series for γ_n exists. Bailey's paper [2] published in the London Mathematical society in 1949, that strengthened the importance of Bailey's transformation. The main result of the paper [2] was recognized as Bailey's lemma during the 20th century. Making use of celebrated transformation, Bailey [1, 2] developed a number of transformations for both ordinary and basic hypergeometric series, and later on he successfully used these transformations to obtain a number of identities of the Rogers-Ramanujan type. The extensive use of Bailey transformation appeared in the papers of Slater [3, 4] and these papers were published in 1951 and 1952, respectively. Slater established 130 identities of the Rogers-Ramanujan type in [3, 4]. The platform provided by Bailey and Slater motivated a number of mathematicians namely Agarwal [5, 6], Andrews [7–9], Andrews and Warner [10], Bressoud et al. [11, 12], Denis et al. [13], Joshi and Vyas [14], Schilling and Warnaar [15], Singh [16], Srivastava [17], Verma and Jain [18, 19] and due to the contribution of these mathematicians, literatures of ordinary and basic hypergeometric series were enriched. In the present paper, making use of certain known summations of truncated series, an attempt has been made to establish transformation formulae involving poly-basic hypergeometric series.

2. Definitions and Notations

For real or complex q ($|q| < 1$), put

$$(\lambda;q)_\infty = \prod_{n=0}^{\infty}(1-\lambda q^n). \tag{2.1}$$

Let $(\lambda;q)_\mu$ be defined by

$$(\lambda;q)_\mu = \frac{(\lambda;q)_\infty}{(\lambda q^\mu;q)_\infty}. \tag{2.2}$$

For arbitrary parameters λ and μ, so that

$$(\lambda;q)_n = \begin{cases} 1, & n = 0, \\ (1-\lambda)(1-\lambda q)\cdots(1-\lambda q^{n-1}), & n\varepsilon(1,2,3\ldots), \end{cases} \tag{2.3}$$

the generalized basic hypergeometric series is defined by:

$${}_r\phi_s\left[\begin{matrix} a_1,a_2,\ldots,a_r;q;z \\ b_1,b_2,\ldots,b_s \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n z^n}{(q,b_1,b_2,\ldots,b_s;q)_n}, \tag{2.4}$$

where $(a_1,a_2,\ldots,a_r;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n$ and $\max(|q|,|z|<1)$ for convergence.

The truncated basic hypergeometric series is defined by

$${}_r\phi_s\left[\begin{matrix} a_1,a_2,\ldots,a_r;q;z \\ b_1,b_2,\ldots,b_s \end{matrix}\right]_N = \sum_{n=0}^{N}\frac{(a_1,a_2,\ldots,a_r;q)_n z^n}{(q,b_1,b_2,\ldots,b_s;q)_n}. \tag{2.5}$$

The polybasic hypergeometric series is defined by (cf. Gasper and Rahman [20, (3.9.1) page 85]):

$$\Phi\begin{bmatrix} a_1, a_2, \ldots, a_r : c_{1,1}, \ldots, c_{1,r_1}; \ldots; c_{m,1}, \ldots, c_{m,r_m}; q, q_1, \ldots, q_m; z \\ b_1, b_2, \ldots, b_{r-1} : d_{1,1}, \ldots, d_{1,r_1}; \ldots; d_{m,1}, \ldots, d_{m,r_m} \end{bmatrix}$$
$$= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n z^n}{(q, b_1, b_2, \ldots, b_{r-1}; q)_n} \prod_{j=1}^{m} \frac{\left(c_{j,1}, \ldots, c_{j,r_j}; q_j\right)_n}{\left(d_{j,1}, \ldots, d_{j,r_j}; q_j\right)_n}, \tag{2.6}$$

where $\max(|z|, |q|, |q_1|, \ldots, |q_m|) < 1$ for convergence.

