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Research Article

The EA-Dimension of a Commutative Ring

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An elementary annihilator of a ring A is an annihilator that has the form $(0:a)_A$; $a \in R \setminus (0)$. We define the elementary annihilator dimension of the ring A, denoted by $\mathrm{EAdim}(A)$, to be the upper bound of the set of all integers n such that there is a chain $(0: a_0) \subset \cdots \subset (0:a_n)$ of annihilators of A. We use this dimension to characterize some zero-divisors graphs.

1. Introduction

In this paper, all rings are considered to be commutative and unitary.

Let A be a ring and S be a nonempty subset of A. We call the annihilator of S in A denoted by $(0:S)_A$ or $(0:S)$ the set $\{a \in A / aS = (0)\}$. If $S = \{a\}$ is a singleton then $(0:S)$ will be denoted by $(0:a)$. If $a \neq 0$ then $(0:a)$ is called an elementary annihilator. An annihilator is said to be maximal if it is maximal in the set of all proper annihilators of A. It is well known that all maximal annihilators are elementary. For $n \in \mathbb{N}$ an elementary annihilator chain $(0:a_1) \subset (0:a_2) \subset \cdots \subset (0:a_{n+1})$ is said to be a chain of elementary annihilators with length n ending in $(0:a_{n+1})$. The upper bound of the set of all lengths of elementary annihilator chains ending in $(0:a)$ is called the elementary annihilator height of a_{n+1} (or $(0:a_{n+1})$). In this paper, we introduce a dimension of a ring R using elementary annihilator chains called *elementary annihilator dimension*, denoted by $\mathrm{EAdim}(R)$. The $\mathrm{EAdim}(R)$ is the upper bound of the set of elementary annihilator heights. We use this dimension to study zero-divisor graphs.

We introduce a class of rings called *isometric maximal elementary annihilator* rings, in short *IMEA*-rings. That is the class of rings with finite EAdimension whose all maximal annihilators have the same height.

2. Elementary Annihilator Dimension of a Ring

Definition 1. (1) Let $n \in \mathbb{N}$ and $(0:a_1) \subset (0:a_2) \subset \cdots \subset (0:a_{n+1})$ be chain of elementary annihilators in the ring A. One says that this chain is an elementary annihilator chain of length n ending in $(0:a_{n+1})$.

(2) Let a be a nonzero element of A. One defines the elementary annihilator height of a, denoted by $\mathrm{EAht}(a)$, as the upper bound of the set of all lengths of elementary annihilator chains ending in $(0:a)$.

(3) One calls elementary annihilator dimension of A, denoted by $\mathrm{EAdim}(A)$, the upper bound of the set $\{\mathrm{EAht}(a); a \in A \setminus \{0\}\}$.

Example 2. (1) $\mathrm{EAdim}(\mathbb{Z}/4\mathbb{Z}) = 1$. Indeed, $(0:1) \subset (0:2)$ is the longest chain of elementary annihilators in $\mathbb{Z}/4\mathbb{Z}$.

(2) $\mathrm{EAht}(1) = 0$.

(3) All nonzero zero-divisors a satisfy $\mathrm{EAht}(a) \geq 1$. Indeed, $(0:1) \subset (0:a)$ is a chain of length one.

It is easy to check the following results.

Remark 3. (1) Let $a \in A \setminus \{0\}$, $\mathrm{EAht}(a) = 0$ if and only if a is regular.

(2) $\mathrm{EAdim}(A) = 0$ if and only if A is a domain.

(3) For an ideal I of A, $\mathrm{EAdim}(A/I) = 0$ if and only if I is prime.

(4) If a is a nonzero noninvertible element $\mathrm{EAdim}(A/(a)) = 0$ if and only if a is prime.

We denote by $\mathrm{nil}(A)$ the set of all nilpotent elements of A. A is said to be reduced if it has no nilpotents other then zero.

Theorem 4. *Let $a \in \mathrm{nil}(A) \setminus \{0\}$ and $n(a)$ be its index of nilpotency; one has:* $\mathrm{EAht}(a) + n(a) \leq \mathrm{EAdim}(A) + 2$.

Proof. If $\mathrm{EAht}(a)+n(a)$ or $\mathrm{EAdim}(A)$ is infinite the result is obvious. Otherwise, there exists a chain whose length is $\mathrm{EAht}(a)$ and it ends in $(0:a)$. Let $(0:1)\subset(0:a_1)\subset\cdots\subset(0:a_{r-1})\subset(0:a)$ be this chain. Moreover, we have $(0:a)\subset\cdots\subset(0:a^{n(a)-1})$. So we obtain the following chain: $(0:1)\subset(0:a_1)\subset\cdots\subset(0:a_{r-1})\subset(0:a)\subset(0:a^2)\subset\cdots\subset(0:a^{n(a)-1})$ whose length is $\mathrm{EAht}(a)+n(a)-2$. Consequently, $\mathrm{EAht}(a)+n(a)-2\leq\mathrm{EAdim}(A)$. □

Corollary 5. *If $a\in\mathrm{nil}(A)\setminus\{0\}$ satisfies $\mathrm{EAht}(a)=\mathrm{EAdim}(A)$ is finite then $n(a)=2$. In particular, if $\mathrm{EAdim}(A)=1$ then for all $a\in\mathrm{nil}(A)\setminus\{0\}$, $n(a)=2$.*

Theorem 6. *Let A_1 and A_2 be two rings; then;*

(1) $\mathrm{EAdim}(A_1\times A_2)$ *is finite if and only if* $\mathrm{EAdim}(A_1)$ *and* $\mathrm{EAdim}(A_2)$ *are finite.*

(2) $\mathrm{EAdim}(A_1\times A_2)=\mathrm{EAdim}(A_1)+\mathrm{EAdim}(A_2)+1$.

