

Research Article

Tensor Products of Noncommutative L^p -Spaces

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We consider the notion of tensor product of noncommutative L^p spaces associated with finite von Neumann algebras and define the notion of tensor product of Haagerup noncommutative L^p spaces associated with σ -finite von Neumann algebras.

1. Introduction and Preliminaries

The main goal of this paper is explanation of the notion of tensor products of noncommutative L^p -spaces associated with von Neumann algebras. The notion of tensor products of noncommutative probability spaces was considered by Xu in [1]. We will generalize that notations to the cases of noncommutative L^p -spaces associated with von Neumann algebras.

In this section, we also give some necessary preliminaries on noncommutative L^p -spaces associated with von Neumann algebras and tensor product of von Neumann algebras.

1.1. Noncommutative L^p -Spaces Associated with Semifinite von Neumann Algebras

We denote by \mathcal{M} an infinite-dimensional von Neumann algebra acting on a separable Hilbert space \mathcal{H} . Let us define a trace on \mathcal{M}^+ , the set of all positive elements of \mathcal{M} .

Definition 1.1. Let \mathcal{M} be a von Neumann algebra.

(i) A trace on \mathcal{M}^+ is a function $\tau : \mathcal{M}^+ \rightarrow [0, \infty]$ satisfying the following.

- (a) $\tau(x + \lambda y) = \tau(x) + \lambda \tau(y)$ for any $x, y \in \mathcal{M}^+$ and any $\lambda \in \mathbb{R}^+$.
- (b) $\tau(xx^*) = \tau(x^*x)$ for any $x \in \mathcal{M}^+$ (tracial property).

- (ii) A trace τ is faithful if $\tau(x) = 0$ implies $x = 0$.
- (iii) A trace τ is normal if $\sup_i \tau(x_i) = \tau(\sup_i x_i)$ for any bounded increasing net (x_i) in \mathcal{M}^+ .
- (iv) A trace τ is semifinite if for any nonzero $x \in \mathcal{M}^+$ there exists a nonzero $y \in \mathcal{M}^+$ such that $y \leq x$ and $\tau(y) < \infty$.
- (v) A trace τ is finite if $\tau(1) < \infty$. In this case, we will often assume that it is normalized.

Recall that a von Neumann algebra \mathcal{M} is called semifinite if any nonzero central projection contains a nonzero finite projection. The following theorem will always be used in our construction and can be found in many references (see, e.g., [2–4]).

Theorem 1.2. *A von Neumann algebra \mathcal{M} is semifinite von Neumann algebra if and only if there exists a faithful normal semifinite trace.*

Proof. Let \mathcal{M} be a von Neumann algebra and τ a faithful normal semifinite trace. For any nonzero central projection $p \in \mathcal{M}$, there exist $x \in \mathcal{M}^+$, $0 \neq x \leq p$ such that $\tau(x) < \infty$. Then, there exists a nonzero projection $e \in \mathcal{M}$ and a positive number ε such that $xe = ex \geq \varepsilon e$. Thus, e is a finite projection. Hence, \mathcal{M} is semifinite.

Conversely, let \mathcal{M} be a semifinite von Neumann algebra. We can assume that \mathcal{M} is a uniform von Neumann algebra, that is, there exists a family $\{e_i\}_{i \in I}$ of equivalent finite mutually orthogonal projections such that $\sum_{i \in I} e_i = 1$. For each e_i , the von Neumann algebra $e_i \mathcal{M} e_i$ is finite and it then possesses a finite normal trace τ_i . Define a mapping by

$$\tau(x) = \sum_{i \in I} \tau_i(v_i^* x v_i), \quad x \in \mathcal{M}^+, \quad (1.1)$$

where $v_i \in \mathcal{M}$ is a partial isometry such that $v_i^* v_i = e_i = v_i v_i^*$. Then, τ is a semifinite normal trace on \mathcal{M}^+ . Since the set of all semifinite normal traces on \mathcal{M}^+ , obtained in this manner, is sufficient. Then, \mathcal{M} possesses a faithful normal semifinite trace. \square

Let \mathcal{M} be a von Neumann algebra equipped with a faithful normal semifinite trace τ . For $0 < p < \infty$, let

$$\|x\|_p = [\tau(|x|^p)]^{1/p}, \quad \text{where } |x| = (x^* x)^{1/2}. \quad (1.2)$$

The noncommutative L^p -space $L^p(\mathcal{M}, \tau)$ associated with (\mathcal{M}, τ) is defined as the Banach space completion of $(\mathcal{M}, \|\cdot\|_p)$. We set $L^\infty(\mathcal{M}, \tau) = \mathcal{M}$ equipped with the norm $\|x\|_\infty = \|x\|$, the operator norm. Note that the usual (commutative) L^p -space is also in the family of noncommutative L^p -space (see, e.g., [1, 5]).