The other notations appearing in this paper have their usual meaning. We will use the following summation formulae in our analysis:

$$_2\phi_1\begin{bmatrix} a, y; & \\ ayq; & q, q \end{bmatrix}_n = \frac{(aq, yq; q)_n}{(q, ayq; q)_n}, \tag{2.7}$$

see [5, App.II(8)]

$$_4\phi_3\begin{bmatrix} \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, e; & \\ \sqrt{\alpha}, -\sqrt{\alpha}, \dfrac{\alpha q}{e}; & q, \dfrac{1}{e} \end{bmatrix}_n = \frac{(\alpha q, eq; q)_n}{(q, \alpha q/e; q)_n e^n}, \tag{2.8}$$

see [5, App.II(8)]

$$_6\phi_5\begin{bmatrix} \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta, \gamma, \delta; & \\ \sqrt{\alpha}, -\sqrt{\alpha}, \dfrac{\alpha q}{\beta}, \dfrac{\alpha q}{\gamma}, \dfrac{\alpha q}{\delta}; & q, q \end{bmatrix}_n = \frac{(\alpha q, \beta q, \gamma q, \delta q; q)_n}{(q, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta; q)_n}, \tag{2.9}$$

see [5, App.II(25)] provided $\alpha = \beta\gamma\delta$,

$$\sum_{r=0}^{n} \frac{(1 - ap^r q^r)(a; p)_r (c; q)_r c^{-r}}{(1 - a)(q; q)_r (ap/c; p)_r} = \frac{(ap; p)_n (cq; q)_n}{(q; q)_n (ap/c; p)_n c^n}, \tag{2.10}$$

see [20, App.II(II.34)]

$$\sum_{r=0}^{n} \frac{(1 - ap^r q^r)(1 - bp^r q^{-r})(a, b; p)_r (c, a/bc; q)_r q^r}{(1 - a)(1 - b)(q, aq/b; q)_r (ap/c, bcp; p)_r} = \frac{(ap, bp; p)_n (cq, aq/bc; q)_n}{(q, aq/b; q)_n (ap/c, bcp; p)_n}, \tag{2.11}$$

see [20, App.II(II.35)]

$$\sum_{r=0}^{n}\frac{(1-adp^rq^r)(1-bp^r/dq^r)(a,b;p)_r(c,ad^2/bc;q)_rq^r}{(1-ad)(1-b/d)(dq,adq/b;q)_r(adp/c,bcp/d;p)_r}$$
$$=\frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}$$
$$\times\left\{\frac{(ap,bp;p)_n(cq,ad^2q/bc;q)_n}{(dq,adq/b;q)_n(adp/c,bcp/d;p)_n}-\frac{(c/ad,d/bc;p)_1(1/d,b/ad;q)_1}{(1/c,bc/ad^2;q)_1(1/a,1/b;p)_1}\right\}, \tag{2.12}$$

which is $m = 0$, case of [20, App. II (II. 36)].

3. Main Results

In this section we have established the following main results.

$$\Phi\begin{bmatrix}\alpha q,\beta q : a,y; & \\ \alpha\beta q : p,ayp; & q,p;p\end{bmatrix}=\frac{[ap,yp;p]_\infty}{[p,ayp;p]_\infty}\frac{[\alpha q,\beta q;q]_\infty}{[q,\alpha\beta q;q]_\infty}$$
$$-\frac{q(1-\alpha)(1-\beta)}{(1-q)(1-\alpha\beta q)}\Phi\begin{bmatrix}ap,yp : \alpha q,\beta q; & \\ ayp : q^2,\alpha\beta q^2; & p,q;q\end{bmatrix}, \tag{3.1}$$

$$\Phi\begin{bmatrix}\alpha q,eq : a,y; & \\ \dfrac{\alpha q}{e} : p,ayp; & q,p;\dfrac{p}{e}\end{bmatrix}=-\frac{(1-\alpha q^2)(1-e)}{e(1-q)(1-\alpha q/e)}$$
$$\times\Phi\begin{bmatrix}ap,yp : \alpha q,q^2\sqrt{\alpha},-q^2\sqrt{\alpha},eq; & \\ ayp : q^2,q\sqrt{\alpha},-q\sqrt{\alpha},\dfrac{\alpha q^2}{e}; & p,q;\dfrac{1}{e}\end{bmatrix}, \tag{3.2}$$