Proof. Let $(a,b)\in A_1\times A_2$ be a nonzero zero-divisor. If a and b are nonzero then $(0:(a,b))_{A_1\times A_2}=(0:a)_{A_1}\times(0:b)_{A_2}$.

If one of them is zero, for example, $a=0$ then $(0:(a,b))_{A_1\times A_2}=A_1\times(0:b)_{A_2}$.

(1) "⇒" Let $\mathrm{EAdim}(A_1\times A_2)=n$, suppose that $\mathrm{EAdim}(A_1)$ or $\mathrm{EAdim}(A_2)$ is infinite; for example, $\mathrm{EAdim}(A_1)$ is infinite. Then there exists $r\geq n+1$ and $(0:1)\subset(0:a_1)\subset\cdots\subset(0:a_r)$ a chain in A_1; then $(0:(1,0))\subset(0:(a_1,0)\subset\cdots\subset(0:(a_r,0))$ is a chain of elementary annihilators in $A_1\times A_2$ whose length is $r>n$, contradiction.

"⇐" If we assume that $n=\mathrm{EAdim}(A_1)$, then there is a chain of length n in A_1; let $(0:1)_{A_1}\subset(0:a_1)_{A_1}\subset\cdots\subset(0:a_n)_{A_1}$ be this chain. In the same way we put $m=\mathrm{EAdim}(A_2)$ and we take $(0:1)_{A_2}\subset(0:b_1)_{A_2}\subset\cdots\subset(0:b_m)_{A_2}$ as a chain of length m. Then $(0:(1,1))_{A_1\times A_2}\subset(0:(a_1,1))_{A_1\times A_2}\subset\cdots\subset(0:(a_n,1))_{A_1\times A_2}\subset(0:(a_n,b_1))_{A_1\times A_2}\subset\cdots\subset(0:(a_n,b_m))_{A_1\times A_2}\subset(0:(0,b_m))_{A_1\times A_2}$ is an elementary annihilator chain of $A_1\times A_2$ whose length is $n+m+1$ that is maximal, because of the inclusion $(0:(a,b))_{A_1\times A_2}\subset(0:(c,d))_{A_1\times A_2}\Leftrightarrow(0:a)_{A_1}\subset(0:c)_{A_1}$ or $(0:b)_{A_2}\subset(0:d)_{A_2}$. Then $\mathrm{EAdim}(A_1\times A_2)$ is finite and $\mathrm{EAdim}(A_1\times A_2)=n+m+1$.

(2) If $\mathrm{EAdim}(A_1\times A_2)$ is infinite (that is $\mathrm{EAdim}(A_1)$ or $\mathrm{EAdim}(A_2)$ is infinite, by (1)) then the result is obvious. The finite case is shown in the proof of (1) "⇐". □

By induction, we have the following result.

Corollary 7. (1) *Let $A_1,\ldots,A_r$ be some rings, one has* $\mathrm{EAdim}(A_1\times\cdots\times A_r)=\sum_{i=1}^{r}\mathrm{EAdim}(A_i)+r-1$.

(2) *If A is a domain and $n\in\mathbb{N}^*$ then* $\mathrm{EAdim}(A^n)=n-1$.

3. The EAdimension and the Zero-Divisor Graph

Let A be a ring. The zero-divisor graph of A is defined to be the graph whose vertices are the nonzero zero-divisors of A and its edges are the pairs $\{a,b\}$ satisfying $ab=0$. We denote this graph by $\Gamma(A)$. For the simplicity of writing we still denote by $\Gamma(A)$ the set of nonzero zero-divisors of A.

$\Gamma(A)$ is said to be connected if for every two different vertices a and b of $\Gamma(A)$ there is a sequence $a_1,\ldots,a_n\in\Gamma(A)$ such as $a=a_1$, $b=a_n$ and $\{a_i,a_{i+1}\}$ is an edge, $\forall 1\leq i\leq n-1$. This sequence is called a path connecting a and b with length $n-1$. $\Gamma(A)$ is said to be complete if each two distinct vertices form an edge. We call the distance between a and b the least length of a path connecting them, denoted by $d_A(a,b)$ or $d(a,b)$. We call the diameter of $\Gamma(A)$, denoted $\mathrm{diam}(\Gamma(A))$, the supremum of the set $\{d(a,b);\ a,b\in\Gamma(A)\}$. In [1], Anderson and Livingston showed that $\Gamma(A)$ is connected and $\mathrm{diam}(\Gamma(A))\in\{0,1,2,3\}$.

For an integer $r\geq 2$, Anderson and Livingston defined $\Gamma(A)$ to be r-partite complete if $\Gamma(R)=\Gamma_1\cup\cdots\cup\Gamma_r$, where the Γ_is are nonempty disjoined sets and for all $x\neq y$ in $\Gamma(A)$ satisfy $xy\neq 0$ if and only if there exists $1\leq i\leq r$ such that $x,y\in\Gamma_i$. In this paper we extend the definition of r-partite complete graph to the case when r is infinite.

