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Research Article

Acceptable Morita Contexts for Semigroups

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This short note deals with Morita equivalence of (arbitrary) semigroups. We give a necessary and sufficient condition for a Morita context containing two semigroups S and T to induce an equivalence between the category of closed right S-acts and the category of closed right T-acts.

1. Preliminaries

Morita equivalence of two semigroups is usually defined by requiring that certain categories of acts over these semigroups are equivalent. If the semigroups are sufficiently “good” (in particular, if they are monoids) then the equivalence of those categories is equivalent to the existence of a unitary Morita context with surjective mappings (see, e.g., Theorems 7.3 and 8.3 of [1] or Theorem 1.1 of [2]). It is known (see Proposition 1 of [3]) that if two semigroups are contained in a unitary Morita context with surjective mappings (in such case they are called *strongly Morita equivalent* [4]), then these semigroups have to be factorisable, meaning that each of their elements can be written as a product of two elements. This suggests that perhaps the class of factorisable semigroups is the largest one to admit a meaningful Morita theory, and that theory is developed in [5]. Still in ring theory there are some articles that go beyond that limit and consider Morita equivalence of arbitrary associative rings. Our inspiration is taken from [6], where a connection between the equivalence of certain module categories over two associative rings and the existence of a certain Morita context is established. We shall prove an analogue of that result for semigroups.

Let S be a semigroup. A *left S-act* is a set A equipped with a left S-action $S \times A \to A, (s, a) \mapsto sa$, such that $s(s'a) = (ss')a$ for all $s, s' \in S$ and $a \in A$. We write ${}_SA$ if A is a left S-act. Similarly right acts over semigroups are defined. If S and T are semigroups then an *(S,T)-biact* ${}_SA_T$ is a set A which is both left S-act and right T-act and $(sa)t = s(at)$ for all

$s \in S, t \in T, a \in A$. Clearly S is an (S, S)-biact with respect to actions defined by multiplication in S. A *left S-act morphism* $\psi : {}_S A \to {}_S B$ is a mapping $\psi : A \to B$ such that $\psi(sa) = s\psi(a)$ for all $s \in S$ and $a \in A$. Similarly morphisms of right acts are defined. A *biact morphism* has to preserve both actions. Left S-acts (right T-acts, (S, T)-biacts) and their morphisms form a category.

The tensor product $A \otimes_T B$ of acts A_T and ${}_T B$ is the quotient set $(A \times B)/\sigma$ by the equivalence relation σ generated by the set $\{((at, b), (a, tb)) \mid a \in A, b \in B, t \in T\}$. The σ-class of $(a, b) \in A \times B$ is denoted by $a \otimes b$. If ${}_S A_T$ is an (S, T)-biact and ${}_T B_S$ is a (T, S)-biact then the tensor product $A \otimes_T B$ can be turned into an (S, T)-biact by setting $s(a \otimes b) := (sa) \otimes b$ and $(a \otimes b)t := a \otimes (bt)$, $a \in A, b \in B, s \in S, t \in T$.

A left S-act ${}_S A$ over a semigroup S is *unitary* if $SA = A$. A biact ${}_S A_T$ over semigroups S and T is *unitary* if $SA = A$ and $AT = A$.

Definition 1.1. A *Morita context* is a sextuple $(S, T, {}_S P_T, {}_T Q_S, \theta, \phi)$, where S and T are semigroups, ${}_S P_T$ and ${}_T Q_S$ are biacts, and

$$\theta : {}_S(P \otimes_T Q)_S \longrightarrow {}_S S_S, \qquad \phi : {}_T(Q \otimes_S P)_T \longrightarrow {}_T T_T, \tag{1.1}$$

are biact morphisms such that, for every $p, p' \in P$ and $q, q' \in Q$,

$$\theta(p \otimes q)p' = p\phi(q \otimes p'), \qquad q\theta(p \otimes q') = \phi(q \otimes p)q'. \tag{1.2}$$

Such a Morita context is called *unitary* if ${}_S P_T$ and ${}_T Q_S$ are unitary biacts.

2. The Result

Let S be a semigroup. We consider the category FAct_S of unitary closed right S-acts A_S that is acts for which the canonical right S-act morphism $\mu_A : A \otimes_S S \to A$, defined by

$$\mu_A(a \otimes s) := as, \tag{2.1}$$

$a \in A$, $s \in S$, is an isomorphism. Since for a unitary act A_S the mapping μ_A is obvioulsy surjective, the closedness of such A_S is equivalent to injectivity of μ_A.