$$\Phi\begin{bmatrix}\alpha q,\beta q,\gamma q,\delta q : a,y; & \\ \dfrac{\alpha q}{\beta},\dfrac{\alpha q}{\gamma},\dfrac{\alpha q}{\delta} : p,ayp; & q,p;p\end{bmatrix}$$
$$=\frac{[ap,yp;p]_\infty[\alpha q,\beta q,\gamma q,\delta q;q]_\infty}{[p,ayp:p]_\infty[q,\alpha q/\beta,\alpha q/\gamma,\alpha q/\delta;q]_\infty}$$
$$-\frac{(1-q^2\alpha)(1-\beta)(1-\gamma)(1-\delta)q}{(1-q)(1-\alpha q/\beta)(1-\alpha q/\gamma)(1-\alpha q/\delta)} \tag{3.3}$$
$$\times\Phi\begin{bmatrix}ap,yp : \alpha q,q^2\sqrt{\alpha},-q^2\sqrt{\alpha},\beta q,\gamma q,\delta q; & \\ ayp : q^2,q\sqrt{\alpha},-q\sqrt{\alpha},\dfrac{\alpha q^2}{\beta},\dfrac{\alpha q^2}{\gamma},\dfrac{\alpha q^2}{\delta}; & p,q;q\end{bmatrix},$$

$$\Phi\left[\begin{matrix} x,y:ap:cp; \\ xyP:\dfrac{ap}{c}:q; \end{matrix}\quad P,p,q;\frac{P}{c}\right]
= \frac{(1-apq)(1-c)}{(1-q)(1-ap/c)c}
\times\Phi\left[\begin{matrix} xP,yP:ap:cq:ap^2q^2; \\ xyP:\dfrac{ap^2}{c}:q^2:apq; \end{matrix}\quad P,p,q,pq;\frac{1}{c}\right], \tag{3.4}$$

$$\begin{aligned}\Phi&\left[\begin{matrix} x,y:ap,bp:cq,\dfrac{aq}{bc}; \\ xyP:\dfrac{ap}{c},bcp:q,\dfrac{aq}{b}; \end{matrix}\quad P,p,q;P\right]\\
&= \frac{[xP,yP;P]_\infty[ap,bp;p]_\infty[cq,aq/bc;q]_\infty}{[P,xyP;P]_\infty[q,aq/b;q]_\infty[ap/c,bcp;p]_\infty}\\
&\quad-\frac{(1-apq)(1-bp/q)(1-c)(1-a/bc)q}{(1-q)(1-aq/b)(1-ap/c)(1-bcp)}\\
&\quad\times\Phi\left[\begin{matrix} xP,yP:ap^2q^2:\dfrac{bp^2}{q^2}:ap,bp:cq,\dfrac{aq}{bc}; \\ xyP:apq:\dfrac{bp}{q}:\dfrac{ap^2}{c},bcp^2:q^2,\dfrac{aq^2}{b}; \end{matrix}\quad P,pq,\frac{p}{q},p,q;q\right],\end{aligned} \tag{3.5}$$

$$\begin{aligned}\Phi&\left[\begin{matrix} x,y:ap,bp:cq,\dfrac{ad^2q}{bc}; \\ xyP:\dfrac{adp}{c},\dfrac{bcp}{d}:dq,\dfrac{adq}{b}; \end{matrix}\quad P,p,q;P\right]\\
&= \frac{[xP,yP;P]_\infty[ap,bp;p]_\infty[cq,ad^2q/bc;q]_\infty}{[P,xyP;P]_\infty[dq,adq/b;q]_\infty[adp/c,bcp/d;p]_\infty}\\
&\quad-\frac{dq(1-adpq)(1-bp/dq)(1-c/d)(1-ad/bc)}{(1-dq)(1-adq/b)(1-adp/c)(1-bcp/d)}\\
&\quad\times\Phi\left[\begin{matrix} xP,yP:adp^2q^2:\dfrac{bp^2}{dq^2}:ap,bp:cq,\dfrac{ad^2q}{bc}; \\ xyP:adpq:\dfrac{bp}{dq}:\dfrac{adp^2}{c},\dfrac{bcp^2}{d}:dq^2,\dfrac{adq^2}{b}; \end{matrix}\quad P,pq,\frac{p}{q},p,q;q\right].\end{aligned} \tag{3.6}$$