Lemma 8 (see [2], Theorem 6). (1) *If $(0:a)$ is an elementary annihilator that is maximal (in the set of proper annihilators of A) then it is prime.*

(2) *Let $a,b\in A\setminus\{0\}$; if $(0:a)$ is maximal and $(0:b)\not\subseteq(0:a)$ then $b\in(0:a)$.*

Proposition 9. *Let A be a reduced ring that is not a domain and $a\neq b$ be two nonzero zero-divisors such that $(0:a)$ and $(0:b)$ are maximal; then $ab\neq 0$ if and only if $(0:a)=(0:b)$.*

Proof. "⇒" Immediately, by Lemma 8 "⇐" If $(0:a)=(0:b)$, suppose that $ba=0$. $ba=0\Rightarrow b\in(0:a)=(0:b)\Rightarrow b^2=0$, contradiction. Then $ab\neq 0$. □

Theorem 10. *If A is a nonreduced ring then* $\mathrm{EAdim}(A)=1$ *if and only if $\Gamma(A)$ is complete.*

Proof. "⇐" if $\Gamma(A)$ is complete then, according to [1, Theorem 2.8], we have for all $x,y\in Z(A)$, $xy=0$. Then $Z(A)=(0:c)$, $\forall c\in Z(A)\setminus\{0\}$. So all nonzero elementary annihilators are equal, and then $\mathrm{EAdim}(A)=1$.

"⇒" Let $a\in\mathrm{nil}(A)\setminus\{0\}$; then $a^2=0$, according to Corollary 5. If $Z(A)=\{0,a\}$ then $\Gamma(A)$ is complete. Otherwise, for all $b\in Z(A)\setminus\{0,a\}$ we have $(0:a)$ and $(0:b)$ are maximal. Suppose that $(0:a)\neq(0:b)$, according to Lemma 8(2), $ab=0$ and there, for example, $y\in(0:a)\setminus(0:b)$; then $(0:b)\subset(0:yb)$, contradiction to the maximality of $(0:b)$. Then $(0:a)=(0:b)$ and $ab=0$. Consequently, $Z(A)=\mathrm{nil}(A)$ and all nonzero zero-divisors b satisfy $b^2=0$. It follows that for all $a,b\in Z(A)$, $ab=0$. According to [1, Theorem 2.8], $\Gamma(A)$ is complete. □

Theorem 11. *If $\Gamma(A)=\bigcup_{i\in I}\Gamma_i$ is r-partite complete graph with $r=\mathrm{card}(I)\in\mathbb{N}\cup\{\infty\}$, $r\geq 2$ then $r=2$ under one of the following conditions:*

(1) *two of the Γ_i's contain, each one, more than one element;*

(2) *A is reduced.*

Proof. Suppose that $r \geq 3$.

First case: If two among the Γ_i's contain, each one, more then one element. Assume that $\Gamma(A) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \cdots$, where Γ_1 and Γ_2 have, each one, at least two elements. Let $a_i \in \Gamma_i$, $1 \leq i \leq 3$ and $a_i \neq b_i \in \Gamma_i$, $1 \leq i \leq 2$. We have $(a_1 + a_2)a_3 = 0$; then $a_1 + a_2$ is a divisor of zero in A. Suppose that $a_1 + a_2 = 0$; then $b_1(a_1+a_2) = 0 \Rightarrow b_1 a_1 = 0$, contradiction. Then $a_1+a_2 \in \Gamma(A)$. Suppose that $a_1+a_2 \in \Gamma_3 \cup \cdots$; then $b_1(a_1+a_2) = 0 \Rightarrow b_1 a_1 = 0$, contradiction; then $a_1 + a_2 \in \Gamma_1 \cup \Gamma_2$. If $a_1 + a_2 \in \Gamma_1$ then $b_2(a_1 + a_2) = 0 \Rightarrow b_2 a_2 = 0$, contradiction. Then $a_1 + a_2 \in \Gamma_2$ and $b_1(a_1 + a_2) = 0 \Rightarrow b_1 a_1 = 0$, contradiction.

Second Case: If A Is Reduced. Assume that $\Gamma(A) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \cdots$. Let $a_i \in \Gamma_i$, $1 \leq i \leq 3$, we have $(a_1 + a_2)a_3 = 0$ then $a_1 + a_2$ is a divisor of zero in A. Suppose that $a_1 + a_2 = 0$, then $a_1(a_1 + a_2) = 0 \Rightarrow a_1^2 = 0$, contradiction. Then $a_1 + a_2 \in \Gamma(A)$. Suppose that $a_1 + a_2 \in \Gamma_3 \cup \cdots$ then $a_1(a_1 + a_2) = 0 \Rightarrow a_1^2 = 0$, contradiction. Then $a_1 + a_2 \in \Gamma_1 \cup \Gamma_2$. If $a_1 + a_2 \in \Gamma_1$ then $a_2(a_1+a_2) = 0 \Rightarrow a_2^2 = 0$, contradiction. Then $a_1+a_2 \in \Gamma_2$ and then $a_1(a_1 + a_2) = 0 \Rightarrow a_1^2 = 0$, contradiction. We conclude that $r = 2$. □

Theorem 12. *If A is reduced then* $\mathrm{EAdim}(A) = 1$ *if and only if* $\Gamma(A)$ *is bipartite complete.*