Elements of the noncommutative L^p -space $L^p(\mathcal{M}, \tau)$ may be identified with unbounded operators.

Definition 1.3. Let \mathcal{M} be a von Neumann algebra equipped with a faithful normal semifinite trace τ .

- (i) A linear operator $x : \text{dom}(x) \rightarrow \mathcal{H}$ is called affiliated with \mathcal{M} if $xu = ux$ for all unitary u in the commutant \mathcal{M}' of \mathcal{M} .

- (ii) A closed densely defined operator x , affiliated with \mathcal{M} , is called τ -measurable if for every $\varepsilon > 0$ there exists an orthogonal projection $p \in \mathcal{M}$ such that $p\mathcal{H} \subseteq \text{dom}(x)$ and $\tau(1-p) < \varepsilon$.

For $0 < p < \infty$, we have

$$L^p(\mathcal{M}, \tau) \cong \{x \mid x \text{ is } \tau\text{-measurable, } \tau(|x|^p) < \infty\}. \quad (1.3)$$

Note that $L^2(\mathcal{M}, \tau)$ is a Hilbert space with respect to the scalar product $\langle x, y \rangle = \tau(y^*x)$.

If τ is a normal faithful finite trace, then it is normalized, that is, $\tau(1) = 1$. In this case, (\mathcal{M}, τ) is called a *noncommutative probability space*.

1.2. Noncommutative L^p -Spaces Associated with Arbitrary von Neumann Algebras

In this subsection, we will recall the definitions of cross product (see [2]) and Haagerup noncommutative L^p -spaces. For details of the following results in Haagerup noncommutative L^p -spaces, we refer to [1, 5].

Let \mathcal{M} be a von Neumann algebra on a Hilbert space \mathcal{H} , $\text{Aut}(\mathcal{M})$ the group of all $*$ -automorphism of \mathcal{M} , G a locally compact group equipped with its left Haar measure dg and

$$G \ni g \mapsto \pi_g \in \text{Aut}(\mathcal{M}) \quad (1.4)$$

a homomorphism of group, such that for any $x \in \mathcal{M}$, the mapping

$$G \ni g \mapsto \pi_g(x) \in \mathcal{M} \quad (1.5)$$

is continuous for the weak operator topology in \mathcal{M} . Let $C_c(G, \mathcal{H})$ be the space of all norm continuous functions defined on G and taking values in \mathcal{H} which have compact supports. We endow it with the inner product:

$$\langle f_1, f_2 \rangle = \int_G \langle f_1(g), f_2(g) \rangle dg, \quad (1.6)$$

and we denote by $L^2(G, \mathcal{H})$ the Hilbert space obtained by completion.

For any $x \in \mathcal{M}$, the operator $\lambda_x \in \mathcal{B}(L^2(G, \mathcal{H}))$ is defined by the relations:

$$(\lambda_x(f))(g) = \pi_g^{-1}(x)(f(g)), \quad f \in C_c(G, \mathcal{H}), \quad g \in G, \quad (1.7)$$

whereas for any $g \in G$ one defines the unitary operator $u_g \in \mathcal{B}(L^2(G, \mathcal{H}))$ by the relations

$$(u_g(f))(g') = f(g^{-1}g'), \quad f \in C_c(G, \mathcal{H}), \quad g' \in G. \quad (1.8)$$

The von Neumann algebra generated in $\mathcal{B}(L^2(G, \mathcal{H}))$ by the operators λ_x , $x \in \mathcal{M}$ and u_g , $g \in G$, is called the *cross-product* of \mathcal{M} by the action π of G and it is denoted by $\mathcal{M} \rtimes_{\pi} G$ or simply by $\mathcal{M} \rtimes G$.