Definition 2.1 (cf. Definition 3.6 of [6]). We say that a Morita context $(S, T, {}_S P_T, {}_T Q_S, \theta, \phi)$ is *right acceptable* if

(1) for every sequence $(s_m)_{m \in \mathbb{N}} \in S^{\mathbb{N}}$ there exists $m_0 \in \mathbb{N}$ such that $s_{m_0} \cdots s_1 \in \mathrm{im}\,\theta$,

(2) for every sequence $(t_n)_{n \in \mathbb{N}} \in T^{\mathbb{N}}$ there exists $n_0 \in \mathbb{N}$ such that $t_{n_0} \cdots t_1 \in \mathrm{im}\,\phi$.

Lemma 2.2. *Let S and T be factorisable semigroups. Then a Morita context* $(S, T, {}_S P_T, {}_T Q_S, \theta, \phi)$ *is right acceptable if and only if* θ *and* ϕ *are surjective.*

Proof. Necessity. Let $(S, T, {}_S P_T, {}_T Q_S, \theta, \phi)$ be a right acceptable Morita context. Suppose that θ is not surjective. Then there is $s \in S \setminus \mathrm{im}\,\theta$. We may factorise it as $s = s'_1 s_1$. We also may find for every $i \in \mathbb{N}$ elements $s'_{i+1}, s_{i+1} \in S$ such that $s'_i = s'_{i+1} s_{i+1}$. In particular, $s = s'_k s_k s_{k-1} \cdots s_1$

for every $k \in \mathbb{N}$. By right acceptability, for the sequence $(s_m)_{m\in\mathbb{N}}$ there exists $m_0 \in \mathbb{N}$ such that $s_{m_0} \cdots s_1 = \theta(p \otimes q)$ for some $p \in P$ and $q \in Q$. But then

$$s = s'_{m_0} s_{m_0} \cdots s_1 = s'_{m_0} \theta(p \otimes q) = \theta(s'_{m_0} p \otimes q) \in \operatorname{im}\theta, \tag{2.2}$$

a contradiction. Similarly we obtain a contradiction if ϕ is not surjective.

Sufficiency is clear. □

Definition 2.3 (cf. [7], page 289). We call a Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ **right FAct-pure** if for every $A_S \in \mathrm{FAct}_S$ and $B_T \in \mathrm{FAct}_T$ the mappings

$$\begin{aligned} 1_A \otimes \theta : A \otimes P \otimes Q \longrightarrow A \otimes S, &\qquad a \otimes p \otimes q \longmapsto a \otimes \theta(p \otimes q), \\ 1_B \otimes \phi : B \otimes Q \otimes P \longrightarrow B \otimes T, &\qquad b \otimes q \otimes p \longmapsto b \otimes \phi(q \otimes p), \end{aligned} \tag{2.3}$$

are injective.

Next we show how to construct closed acts over semigroups.

Construction 1. Let S be a semigroup and $(s_m)_{m\in\mathbb{N}} \in S^{\mathbb{N}}$. We consider the free right S-act

$$F_S := \coprod_{\mathbb{N}} S = \bigcup_{n\in\mathbb{N}} (\{n\} \times S), \tag{2.4}$$

with the right S-action

$$(n, s)z := (n, sz), \tag{2.5}$$

and its quotient act

$$M_S := F/\sim = \{[k, s] \mid k \in \mathbb{N}, s \in S\}, \tag{2.6}$$

where the right S-act congruence $\sim$ on F is defined by

$$(k, s) \sim (l, z) \Longleftrightarrow (\exists n \in \mathbb{N})(n \geq k, l \wedge s_n \cdots s_{k+1} s = s_n \cdots s_{l+1} z), \tag{2.7}$$

$k, l \in \mathbb{N}$, $s, z \in S$, and the equivalence class of a pair (k, s) by $\sim$ is denoted by $[k, s]$. Take $[k, s] \in M$, where $k \in \mathbb{N}$, $s \in S$. Since $(s_{k+2} s_{k+1}) s = s_{k+2}(s_{k+1} s)$, we have $(k, s) \sim (k+1, s_{k+1} s)$, and hence

$$[k, s] = [k+1, s_{k+1} s] = [k+1, s_{k+1}] s \in MS. \tag{2.8}$$

Consequently, $MS = M$ and M_S is unitary.

To prove the injectivity of the mapping $\mu_M : M\otimes_S S \to M$, suppose that $\mu_M([k,s]\otimes u) = \mu_M([l,z]\otimes v)$ where $k,l\in\mathbb{N}$, $s,z,u,v\in S$. Then $[k,su]=[l,zv]$ and hence $s_n\cdots s_{k+1}su = s_n\cdots s_{l+1}zv$ for some $n\geq k,l$. Consequently,

$$\begin{aligned}
[k,s]\otimes u &= [n+1, s_{n+1}s_n\cdots s_{k+1}s]\otimes u = [n+1,s_{n+1}]s_n\cdots s_{k+1}s\otimes u\\
&= [n+1,s_{n+1}]\otimes s_n\cdots s_{k+1}su = [n+1,s_{n+1}]\otimes s_n\cdots s_{l+1}zv\\
&= [n+1,s_{n+1}]s_n\cdots s_{l+1}z\otimes v = [n+1,s_{n+1}s_n\cdots s_{l+1}z]\otimes v\\
&= [l,z]\otimes v,
\end{aligned}\tag{2.9}$$

in $M\otimes_S S$, so μ_M is an isomorphism and $M_S\in \mathrm{FAct}_S$.