Proof. "$\Rightarrow$" In $\Gamma(A)$, we define the relation "$\sim$" by the following: $x \sim y$ if $(0 : x) = (0 : y)$. $\sim$ is a relation of equivalence. For all $x \in \Gamma(A)$, we denote by Γ_x its equivalence class. The different classes Γ_x form a partition of $\Gamma(A)$ and we write $\Gamma(A) = \bigcup_{i\in I} \Gamma_{x_i}$. Since A is reduced and not a domain, then there exist nonzero elements $b \neq a$ satisfying $ba = 0$. Now, $\mathrm{EAdim}(A) = 1$; then $(0 : a)$ and $(0 : b)$ are maximal. According to Proposition 9, $(0 : a) \neq (0 : b)$; then $\Gamma_a \neq \Gamma_b$; then $r = \mathrm{card}(I) \geq 2$.

If $y \neq z \in \Gamma_x$ then $(0 : z) = (0 : y)$ then $yz \neq 0$, by Proposition 9.

Let $\Gamma_x \neq \Gamma_y$; then $(0 : x) \neq (0 : y)$; then, by Lemma 8, $xy = 0$. And we conclude that $\Gamma(A)$ is r-partite complete. According to Theorem 11, $\Gamma(A)$ is bipartite complete.

"$\Leftarrow$" Assume that $\Gamma(A) = \Gamma_1 \cup \Gamma_2$ is bipartite complete. If $a \in \Gamma_i$ then $(0 : a) = \Gamma_j \cup \{0\}$, for $i \neq j \in \{1, 2\}$. Let $a, b \in \Gamma(A)$, if $a, b \in \Gamma_i$ then $(0 : a) = (0 : b) = \Gamma_j \cup \{0\}$. Otherwise $(0 : a)$ and $(0 : b)$ are incomparable. Then for all $a \in \Gamma(A)$, $\mathrm{EAht}(a) = 1$. It follows that $\mathrm{EAdim}(A) = 1$. □

Theorem 13. *Let R be a ring.*

(1) *If* $\mathrm{EAdim}(R) = 1$ *then* $\mathrm{diam}(R) \leq 2$.

(2) *If* $\mathrm{diam}(R) \leq 1$ *or* $\mathrm{diam}(R) = 2$ *and R is reduced then* $\mathrm{EAdim}(R) = 1$.

(3) *If* $\mathrm{diam}(R) = 2$ *and R is not reduced or* $\mathrm{diam}(R) = 3$ *then* $\mathrm{EAdim}(R) \geq 2$.

Proof. (1) If R is reduced then, by Theorem 12, $\Gamma(R)$ is bipartite complete; then $\mathrm{diam}(R) \leq 2$.

If R is not reduced then, by Theorem 10, $\Gamma(R)$ is complete then, by Theorem 2.8 of [1], for all $x, y \in Z(R)$, $xy = 0$. Then, by Theorem 2.6 of [3], $\mathrm{diam}(R) \leq 1$.

(2) If $\mathrm{diam}(R) = 2$ and R is reduced then, by Theorem 2.6 of [3], R is reduced with exactly two minimal primes and at least three nonzero zero-divisors. Then $Z(R) = P_1 \cup P_2$, where P_1, P_2 are the two minimal primes of R; they satisfy $P_1 \cap P_2 = (0)$. Then for all $p_1 \in P_1$ and $p_2 \in P_2$, $p_1, p_2 = 0$ and for $x \neq y \in P_i \setminus \{0\}$, $xy \neq 0$. Consequently, $\Gamma(R) = [P_1 \setminus \{0\}] \cup [P_2 \setminus \{0\}]$ is bipartite complete graph. According to Theorem 12, $\mathrm{EAdim}(R) = 1$.

If $\mathrm{diam}(R) \leq 1$: if $\mathrm{diam}(R) = 0$, by Theorem 2.6 of [3], R is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[y]/(y^2)$ and in both cases R has a unique nonzero elementary annihilator then $\mathrm{EAdim}(R) = 1$.

Now, if $\mathrm{diam}(R) = 1$, using Theorem 2.6 of [3], $xy = 0$ for each distinct pair of zero-divisors and R has at least two nonzero zero-divisors. According to Theorem 2.8 of [1], $\Gamma(R)$ is complete and R is not reduced. By Theorem 10, $\mathrm{EAdim}(R) = 1$.

(3) If $\mathrm{diam}(R) = 3$ then, by (1), $\mathrm{EAdim}(R) \neq 1$. Since $\mathrm{diam}(R) = 3$ then R is not a domain then $\mathrm{EAdim}(R) \neq 0$. Consequently, $\mathrm{EAdim}(R) \geq 2$. If $\mathrm{diam}(R) = 2$ and R is not reduced: $\mathrm{diam}(R) = 3$ then R is not a domain then $\mathrm{EAdim}(R) \neq 0$. Suppose that $\mathrm{EAdim}(R) = 1$; then by Theorem 10, $\Gamma(R)$ is complete then $\mathrm{diam}(R) = 1$, contradiction. Then $\mathrm{EAdim}(R) \geq 2$. □

Lemma 14. *Let A be a ring, and $F = \{P_1, \ldots, P_r\}$, $r > 1$ a set of distinct prime ideals which are incomparable. For a an element of R we denote $E(a) = \{P \in F; a \notin P\}$. Let $R = A/(P_1 \cap \cdots \cap P_r)$.*