Remark 1.4. If \mathcal{M} is a von Neumann algebra on a separable Hilbert space \mathcal{H} and G is a separable abelian locally compact group acting by $*$ -automorphisms of \mathcal{M} , then the group \widehat{G} of the character of G acts by $*$ -automorphisms of $\mathcal{M} \rtimes G$. M. Takesaki has proved that

$$(M \rtimes G) \rtimes \widehat{G} \cong \mathcal{M} \overline{\otimes} \mathcal{B}(L^2(G, \mathcal{H})). \quad (1.9)$$

In particular, if \mathcal{M} is properly infinite, then $(M \rtimes G) \rtimes \widehat{G} \cong \mathcal{M}$.

Let \mathcal{M} be a von Neumann algebra on a Hilbert space \mathcal{H} with a faithful normal semifinite weight φ . Let us recall the noncommutative L^p -space associated with (\mathcal{M}, φ) constructed by Haagerup (see, e.g., [1, 5]).

Let $\sigma_t = \sigma_t^\varphi$, $t \in \mathbb{R}$ denote the one parameter modular automorphism group of \mathbb{R} on \mathcal{M} associated with φ . The group $\{\sigma_t^\varphi\}$ is the only group of $*$ -automorphisms of \mathcal{M} , with respect to φ which satisfies the KMS-conditions. We consider the cross-product $\mathcal{N} = \mathcal{M} \times_{\sigma} \mathbb{R}$, that is, a von Neumann algebra acting on $L^2(\mathbb{R}, \mathcal{H})$, generated by the operators π_x , $x \in \mathcal{M}$, and the operators λ_s , $s \in \mathbb{R}$, defined by

$$\pi_x(f(t)) = \sigma_{-t}(x)f(t), \quad \lambda_s(f(t)) = f(t-s) \quad \text{for any } f \in L^2(\mathbb{R}, \mathcal{H}), \quad t \in \mathbb{R}. \quad (1.10)$$

It is well known that cross product \mathcal{N} is semifinite (see [5]). By Theorem 10.29 of [2], there exists a strong operator continuous group $\{u_t\}_{t \in \mathbb{R}}$ of unitary operators in \mathcal{M} such that

$$\widehat{\sigma}_t^\varphi(x) = u_t x u_t^*, \quad t \in \mathbb{R}. \quad (1.11)$$

Let τ be its (unique) faithful normal semifinite trace satisfying

$$\tau \circ \widehat{\sigma}_t = e^{-t} \tau, \quad \forall t \in \mathbb{R}, \quad (1.12)$$

The $*$ -algebra of all τ -measurable operators on $L^2(\mathbb{R}, \mathcal{H})$ affiliated with \mathcal{N} is denoted by $\widetilde{\mathcal{N}}$. For each $0 < p \leq \infty$, we define the *Haagerup noncommutative L^p -spaces* by

$$L^p(\mathcal{M}, \varphi) = \left\{ x \in \widetilde{\mathcal{N}} \mid \widehat{\sigma}_t(x) = e^{-t/p} x, \quad \forall t \in \mathbb{R} \right\}. \quad (1.13)$$

We have

$$L^\infty(\mathcal{M}, \varphi) = \mathcal{M}, \quad L^1(\mathcal{M}, \varphi) = \mathcal{M}_*. \quad (1.14)$$

For $0 < p < \infty$, $x \in L^p(\mathcal{M}, \varphi)$ if and only if $|x^p| \in L^1(\mathcal{M}, \varphi)$, we then define

$$\|x\|_p = \| |x|^p \|_1^{1/p}, \quad x \in L^p(\mathcal{M}, \varphi). \quad (1.15)$$

For $1 \leq p < \infty$, $L^p(\mathcal{M}, \varphi)$ is a Banach space equipped with a norm $\|\cdot\|_p$. For $0 < p < 1$, $L^p(\mathcal{M}, \varphi)$ is a quasi-Banach space equipped with a p -norm $\|\cdot\|_p$.

It is well known that $L^p(\mathcal{M}, \varphi)$ is independent of φ up to isometric isomorphism preserving the order and modular structure of $L^p(\mathcal{M}, \varphi)$ (see [6–8]). Sometimes, we denote $L^p(\mathcal{M}, \varphi)$ simply by $L^p(\mathcal{M})$.