4. Proof of Main Results

Taking $u_r = v_r = 1$ in (1.1), Bailey's transformation takes the following form:

$$\text{If } \beta_n = \sum_{r=0}^{n} \alpha_r, \tag{4.1}$$

$$\gamma_n = \sum_{r=0}^{\infty} \delta_r, \tag{4.2}$$

$$\text{then } \sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{4.3}$$

Proof of Result (3.1). Taking $\alpha_r = (\alpha, \beta; q)_r q^r / (q, \alpha\beta q; q)_r$ and $\delta_r = (a, y; p)_r p^r / (p, ayp; p)_r$ in (4.1) and (4.2), respectively, and making use of (2.7), we get

$$\beta_n = \frac{(\alpha q, \beta q; q)_n}{(q, \alpha\beta q; q)_n}, \qquad \gamma_n = \frac{(ap, yp; p)_\infty}{(p, ayp; p)_\infty} - \frac{(1-ay)(1-p^n)(a, y; p)_n}{(1-a)(1-y)(p, ay; p)_n}. \tag{4.4}$$

Putting these values in (4.3), we get the following transformation:

$$\begin{aligned} &\Phi\begin{bmatrix} \alpha q, \beta q : a, y; & \\ \alpha\beta q : p, ayp; & q, p; p \end{bmatrix} + \frac{(1-ay)}{(1-a)(1-y)} \Phi\begin{bmatrix} \alpha, \beta : a, y; & \\ \alpha\beta q : p, ay; & q, p; q \end{bmatrix} \\ &\quad = \frac{(ap, yp; p)_\infty}{(p, ayp; p)_\infty} \frac{(\alpha q, \beta q; q)_\infty}{(q, \alpha\beta q; q)_\infty} + \frac{(1-ay)}{(1-a)(1-y)} \Phi\begin{bmatrix} \alpha, \beta : a, y; & \\ \alpha\beta q : p, ay; & q, p; pq \end{bmatrix}, \end{aligned} \tag{4.5}$$

which on simplification gives the result (3.1). □

Proof of Result (3.2). Taking $\alpha_r = (\alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, e; q)_r / (q, \sqrt{\alpha}, -\sqrt{\alpha}, \alpha q/e; q)_r e^r$ and $\delta_r = (a, y; p)_r p^r / (p, ayp; p)_r$ in (4.1) and (4.2), respectively, and making use of (2.8) and (2.7), we get

$$\beta_n = \frac{(\alpha q, eq; q)_n}{(q, \alpha q/e; q)_n e^n}, \qquad \gamma_n = \frac{(ap, yp; p)_\infty}{(p, ayp; p)_\infty} - \frac{(1-ay)(1-p^n)(a, y; p)_n}{(1-a)(1-y)(p, ay; p)_n}. \tag{4.6}$$

Substituting these values in (4.3), we get the following transformation for $|e| > 1$:

$$\begin{aligned} \Phi\begin{bmatrix} \alpha q, eq : a, y; & \\ \dfrac{\alpha q}{e} : p, ayp; & q, p; \dfrac{p}{e} \end{bmatrix} &= \frac{(1-ay)}{(1-a)(1-e)} \times \Phi\begin{bmatrix} \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, e : a, y; & \\ \sqrt{\alpha}, -\sqrt{\alpha}, \dfrac{\alpha q}{e} : p, ay; & q, p; \dfrac{p}{e} \end{bmatrix} \\ &\quad - \frac{(1-ay)}{(1-a)(1-y)} \Phi\begin{bmatrix} \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, e : a, y; & \\ \sqrt{\alpha}, -\sqrt{\alpha}, \dfrac{\alpha q}{e} : p, ay; & q, p; \dfrac{1}{e} \end{bmatrix}, \end{aligned} \tag{4.7}$$

which on simplification gives result (3.2). □

Proof of Result (3.3). Taking $\alpha_r=(\alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta, \gamma, \delta; q)_r q^r/(q, \sqrt{\alpha}, -\sqrt{\alpha}, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta; q)_r$, where $\alpha = \beta\gamma\delta$ and $\delta_r = (a, y; p)_r p^r/(p, ayp; p)_r$ in (4.1) and (4.2), respectively, and making use of (2.9) and (2.7), we get

$$\beta_n = \frac{(\alpha q, \beta q, \gamma q, \delta q; q)_n}{(q, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta; q)_n},$$
$$\gamma_n = \frac{(ap, yp; p)_\infty}{(p, ayp; p)_\infty} - \frac{(1-ay)(1-p^n)(a, y; p)_n}{(1-a)(1-y)(p, ay; p)_n}. \tag{4.8}$$