(1) *For all subsets E of F there is $a \in A$ such that $E(a) = E$.*

(2) *Let $a \in A$, $\bar{a} \in Z(R) \Leftrightarrow E(a) \neq F$. In particular, $\bar{a} = \bar{0} \Leftrightarrow E(a) = \emptyset$.*

(3) *Let E_1 and E_2 be two nonempty subsets of F. If $\bigcap_{P\in E_1} P \subseteq \bigcap_{P\in E_2} P$ then $E_2 \subseteq E_1$.*

(4) *Let $a, b \in A$. $\bar{b} \in (\bar{0} : \bar{a}) \Leftrightarrow b \in \bigcap_{P\in E(a)} P$.*

(5) *The map $\varphi : \{(\bar{0} : \bar{a}), \bar{a} \neq \bar{0}\} \to \mathscr{P}(F) \setminus \{\emptyset\}$, $(\bar{0} : \bar{a}) \longmapsto E(a)$ is a decreasing bijection, here $\mathscr{P}(F)$ denotes the set of F subsets.*

Proof. (1) If $E = \emptyset$, take $a = 0$ then $a \in P$, for all $P \in F$ then $E(a) = \emptyset$ and the result is true in this case.

If $E = F$, take $a = 1$ then $a \notin P$, for all $P \in F$ then $E(a) = F$.

Now if $E \notin \{F, \emptyset\}$: $E(a) = E \Leftrightarrow a \in P, \forall P \notin E$ and $a \notin P, \forall P \in E \Leftrightarrow a \in (\bigcap_{P\in F\setminus E} P) \setminus (\bigcup_{P\in E} P)$. Suppose that $(\bigcap_{P\in F\setminus E} P) \setminus (\bigcup_{P\in E} P) = \emptyset \Rightarrow \bigcap_{P\in F\setminus E} P \subseteq \bigcup_{P\in E} P$; then there exists $P_{i_0} \in E$ such that $\bigcap_{P\in F\setminus E} P \subseteq P_{i_0}$; then there exists $P_{i_1} \in F \setminus E$ such that $P_{i_1} \subseteq P_{i_0}$. Since the P's in F are incomparable under inclusion then $P_{i_1} = P_{i_0}$, then $P_{i_0} \in (F \setminus E) \cap E = \emptyset$, contradiction. Consequently, there exists $a \in A$ such that $E(a) = E$.

(2) Let $a \in A$, $\bar{a} \in Z(R) \Leftrightarrow \exists \bar{b} \neq \bar{0}/\bar{a}\bar{b} = \bar{0} \Leftrightarrow \exists b \in A/b \notin \bigcap_{P\in F} P$ and $ba \in \bigcap_{P\in F} P \Leftrightarrow \exists b \in A/E(b) \neq \emptyset$ and $ba \in P, \forall P \in E(a) \Leftrightarrow \exists b \in A/E(b) \neq \emptyset$ and $b \in P, \forall P \in E(a) \Leftrightarrow \exists b \in A/E(b) \neq \emptyset$ and $E(a) \subseteq F \setminus E(b) \Leftrightarrow E(a) \neq F$. In the last equivalence the indirect sense "$\Leftarrow$" is obtained by (1).

(3) Suppose that $E_1 = \{Q_1, \dots, Q_s\}$ and $E_2 = \{R_1, \dots, R_t\}$, where the Q_i's (resp., P_i's) are pairwise different. $Q_1 \cap \cdots \cap Q_s \subseteq R_1 \cap \cdots \cap R_t \Rightarrow Q_1 \cap \cdots \cap Q_s \subseteq R_1$ then one of the Q_i's is contained in R_1; for example, $Q_1 \subseteq R_1$. Since the elements of F are incomparable under inclusion then $R_1 = Q_1$.

$Q_1 \cap \cdots \cap Q_s \subseteq R_1 \cap \cdots \cap R_t \Rightarrow Q_1 \cap \cdots \cap Q_s \subseteq R_2$ then one of the Q_i's is contained in R_2. Suppose that $Q_1 \subseteq R_2$; then $Q_1 = R_2$ then $R_1 = R_2$, contradiction. Then there exists $i \geq 2$ such that $Q_i \subseteq R_2$; for example, $Q_2 \subseteq R_2$; then $R_2 = Q_2$.

We repeat this process until reaching the stage number $n = \min(s, t)$.

Suppose that $s < t$; then $Q_1 \cap \cdots \cap Q_s = R_1 \cap \cdots \cap R_s \subseteq R_{s+1}$; then there exists $k \leq s$ such that $R_k \subseteq R_{s+1}$ that is, $R_k = R_{s+1}$, contradiction. Consequently, $s \geq t$ and we get $E_1 = \{R_1, \dots, R_t, Q_{t+1}, \dots, Q_s\}$; then $E_2 \subseteq E_1$.

(4) Let $a, b \in A$, $\overline{b} \in (\overline{0} : \overline{a}) \Leftrightarrow ba \in \bigcap_{P \in F} P \Leftrightarrow b \in \bigcap_{P \in E(a)} P$.