1.3. Tensor Products of von Neumann Algebras

Let $\mathcal{H} \bar{\otimes} \mathcal{K}$ be the Hilbert space tensor product of \mathcal{H} and \mathcal{K} . For $x \in \mathcal{M}$ and $y \in \mathcal{N}$, the tensor product $x \bar{\otimes} y$ is the bounded linear operator on $\mathcal{H} \bar{\otimes} \mathcal{K}$ uniquely determined by

$$(x \bar{\otimes} y)(\xi \otimes \eta) = x(\xi) \otimes y(\eta) \quad \forall \xi \in \mathcal{H}, \eta \in \mathcal{K}. \quad (1.16)$$

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, $\mathcal{N} \subset \mathcal{B}(\mathcal{K})$ be two von Neumann algebras. The algebraic tensor product $\mathcal{M} \otimes \mathcal{N}$ of \mathcal{M} and \mathcal{N} ,

$$\mathcal{M} \otimes \mathcal{N} = \left\{ \sum_{k=1}^n x_k \otimes y_k \mid x_k \in \mathcal{M}, y_k \in \mathcal{N}, n = 1, 2, \dots \right\}, \quad (1.17)$$

is a $*$ -subalgebra of operators on $\mathcal{H} \bar{\otimes} \mathcal{K}$. The von Neumann algebra generated by $\mathcal{M} \bar{\otimes} \mathcal{N}$ in $\mathcal{B}(\mathcal{H} \bar{\otimes} \mathcal{K})$ is denoted by $\mathcal{M} \bar{\otimes} \mathcal{N}$ and it is called the tensor product of von Neumann algebras \mathcal{M} and \mathcal{N} . Since the map

$$\mathcal{M} \ni x \mapsto x \bar{\otimes} 1 \in \mathcal{M} \bar{\otimes} \mathcal{N} \quad (1.18)$$

is a $*$ -isomorphism, we can view \mathcal{M} as a von Neumann subalgebra of $\mathcal{M} \bar{\otimes} \mathcal{N}$. Similarly, we can also view \mathcal{N} as a von Neumann subalgebra of $\mathcal{M} \bar{\otimes} \mathcal{N}$. By the Tomita commutation theorem, \mathcal{M} and \mathcal{N} commute and together generate $\mathcal{M} \bar{\otimes} \mathcal{N}$.

Example 1.5. Let \mathbb{T} be the unit circle equipped with the normalized Lebesgue measure dm and (\mathcal{M}, τ) a finite von Neumann algebra. Let $(L^\infty(\mathbb{T}), \mathrm{dm}) \bar{\otimes} (\mathcal{M}, \tau)$ be consisting of all functions f such that

$$\int \tau(xf(z)) \bar{z}^n \mathrm{dm}(z), \quad \forall x \in L^1(\mathcal{M}, \tau), n \in \mathbb{Z}, n > 0. \quad (1.19)$$

Then, $H^\infty(\mathbb{T}, \mathcal{M})$ is a finite subdiagonal algebra of $(L^\infty(\mathbb{T}), \mathrm{dm}) \bar{\otimes} (\mathcal{M}, \tau)$ (see [5]).

2. Tensor Products of Noncommutative L^p -Spaces Associated with von Neumann Algebras

We first consider the simple case: finite von Neumann algebras.

2.1. Tensor Products of Noncommutative L^p -Spaces Associated with Normal Faithful Finite von Neumann Algebras

Theorem 2.1. *Let \mathcal{M} and \mathcal{N} be finite von Neumann algebras equipped with normal faithful normalized traces τ_1 and τ_2 , respectively. Then, there exists a normal faithful trace on the tensor product von Neumann algebra $\mathcal{M} \bar{\otimes} \mathcal{N}$ such that*

$$\tau(x \otimes y) = \tau_1(x)\tau_2(y), \quad x \in \mathcal{M}, y \in \mathcal{N}. \quad (2.1)$$

Proof. Since τ_1 and τ_2 are normal faithful normalized traces, we can view \mathcal{M} and \mathcal{N} as von Neumann algebras acting on $\mathcal{H} = L^2(\mathcal{M}, \tau_1)$ and $\mathcal{K} = L^2(\mathcal{N}, \tau_2)$, respectively, by left multiplication. Then, τ_1 and τ_2 are the vector states associated to the identities $1_{\mathcal{M}}$ of \mathcal{M} and $1_{\mathcal{N}}$ of \mathcal{N} , respectively. That is,

$$\tau_1(x) = \langle x(1_{\mathcal{M}}), 1_{\mathcal{M}} \rangle, \quad \tau_2(y) = \langle y(1_{\mathcal{N}}), 1_{\mathcal{N}} \rangle, \quad x \in \mathcal{M}, y \in \mathcal{N}. \quad (2.2)$$