Given a Morita context $(S,T,{}_SP_T,{}_TQ_S,\theta,\phi)$, one can always consider natural transformations $\gamma : -\otimes_S P\otimes_T Q \Rightarrow 1_{\mathrm{FAct}_S}$ and $\delta : -\otimes_T Q\otimes_S P \Rightarrow 1_{\mathrm{FAct}_T}$ which are defined by

$$\begin{aligned}
\gamma_A &:= \mu_A\circ(1_A\otimes\theta) : A\otimes_S P\otimes_T Q_S \longrightarrow A_S\\
\delta_B &:= \mu_B\circ(1_B\otimes\phi) : B\otimes_T Q\otimes_S P_T \longrightarrow B_T,
\end{aligned}\tag{2.10}$$

$A_S\in\mathrm{FAct}_S$, $B_T\in\mathrm{FAct}_T$. Explicitly:

$$\begin{aligned}
\gamma_A(a\otimes p\otimes q) &= a\theta(p\otimes q),\\
\delta_B(b\otimes q\otimes p) &= b\phi(q\otimes p),
\end{aligned}\tag{2.11}$$

where $a\in A$, $b\in B$, $p\in P$, $q\in Q$. We say that the natural transformations γ and δ are induced by the context. The following is an analogue of Theorem 3.10 of [6].

Theorem 2.4. *Let $(S,T,{}_SP_T,{}_TQ_S,\theta,\phi)$ be a Morita context. The following assertions are equivalent.*

(1) *$\mathit{FAct}_S \underset{-\otimes_T Q}{\overset{-\otimes_S P}{\rightleftarrows}} \mathit{FAct}_T$ are inverse equivalence functors with the natural isomorphisms induced by the context.*

(2) *The context $(S,T,{}_SP_T,{}_TQ_S,\theta,\phi)$ is right acceptable and right FAct-pure.*

Proof. (1) $\Rightarrow$ (2). If γ_A,δ_B are isomorphisms then clearly $1_A\otimes\theta$ and $1_B\otimes\phi$ have to be injective, so the context is right FAct-pure.

Consider now a sequence $(s_m)_{m\in\mathbb{N}}\in S^{\mathbb{N}}$ and M_S as in Construction 1. Using surjectivity of $\gamma_M : M\otimes P\otimes Q \to M$ we can find $p\in P$, $q\in Q$, $k\in\mathbb{N}$, $s\in S$ such that

$$[1,s_1] = \gamma_M([k,s]\otimes p\otimes q) = [k,s]\theta(p\otimes q) = \bigl[k, s\theta(p\otimes q)\bigr].\tag{2.12}$$

Hence

$$s_n\cdots s_1 = s_n\cdots s_{k+1}s\theta(p\otimes q) = \theta(s_n\cdots s_{k+1}sp\otimes q)\in \operatorname{im}\theta\tag{2.13}$$

for some $n\in\mathbb{N}$, $n\geq k$.

(2) $\Rightarrow$ (1). Obviously γ_A is injective if $A_S \in \mathrm{FAct}_S$ and the context is right FAct-pure. To prove that γ_A is surjective, take $a \in A$. By unitarity of A_S, there exist $(a_m)_{m\in\mathbb{N}} \in A^{\mathbb{N}}$, $(s_m)_{m\in\mathbb{N}} \in S^{\mathbb{N}}$ such that $a = a_1 s_1$ and $a_k = a_{k+1}s_{k+1}$ for every $k \in \mathbb{N}$. Then for the sequence $(s_m)_{m\in\mathbb{N}}$ there exists $n \in \mathbb{N}$, $p \in P$, $q \in Q$ such that $s_n \cdots s_1 = \theta(p \otimes q) \in \mathrm{im}\theta$. Hence

$$a = a_1 s_1 = a_2 s_2 s_1 = \cdots = a_n s_n s_{n-1} \cdots s_1 = a_n \theta(p \otimes q) = \gamma_A(a_n \otimes p \otimes q). \tag{2.14}$$

For δ_B the proof is similar. □

Corollary 2.5. *If for semigroups S and T there exists a right acceptable and right FAct-pure Morita context then the categories FAct_S and FAct_T are equivalent.*

This corollary may be considered as a generalisation of Theorem 3 of [4], which states that if factorisable semigroups S and T are strongly Morita equivalent then they are Morita equivalent.

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