Substituting these values in (4.3), we get the following transformation for $\alpha = \beta\gamma\delta$:

$$\Phi\left[\begin{matrix} \alpha q, \beta q, \gamma q, \delta q : a, y; \\ \dfrac{\alpha q}{\beta}, \dfrac{\alpha q}{\gamma}, \dfrac{\alpha q}{\delta} : p, ayp; \end{matrix} \quad q, p; p\right] + \frac{(1-ay)}{(1-a)(1-y)} \times \Phi\left[\begin{matrix} \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta, \gamma, \delta : a, y; \\ \sqrt{\alpha}, -\sqrt{\alpha}, \dfrac{\alpha q}{\beta}, \dfrac{\alpha q}{\gamma}, \dfrac{\alpha q}{\delta} : p, ay; \end{matrix} \quad q, p; q\right]$$
$$= \frac{(ap, yp; p)_\infty}{(p, ayp; p)_\infty} \times \frac{(\alpha q, \beta q, \gamma q, \delta q; q)_\infty}{(q, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta; q)_\infty} + \frac{(1-ay)}{(1-a)(1-y)}$$
$$\times \Phi\left[\begin{matrix} \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta, \gamma, \delta : a, y; \\ \sqrt{\alpha}, -\sqrt{\alpha}, \dfrac{\alpha q}{\beta}, \dfrac{\alpha q}{\gamma}, \dfrac{\alpha q}{\delta} : p, ay; \end{matrix} \quad q, p; pq\right], \tag{4.9}$$

which on simplification gives result (3.3). □

Proof of Result (3.4). Taking $\alpha_r = (apq; pq)_r (a; p)_r (c; q)_r c^{-r}/((a; pq)_r (q; q)_r (ap/c; p)_r)$ and $\delta_r = (x, y; P)_r P^r/(P, xyP; P)_r$ in (4.1) and (4.2), respectively and making use of (2.10) and (2.7), we get

$$\beta_n = \frac{(ap; p)_n (cq; q)_n c^{-n}}{(q; q)_n (ap/c; p)_n}, \qquad \gamma_n = \frac{(xP, yP; P)_\infty}{(P, xyP; P)_\infty} - \frac{(1-xy)(1-P^n)(x, y; P)_n}{(1-x)(1-y)(P, xy; P)_n}. \tag{4.10}$$

Putting these values in (4.3), we get the following transformation for $|c| > 1$:

$$\Phi\left[\begin{matrix} x, y : ap : cq; \\ xyP : \dfrac{ap}{c} : q; \end{matrix} \quad P, p, q; \frac{P}{c}\right] = \frac{(1-xy)}{(1-x)(1-y)} \times \Phi\left[\begin{matrix} x, y : apq : a : c; \\ xy : a : \dfrac{ap}{c} : q; \end{matrix} \quad P, pq, p, q; \frac{P}{c}\right]$$
$$- \frac{(1-xy)}{(1-x)(1-y)} \times \Phi\left[\begin{matrix} x, y : apq : a : c; \\ xy : a : \dfrac{ap}{c} : q; \end{matrix} \quad P, pq, p, q; \frac{1}{c}\right], \tag{4.11}$$

which on simplification gives result (3.4). □

Proof of Result (3.5). Taking $\alpha_r = (apq;pq)_r(bp/q;p/q)_r(a,b;p)_r(c,a/bc;q)_r q^r/((a;pq)_r(b;p/q)_r(q,aq/b;q)_r(ap/c,bcp;p)_r)$ and $\delta_r = (x,y;P)_r P^r/(P,xyP;P)_r$ in (4.1) and (4.2), respectively, and making use of (2.11) and (2.7), we get

$$\beta_n = \frac{(ap,bp;p)_n(cq,aq/bc;q)_n}{(q,aq/b;q)_n(ap/c,bcp;p)_n}, \qquad \gamma_n = \frac{(xP,yP;P)_\infty}{(P,xyP;P)_\infty} - \frac{(1-xy)(1-P^n)(x,y;P)_n}{(1-x)(1-y)(P,xy;P)_n}. \tag{4.12}$$