Let τ be the vector state associated to $1_{\mathcal{M}} \otimes 1_{\mathcal{N}}$ on $\mathcal{M} \bar{\otimes} \mathcal{N}$. Then, τ is uniquely determined by $\tau(x \otimes y) = \tau_1(x)\tau_2(y)$ for all $x \in \mathcal{M}, y \in \mathcal{N}$. Therefore, τ is tracial and faithful. \square

τ is called *the tensor product trace* of τ_1 and τ_2 , and we denote it by $\tau_1 \bar{\otimes} \tau_2$. Then, we can define the noncommutative L^p -spaces $L^p(\mathcal{M} \bar{\otimes} \mathcal{N}, \tau_1 \bar{\otimes} \tau_2)$ and called it the *noncommutative L^p -tensor product* of (\mathcal{M}, τ_1) and (\mathcal{N}, τ_2) .

Example 2.2. Let us consider two cases (see [1, 5]).

- (1) Let (Ω, P) be a probability space. We can represent $L^\infty(\Omega)$ as a von Neumann algebra on $\mathcal{H} = L^2(\Omega)$ by multiplication and the integral against P is a normal faithful normalized trace on $L^\infty(\Omega)$. Let (\mathcal{M}, τ) be a noncommutative probability space. Then, $L^p(L^\infty(\Omega) \bar{\otimes} \mathcal{M}, \int \bar{\otimes} \tau)$ is isometric to $L^p(\Omega, L^p(\mathcal{M}))$, the usual L^p -space of p -integrable functions from Ω to $L^p(\mathcal{M})$.
- (2) Let $\mathcal{B}(l^2)$ be equipped with the usual trace Tr and let (\mathcal{M}, τ) be a noncommutative probability space. Then, the element of $L^p(\mathcal{B}(l^2) \bar{\otimes} \mathcal{M}, \text{Tr} \bar{\otimes} \tau)$, the noncommutative L^p -tensor product of $(\mathcal{B}(l^2), \text{Tr})$ and (\mathcal{M}, τ) can be identified with an infinite matrix with entries in $L^p(\mathcal{M}, \tau)$.

2.2. Infinite Tensor Products of Noncommutative L^p -Spaces Associated with Finite von Neumann Algebras

For $n \in \mathbb{N}$, let \mathcal{M}_n be a von Neumann algebras. The *infinite algebraic tensor product* $\otimes_{n \geq 1} \mathcal{M}_n$ of \mathcal{M}_n is the set of all finite linear combinations of elementary tensors $\otimes_{n \geq 1} x_n$, where $x_n \in \mathcal{M}_n$ and all but finitely many x_n are 1, that is,

$$\bigotimes_{n \geq 1} \mathcal{M}_n = \left\{ \sum_{k=1}^m \left(\bigotimes_{n \geq 1} x_n^{(k)} \right) \mid x_n^{(k)} \in \mathcal{M}_n \text{ and all but finitely many } x_n \text{ are } 1, m \in \mathbb{N} \right\}. \quad (2.3)$$

First, let us consider infinite tensor products of noncommutative L^p -spaces associated with finite factors.

For $n \in \mathbb{N}$, let \mathcal{M}_n be a finite factor equipped with a unique normal faithful normalized trace τ_n . We have the product state τ on $\otimes_{n \geq 1} \mathcal{M}_n$, defined by

$$\tau\left(\bigotimes_{n \geq 1} x_n\right) = \prod_{n \geq 1} \tau_n(x_n), \quad x_n \in \mathcal{M}_n. \quad (2.4)$$

The *infinite von Neumann tensor product* $\overline{\otimes}_{n \geq 1} \mathcal{M}_n$ is the weak-closure of the image of the representation of $\otimes_{n \geq 1} \mathcal{M}_n$ by the left multiplication on the Hilbert space $L^2(\otimes_{n \geq 1} \mathcal{M}_n)$. It is a finite factor with the trace $\bar{\tau}$ is the extension of τ , which is the unique normalized trace. $\bar{\tau}$ is called the *infinite tensor product trace* of τ_n and denoted by $\overline{\otimes}_{n \geq 1} \tau_n$ (see [7]). Then, we can define the noncommutative L^p -spaces $L^p(\overline{\otimes}_{n \geq 1} \mathcal{M}_n, \overline{\otimes}_{n \geq 1} \tau_n)$ and called it the *infinite noncommutative L^p -tensor product* of (\mathcal{M}_n, τ_n) .