(5) We check that φ is well defined: put $(\overline{0} : \overline{a}) = (\overline{0} : \overline{b})$. According to (4), $\overline{x} \in (\overline{0} : \overline{a}) \Leftrightarrow x \in \bigcap_{P \in E(a)} P$. Then $x \in \bigcap_{P \in E(a)} P \Leftrightarrow x \in \bigcap_{P \in E(b)} P$; then $\bigcap_{P \in E(b)} P = \bigcap_{P \in E(a)} P$. According to (3), $E(b) = E(a)$.

For $\overline{a} \neq \overline{0}$, $E(a) \neq \emptyset$; then $E(a) \in \mathscr{P}(F) \setminus \{\emptyset\}$. Then φ is well defined.

φ is injective, by (4).

We show that φ is surjective: let $E \in \mathscr{P}(F) \setminus \{\emptyset\}$. According to (1), there exists $a \in A$ such that $E(a) = E$. Since $E \neq \emptyset$ then $\overline{a} \neq \overline{0}$ and $\varphi(\overline{0} : \overline{a}) = E$.

$(\overline{0} : \overline{a}) \subseteq (\overline{0} : \overline{b}) \Rightarrow \bigcap_{P \in E(a)} P \subseteq \bigcap_{P \in E(b)} P$. Then, according to (3), $E(b) \subseteq E(a)$. Thus φ is a decreasing bijection. □

Theorem 15. *Let A be a ring and $P_1, \dots, P_r, r \geq 1$ be different incomparable prime ideals of A. Then* $\mathrm{EAdim}\,(A/(P_1 \cap \cdots \cap P_r)) = r - 1$.

Proof. If $r = 1$ then A/P_1 is a domain and the result is checked.

Now let $r \geq 2$. $\{P_1\} \subset \{P_1, P_2\} \subset \cdots \subset \{P_1, \dots, P_{r-1}\} \subset \{P_1, \dots, P_r\}$ is decreasing sequence in $\mathscr{P}(F) \setminus \{\emptyset\}$ with maximal length. Using the bijection φ defined in Lemma 14, $\varphi^{-1}(\{P_1, \dots, P_r\}) \subset \varphi^{-1}(\{P_1, \dots, P_{r-1}\}) \subset \cdots \subset \varphi^{-1}(\{P_1, P_2\}) \subset \varphi^{-1}(\{P_1\})$ is a chain of elementary annihilators in $A/(P_1 \cap \cdots \cap P_r)$ that has a maximal length. Then $\mathrm{EAdim}(A/(P_1 \cap \cdots \cap P_r)) = r - 1$. □

Example 16. A ring is said to be semilocal ring if it has a finite number of maximal ideals. Let A be a semilocal ring with n maximal ideal; then $\mathrm{EAdim}\,(A/J) = n - 1$, where J is the Jacobson radical of A. We obtain this result by using the previous theorem.

A ring is called a noetherian spectrum ring if it satisfies the ascending chain condition (acc) on radical ideals; equivalently each radical ideal is a radical of finitely generated ideal. The set of prime ideals of a ring A which are minimal over an ideal I, denoted by $\min_I(A)$, is finite in the case when A is a noetherian spectrum ring. If $I = (0)$, we denote by $\min(A)$ instead of $\min_I(A)$. For more information about noetherian spectrum rings see [4, Chapter 2].

Proposition 17. *Let A be a noetherian spectrum ring, for all ideals I,* $\mathrm{EAdim}\,(A/\sqrt{I}) = |\min_I(A)| - 1$.

Proof. Since A is a noetherian spectrum ring then $\min_I(A)$ is finite ([4], Chapter 2, Corollary 2.1.10). Assume that $\min_I(A) = \{P_1, \dots, P_r\}$, $r = |\min_I(A)| \in \mathbb{N}^*$. The P_i's are incomparable, then we get the result by using Theorem 15. □

Definition 18 (according to [5]). Let R be a ring.

(1) One calls the chromatic number of R the minimal number of colors used to color the elements of R such that each two adjacent elements (with zero product) have different colors, denoted by $\chi(R)$.

(2) One says that the ring R is a coloring if its chromatic number is finite.

Theorem 19. *If R is a reduced coloring then* $\mathrm{EAdim}\,(R) = \chi(R) - 2$.

Proof. If R is a reduced coloring, according to [5, Theorem 3.8], $\min(R)$ is finite. And if $|\min(R)| = n$ then $\chi(R) = n + 1$. Let $\min(R) = \{P_1, \dots, P_n\}$; then $R = R/(P_1 \cap \cdots \cap P_n)$. According to Theorem 15, $\mathrm{EAdim}(R) = n - 1 = \chi(R) - 2$.

Let A be a ring such that $\mathrm{EAdim}(A) = n \geq 1$. For all vertices a in $\Gamma(A)$, we denote $i(a) = \mathrm{EAht}(a)$ and $j(a) = \max\{\mathrm{EAht}(b); (0 : a) \subseteq (0 : b)\}$. For all $i \leq j \leq n$, we denote by $\Gamma_{i,j}(A)$ the subgraph of $\Gamma(A)$ whose vertices form the following set $\{a; i(a) = i \text{ et } j(a) = j\}$.