Putting these values in (4.3), we get the following transformation:

$$\begin{aligned}
&\Phi\left[\begin{matrix} x,y : ap,bp : cq,\dfrac{aq}{bc}; \\ xyP : \dfrac{ap}{c},bcp : q,\dfrac{aq}{b}; \end{matrix}\; P,p,q;P\right] + \frac{(1-xy)}{(1-x)(1-y)} \\
&\quad\times\Phi\left[\begin{matrix} x,y : apq : \dfrac{bp}{q} : a,b : c,\dfrac{a}{bc}; \\ xy : a : b : \dfrac{ap}{c},bcp : q,\dfrac{aq}{b}; \end{matrix}\; P,pq,\frac{p}{q},p,q;q\right] \\
&= \frac{(xP,yP;P)_\infty}{(P,xyP;P)_\infty}\frac{(ap,bp;p)_\infty}{(q,aq/b;q)_\infty}\frac{(cq,aq/bc;q)_\infty}{(ap/c,bcp;p)_\infty} + \frac{(1-xy)}{(1-x)(1-y)} \\
&\quad\times\Phi\left[\begin{matrix} x,y : apq : \dfrac{bp}{q} : a,b : c,\dfrac{a}{bc}; \\ xy : a : b : \dfrac{ap}{c},bcp : q,\dfrac{aq}{b}; \end{matrix}\; P,pq,\frac{p}{q},p,q;Pq\right],
\end{aligned} \tag{4.13}$$

which on simplification gives result (3.5). □

Proof of Result (3.6). Taking $\alpha_r = (adpq;pq)_r(bp/dq;p/q)_r(a,b;p)_r(c,ad^2/bc;q)_r q^r/((ad;pq)_r(b/d;p/q)_r(dq,adq/b;q)_r(adp/c,bcp/d;p)_r)$ and $\delta_r = (x,y;P)_r P^r/(P,xyP;P)_r$ in (4.1) and (4.2), respectively, and making use of (2.12) and (2.7), we get

$$\begin{aligned}
\beta_n &= \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \\
&\quad\times\left\{\frac{(ap,bp;p)_n(cq,ad^2q/bc;q)_n}{(dq,adq/b;q)_n(adp/c,bcp/d;p)_n} - \frac{(b-ad)(c-ad)(d-bc)(1-d)}{d(1-a)(1-b)(1-c)(bc-ad^2)}\right\}, \\
\gamma_n &= \frac{(xP,yP;P)_\infty}{(P,xyP;P)_\infty} - \frac{(1-xy)(1-P^n)(x,y;P)_n}{(1-x)(1-y)(P,xy;P)_n}.
\end{aligned} \tag{4.14}$$

Putting these values in (4.3), we get the following transformation:

$$\frac{(1-a)(1-b)(1-c)\left(1-ad^2/bc\right)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}\times\Phi\left[\begin{matrix} x,y : ap,bp : cq,\dfrac{ad^2q}{bc}; \\ xyP : \dfrac{adp}{c},\dfrac{bcp}{d} : dq,\dfrac{adq}{b}; \end{matrix}\; P,p,q;P\right]$$
$$+\frac{(1-xy)}{(1-x)(1-y)}\times\Phi\left[\begin{matrix} x,y : adpq : \dfrac{bp}{dq} : a,b : c,\dfrac{ad^2}{bc}; \\ xy : ad : \dfrac{b}{d} : \dfrac{adp}{c},\dfrac{bcp}{d} : dq,\dfrac{adq}{b}; \end{matrix}\; P,pq,\frac{p}{q},p,q;q\right]$$
$$=\frac{(1-a)(1-b)(1-c)\left(1-ad^2/bc\right)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}$$
$$\times\frac{(xP,yP;P)_\infty(ap,bp;p)_\infty(cq,ad^2q/bc;q)_\infty}{(P,xyP;P)_\infty(dq,adq/b;q)_\infty(adp/c,bcp/d;p)_\infty}$$
$$+\frac{(1-xy)}{(1-x)(1-y)}\times\Phi\left[\begin{matrix} x,y : adpq : \dfrac{bp}{dq} : a,b : c,\dfrac{ad^2}{bc}; \\ xy : ad : \dfrac{b}{d} : \dfrac{adp}{c},\dfrac{bcp}{d} : dq,\dfrac{adq}{b}; \end{matrix}\; P,pq,\frac{p}{q},p,q;Pq\right], \tag{4.15}$$

which on simplification gives result (3.6). □

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