Next, let us consider the infinite tensor products of noncommutative L^p -Spaces associated with normal faithful finite von Neumann algebras.

Theorem 2.3. *Let $(\mathcal{M}_m)_{m \in \mathbb{N}}$ be a sequence of finite von Neumann algebras equipped with normal faithful normalized traces τ_m . Let $\mathcal{A} = \cup_{m \geq 1} (\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_m)$. Let \mathcal{H} be the completion of \mathcal{A} with respect to the inner product*

$$\langle x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m \rangle = \prod_{k=1}^m \tau_k(y_k^* x_k). \quad (2.5)$$

Let $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ be defined by

$$\pi(x)\Lambda(a) = \Lambda(xa), \quad x \in \mathcal{A}, \quad a \in \mathcal{A}, \quad \text{where } \Lambda : \mathcal{A} \longrightarrow \mathcal{H} \text{ is the inclusion.} \quad (2.6)$$

Let \mathcal{N} be the weak*-closure of $\pi(\mathcal{A})$ in $\mathcal{B}(\mathcal{H})$. Then, there exists a normal state ν on \mathcal{N} such that

$$\nu(x_1 \otimes \cdots \otimes x_m) = \prod_{k=1}^m \tau_k(x_k), \quad x_k \in \mathcal{M}_k, \quad m \in \mathbb{N}. \quad (2.7)$$

Proof. Let $\mathcal{H}_m = L^2(\mathcal{M}_m)$ and consider \mathcal{M}_m as a von Neumann algebra on \mathcal{H}_m by left multiplication. Let

$$\begin{aligned} \mathcal{N}_m &= \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_m, \\ \nu_m &= \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_m. \end{aligned} \quad (2.8)$$

We view \mathcal{N}_m as a von Neumann subalgebra of \mathcal{N}_{m+1} via the inclusion:

$$x_1 \otimes \cdots \otimes x_m \longmapsto x_1 \otimes \cdots \otimes x_m \otimes 1_{\mathcal{M}_{m+1}}. \quad (2.9)$$

Since $\tau_{m+1}(1) = 1$, $\nu_{m+1}|_{\mathcal{N}_m} = \nu_m$. Note that \mathcal{A} is a unital $*$ -algebra and the traces ν_m induce a faithful normal state ν_o on \mathcal{A} . Since ν_o is faithful, the representation π is faithful. Therefore, \mathcal{A} , and all \mathcal{N}_m , can be viewed as subalgebras of $\mathcal{B}(\mathcal{H})$. Let ν the restriction to \mathcal{N} of the vector

state given by $\Lambda(1)$. Then, ν is tracial and faithful. The trace $\nu|_{\mathcal{M}_m} = \tau_m$ and ν is the unique normal state on \mathcal{N} such that

$$\nu(x_1 \bar{\otimes} \cdots \bar{\otimes} x_m) = \prod_{k=1}^m \tau_k(x_k), \quad x_k \in \mathcal{M}_k, \quad m \in \mathbb{N}. \quad (2.10)$$

□

(\mathcal{N}, ν) is called *the infinite tensor products of noncommutative L^p -spaces of (\mathcal{M}_m, τ_m)* (see [1]).

Example 2.4. Let $M_2(\mathbb{C})$ be the full algebra of 2×2 matrices. Murray and von Neumann proved that the infinite tensor product

$$\overline{(\otimes_{n \geq 1} M_2(\mathbb{C}))}^{\text{WOT}}, \quad (2.11)$$

produced with respect to the unique normalized trace tr_2 on $M_2(\mathbb{C})$, is the unique AFD II_1 -factor (see, e.g., [7]).

2.3. Tensor Products of Noncommutative L^p -Spaces Associated with σ -Finite von Neumann Algebras

In the case of tensor products of σ -finite von Neumann algebras, we will apply the *reduction theorem*. This theorem was proved by Haaagerup in 1979 and can be used to reduce the problems on general noncommutative L^p -spaces to the corresponding ones on those associated with finite von Neumann algebras (see, e.g., [6, 8]).