Theoretically we can write $\Gamma(A) = \bigcup_{1 \leq i \leq j \leq n} \Gamma_{i,j}(A)$. □

Remark 20. Let A be a ring such that $\mathrm{EAdim}(A) = n \geq 2$. If $\Gamma_{1,1}(A) \neq \emptyset$ then $\mathrm{diam}(\Gamma(A)) = 2$. Indeed, consider $a \in \Gamma_{1,1}(A)$ and $b \in \Gamma(A) \setminus \Gamma_{1,1}(A)$. There exists $i \geq 1$ and $j \geq 2$ such that $b \in \Gamma_{i,j}(A)$. Suppose that $ab \neq 0$ then $b \notin (0 : a)$. According to Lemma 8(2) $(0 : b) \subseteq (0 : a)$, this contradicts the fact that $a \in \Gamma_{1,1}(A)$. Then $ab = 0$ and $\{a, b\}$ is an edge. Now take $x \neq y \in \Gamma(A)$, three cases are possible. If $x, y \in \Gamma_{1,1}(A)$ then take $b \in \Gamma(A) \setminus \Gamma_{1,1}(A)$ and the chain $x - b - y$ is of length 2 then $d(x, y) \leq 2$. If only x is in $\Gamma_{1,1}(A)$ then $x - y$ is an edge. If $x, y \in \Gamma(A) \setminus \Gamma_{1,1}(A)$ then take $a \in \Gamma_{1,1}(A)$ and $x - a - y$ is a chain of length 2. Then, in all cases, $d(x, y) \leq 2$ that is $\mathrm{diam}(\Gamma(A)) \leq 2$. Now $\mathrm{EAdim}(A) = n \geq 2$, then there exists a, b such that $(0 : 1) \subset (0 : a) \subset (0 : b)$. Let $x \in (0 : b) \setminus (0 : a)$ then $xa \neq 0$ then $d(x, a) \geq 2$. Consequently, $\mathrm{diam}(\Gamma(A)) = 2$.

4. Isometric Maximal Elementary Annihilator Rings

Definition 21. Let R be a ring with finite EA dimension; one says that R is an isometric maximal elementary annihilator ring, in short an *IMEA-ring* if its all maximal elementary annihilators have the same height.

Example 22. The ring $\mathbb{Z}/8\mathbb{Z}$ is an IMEA-ring. Indeed, the elementary annihilators of $\mathbb{Z}/8\mathbb{Z}$ are $(\overline{0} : \overline{1})$, $(\overline{0} : \overline{2})$, $(\overline{0} : \overline{4})$

and $(\overline{0} : \overline{6})$. They satisfy $(\overline{0} : \overline{1}) \subset (\overline{0} : \overline{2}) \subset (\overline{0} : \overline{4})$ and $(\overline{0} : \overline{1}) \subset (\overline{0} : \overline{2}) \subset (\overline{0} : \overline{6})$.

Theorem 23. *Let R_1 and R_2 be two rings; then $R_1 \times R_2$ is an IMEA-ring if and only if R_1 and R_2 are two IMEA-rings.*

Proof. By Theorem 6(1), $\mathrm{EAdim}(R_1 \times R_2)$ is finite if and only if $\mathrm{EAdim}(R_1)$ and $\mathrm{EAdim}(R_2)$ are finite.

$(0 : (a, b))$ is a maximal elementary annihilator in $R_1 \times R_2$ if and only if $(0 : (a, b)) = R_1 \times (0 : b)$, and $(0 : b)$ is a maximal elementary annihilator in R_2 or $(0 : (a, b)) = (0 : a) \times R_2$, and $(0 : a)$ is a maximal elementary annihilator in R_1, then all maximal elementary annihilators of $R_1 \times R_2$ have the same hight if and only if all maximal elementary annihilators of R_1 have the same height and also for the maximal elementary annihilators of R_2. □

We get the following result, inductively.

Corollary 24. *Let $R_1, \ldots, R_n$ be some rings. We have $R_1 \times R_2 \times \cdots \times R_n$ is an IMEA-ring if and only if each R_i is an IMEA-ring.*

Let R be a domain, we say that R is atomic if each nonzero nonunit element of R decomposes into a finite product of irreducibles, according to [6]. An atomic domain is called a *half factorial domain*, in short a *HFD* if $x_1 \cdots x_n = y_1 \cdots y_m$ are two decompositions into irreducibles then $n = m$. This concept was introduced by Zaks in [7]. A *HFD* is called a unique factorization domain, in short a *UFD* if $x_1 \cdots x_n = y_1 \cdots y_n$ are two decompositions into irreducibles then the x_i's and the y_i's are associates after reordering them. It is well known that a *UFD* is an atomic domain in which each irreducible is primed by [8, Theorem 1].

Proposition 25. *If R is a UFD, then for all nonzero nonunit a of R we have $R/(a)$ is an IMEA-ring. Moreover if $a = p_1^{m_1} \cdots p_r^{m_r}$ is the decomposition of a into prime elements then* $\mathrm{EAdim}(R/(a)) = m_1 + \cdots + m_r - 1$.