For each $k \in \{1, 2\}$, let \mathcal{M}_k be a σ -finite von Neumann algebra. Let $L^p(\mathcal{M}_k)$ be the Haagerup noncommutative L^p -spaces. By the reduction theorem, there exist a Banach space $(X_p)_k$ (a quasi Banach space if $p < 1$), a sequence $(\mathcal{R}_{k,m})_{m \in \mathbb{N}}$ of finite von Neumann algebras, each equipped with a faithful normal finite trace $\tau_{k,m}$, and for each $m \in \mathbb{N}$ an isometric embedding $J_{k,m} : L^p(\mathcal{R}_{k,m}, \tau_{k,m}) \rightarrow (X_p)_k$ such that

- (1) $J_{k,m_1}(L^p(\mathcal{R}_{k,m_1}, \tau_{k,m_1})) \subset J_{k,m_2}(L^p(\mathcal{R}_{k,m_2}, \tau_{k,m_2}))$ for all $m_1, m_2 \in \mathbb{N}$ such that $m_1 \leq m_2$;
- (2) $\bigcup_{m \in \mathbb{N}} J_{k,m}(L^p(\mathcal{R}_{k,m}, \tau_{k,m}))$ is dense in $(X_p)_k$;
- (3) $L^p(\mathcal{M}_k)$ is isometric to a subspace $(Y_p)_k$ of $(X_p)_k$;
- (4) $(Y_p)_k$ and all $J_k(L^p(\mathcal{R}_{k,m}, \tau_{k,m}))$, $m \in \mathbb{N}$ are 1-complemented in $(X_p)_k$ for $1 \leq p < \infty$.

Here, $L^p(\mathcal{R}_{k,m}, \tau_{k,m})$ is the tracial noncommutative L^p -space associated with $(\mathcal{R}_{k,m}, \tau_{k,m})$.

Thus, we have a sequence $(\mathcal{R}_{k,m}, \tau_{k,m})$ of finite von Neumann algebras. We then have the noncommutative L^p -tensor product $(\mathcal{R}_m, \tau_m) := (\mathcal{R}_{1,m} \bar{\otimes} \mathcal{R}_{2,m}, \tau_{1,m} \bar{\otimes} \tau_{2,m})$. Applying the construction in Section 2.2, we will be able to construct the infinite tensor products of noncommutative L^p -spaces of (\mathcal{R}_m, τ_m) . Hence, we have the tensor products of noncommutative L^p -spaces of $L^p(\mathcal{M}_1)$ and $L^p(\mathcal{M}_2)$.

With this setting, if $\{\mathcal{M}_k\}_{k \in \mathbb{N}}$ be a sequence of σ -finite von Neumann algebra, we will also be able to construct the infinite tensor product of noncommutative L^p -spaces associated with σ -finite von Neumann algebras.

Let \mathcal{M} be an (arbitrary) von Neumann algebra. Then, \mathcal{M} admits the following direct sum decomposition:

$$\mathcal{M} = \bigotimes_{j \in I} \mathcal{N}_j \bar{\otimes} B(K_j), \quad (2.12)$$

where each \mathcal{N}_j is an σ -finite von Neumann algebra. Using the reduction theorem in general case, the approximation theorem can be extended to the general case as follows.

Let \mathcal{M} be a general von Neumann algebra and $0 < p < \infty$. Let $L^p(\mathcal{M})$ be the Haagerup noncommutative L^p -space associated with \mathcal{M} . Then, there exist a Banach space X_p (a quasi Banach space if $p < 1$), a family $(\mathcal{R}_i)_{i \in I}$ of finite von Neumann algebras, each equipped with a normal faithful finite trace τ_i , and, for each $i \in I$, an isometric embedding $J_i : L^p(\mathcal{R}_i, \tau_i) \rightarrow X_p$ such that

- (1) $J_i(L^p(\mathcal{R}_i, \tau_i)) \subset J_j(L^p(\mathcal{R}_j, \tau_j))$ for all $i, j \in I$ such that $i \leq j$;
- (2) $\bigcup_{i \in I} J_i(L^p(\mathcal{R}_i, \tau_i))$ is dense in X_p ;
- (3) $L^p(\mathcal{M})$ is isometric to a subspace Y_p of X_p ;
- (4) Y_p and all $J_i(L^p(\mathcal{R}_i, \tau_i))$, $i \in I$ are 1-complemented in X_p for $1 \leq p < \infty$.

Here, $L^p(\mathcal{R}_i, \tau_i)$ is the tracial noncommutative L^p -space associated with (\mathcal{R}_i, τ_i) .

If we can define the notion of (uncountable) infinite tensor products of noncommutative L^p -spaces associated with finite von Neumann algebras, we should be able to define tensor products of Haagerup noncommutative L^p -spaces.

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