Proof. Let $a = p_1^{m_1} \cdots p_r^{m_r} \in R$. Suppose that $x \in R$; then $\overline{x} \in Z(R/(a)) \setminus \{\overline{0}\}$

$$\Longleftrightarrow \begin{cases} \overline{x} \neq \overline{0} \\ \exists \overline{y} \neq \overline{0} \\ \overline{xy} = \overline{0} \end{cases}$$

$$\Longleftrightarrow \begin{cases} x = p_1^{\alpha_1} \cdots p_r^{\alpha_r} x_1; \alpha_i \in \mathbb{N},\ x_1 \notin (p_i), \\ \qquad \forall i, \quad \dfrac{\exists i}{\alpha_i} < m_i, \\ \exists y = p_1^{\beta_1} \cdots p_r^{\beta_r} y_1; \beta_i \in \mathbb{N}, y_1 \notin (p_i), \\ \qquad \forall i, \quad \dfrac{\exists i}{\beta_i} < m_i, \\ \alpha_i + \beta_i \geq m_i,\ \forall i. \end{cases} \tag{1}$$

Then

$$Z\left(\frac{R}{(a)}\right) = \left\{\overline{p_1^{\alpha_1} \cdots p_r^{\alpha_r} x_1}; (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r \setminus \{(0, \ldots, 0)\},\right. \\ \left. x_1 \in R \setminus \bigcup_i (p_i) \right\}. \tag{2}$$

And for $\overline{x} = \overline{p_1^{\alpha_1} \cdots p_r^{\alpha_r} x_1} \neq \overline{0}$ (one among the α_i's is $< m_i$ and $x_1 \in R \setminus \cup_i (p_i)$), we have the following:

$$\left(\overline{0} : \overline{x}\right) = \left\{\overline{p_1^{\beta_1} \cdots p_r^{\beta_r} y_1}; (\beta_1, \ldots \beta_r) \in \mathbb{N}^r \setminus \{(0, \ldots, 0)\} / \beta_i \right. \\ \left. \geq m_i - \alpha_i, \forall i,\ y_1 \in R \setminus \bigcup_i (p_i) \right\}. \tag{3}$$

It is easy to check that the set of all elementary annihilators of $R/(a)$ is

$$\left\{\left(\overline{0} : \overline{p_1^{\alpha_1} \cdots p_r^{\alpha_r}}\right); \right. \\ \left. (\alpha_1, \ldots, \alpha_r) \in [[0, m_1]] \times \cdots \times [[0, m_r]] \setminus (m_1, \ldots, m_r) \right\}. \tag{4}$$

The maximal ones among them are

$$\left(\overline{0} : \overline{p_1^{m_1 - 1} p_2^{m_2} \cdots p_r^{m_r}}\right), \left(\overline{0} : \overline{p_1^{m_1} p_2^{m_2 - 1} p_3^{m_3} \cdots p_r^{m_r}}\right), \\ \ldots, \left(\overline{0} : \overline{p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r - 1}}\right). \tag{5}$$

A longest chain ending in one of them, for example, $(\overline{0} : \overline{1}) \subset (\overline{0} : \overline{p_1}) \subset \cdots \subset (\overline{0} : \overline{p_1^{m_1 - 1}}) \subset (\overline{0} : \overline{p_1^{m_1 - 1} p_2}) \subset \cdots \subset (\overline{0} : \overline{p_1^{m_1 - 1} p_2^{m_2}}) \subset \cdots \subset (\overline{0} : \overline{p_1^{m_1} p_2^{m_2} \cdots p_{r-1}^{m_r - 1} p_r}) \subset \cdots \subset (\overline{0} : \overline{p_1^{m_1 - 1} p_2^{m_2} \cdots p_r^{m_r}})$, has the length $m_1 + \cdots + m_r - 1$. Thus all maximal elementary annihilators have the same height $m_1 + \cdots + m_r - 1$ then $R/(a)$ is an IMEA-ring and $\mathrm{EAdim}(R/(a)) = m_1 + \cdots + m_r - 1$. □

Proposition 26. *Let R be a HFD. R is a UFD if and only if* $\mathrm{EAdim}\,(R/(a)) = \lambda(a) - 1$, *for all nonzero nonunit a. Where $\lambda(a)$ is the number of factors in a decomposition of a into irreducibles (counted with multiplicities).*

Proof. "⇒" is due to the previous proposition.

"⇐" Let a be an irreducible of R then $\lambda(a) = 1$; then $\mathrm{EAdim}(R/(a)) = \lambda(a) - 1 = 0$. According to Remark 3, $R/(a)$ is a domain; then (a) is prime. According to [8, Theorem 1], R is a *UFD*. □

Question 1. Are all finite EAdimensional rings IMEA-rings?

Question 2. Are all finite rings IMEA-rings?

References

[1] D. F. Anderson and P. S. Livingston, "The zero-divisor graph of a commutative ring," *Journal of Algebra*, vol. 217, no. 2, pp. 434–447, 1999.

[2] I. Kaplansky, *Commutative Rings*, The University of Chicago Press, Chicago, Ill, USA, Revised edition, 1974.

[3] T. G. Lucas, "The diameter of a zero divisor graph," *Journal of Algebra*, vol. 301, no. 1, pp. 174–193, 2006.

[4] A. Benhissi, *Les Anneaux de Séries Formelles*, vol. 124 of *Queen's Papers in Pure and Applied Mathematics*, Kingston, Ontario, Canada, 2003.

[5] I. Beck, "Coloring of commutative rings," *Journal of Algebra*, vol. 116, no. 1, pp. 208–226, 1988.

[6] P. M. Cohn, "Bezout rings and their subrings," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 64, pp. 251–264, 1968.

[7] A. Zaks, "Half factorial domains," *Bulletin of the American Mathematical Society*, vol. 82, no. 5, pp. 721–724, 1976.

[8] P. M. Cohn, "Unique factorization domains," *The American Mathematical Monthly*, vol. 80, no. 1, pp. 1–18, 1973.