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Research Article

Quadruple Fixed Point Theorems in Partially Ordered Metric Spaces Depending on Another Function

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We prove quadruple fixed point theorems in partially ordered metric spaces depending on another function. Also, we state some examples showing that our results are real generalization of known ones in quadruple fixed point theory.

1. Introduction and Preliminaries

Basic topological properties of an ordered set like convergence were introduced by Wolk [1]. In 1981, Monjardet [2] considered metrics on partially ordered sets. Ran and Reurings [3] proved an analog of Banach contraction mapping principle in partially ordered metric spaces. In their pioneering work, they also provide applications to matrix equations. As an extension, Nieto and Rodríguez-López [4] discovered further fixed point theorems in partially ordered metric spaces. For some other related results in ordered metric spaces, see, for example, [5–7].

Bhaskar and Lakshmikantham in [8] introduced the concept of coupled fixed point of a mapping $F : X \times X \to X$ and investigated the existence and uniqueness of a coupled fixed point theorem in partially ordered complete metric spaces. Lakshmikantham and Ćirić in [9] defined mixed g-monotone property and coupled coincidence point in partially ordered metric spaces. They also proved related fixed point theorems. Later, various results on coupled fixed point have been obtained, see, for example, [9–20].

Following this trend, Berinde and Borcut [21] introduced the concept of tripled fixed point in ordered sets. The following two definitions are from [21].

Definition 1.1. Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$. The mapping F is said to have the mixed monotone property if, for any $x, y, z \in X$,

$$\begin{gathered} x_1, x_2 \in X, \quad x_1 \leq x_2 \Longrightarrow F(x_1, y, z) \leq F(x_2, y, z), \\ y_1, y_2 \in X, \quad y_1 \leq y_2 \Longrightarrow F(x, y_1, z) \geq F(x, y_2, z), \\ z_1, z_2 \in X, \quad z_1 \leq z_2 \Longrightarrow F(x, y, z_1) \leq F(x, y, z_2). \end{gathered} \tag{1.1}$$

Definition 1.2. Let $F : X \times X \times X \to X$. An element (x, y, z) is called a tripled fixed point of F if

$$F(x, y, z) = x, \qquad F(y, x, y) = y, \qquad F(z, y, x) = z. \tag{1.2}$$

Also, Berinde and Borcut [21] proved the following theorem.

Theorem 1.3. *Let $(X, \leq, d)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Suppose also that $F : X \times X \times X \to X$ be a mapping such that it has the mixed monotone property and there exist $j, r, l \geq 0$ with $j + r + l < 1$ such that*

$$d(F(x, y, z), F(u, v, w)) \leq j d(x, u) + r d(y, v) + l d(z, w), \tag{1.3}$$

for any $x, y, z \in X$ for which $x \leq u$, $v \leq y$, and $z \leq w$. Additionally suppose that either F is continuous or X has the following properties:

(1) *if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,*

(2) *if a nonincreasing sequence $y_n \to y$, then $y \leq y_n$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, z_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \qquad F(y, x, y) = y, \qquad F(z, y, x) = z, \tag{1.4}$$

that is, F has a tripled fixed point.

The notion of fixed point of order $N \geq 3$ was first introduced by Samet and Vetro [22]. Very recently, Karapınar used the notion of quadruple fixed point and obtained some quadruple fixed point theorems [23] in partially ordered metric spaces. This work motivated the following studies [24–27] which provide further fixed point theorems on quadruple fixed points.

From now on, we denote $X^4 = X \times X \times X \times X$.

Definition 1.4 (see [24]). Let X be a nonempty set and let $F : X^4 \to X$ be a given mapping. An element $(x, y, z, w) \in X \times X^3$ is called a quadruple fixed point of F if

$$F(x, y, z, w) = x, \qquad F(y, z, w, x) = y, \qquad F(z, w, x, y) = z, \qquad F(w, x, y, z) = w. \tag{1.5}$$

Let (X, d) be a metric space. The mapping $\overline{d} : X^4 \to X$, given by

$$\overline{d}((x, y, z, w), (u, v, h, l)) = d(x, y) + d(y, v) + d(z, h) + d(w, l), \tag{1.6}$$

defines a metric on X^4, which will be denoted for convenience by d.

Remark 1.5. In [23, 24, 27], the notion of *quadruple fixed point* is called *quartet fixed point*.

Definition 1.6 (see [24]). Let $(X, \leq)$ be a partially ordered set and $F : X^4 \to X$ be a mapping. We say that F has the mixed monotone property if $F(x, y, z, w)$ is monotone nondecreasing in x and z and is monotone nonincreasing in y and w; that is, for any $x, y, z, w \in X$,

$$\begin{gathered} x_1, x_2 \in X, \quad x_1 \leq x_2 \quad \text{implies } F(x_1, y, z, w) \leq F(x_2, y, z, w), \\ y_1, y_2 \in X, \quad y_1 \leq y_2 \quad \text{implies } F(x, y_2, z, w) \leq F(x, y_1, z, w), \\ z_1, z_2 \in X, \quad z_1 \leq z_2 \quad \text{implies } F(x, y, z_1, w) \leq F(x, y, z_2, w), \\ w_1, w_2 \in X, \quad w_1 \leq w_2 \quad \text{implies } F(x, y, z, w_2) \leq F(x, y, z, w_1). \end{gathered} \tag{1.7}$$

In this paper, we prove some quadruple fixed point theorems in partially ordered metric spaces depended on another function $T : X \to X$.

2. Main Results

We start with the following definition (see, e.g., [28–31]).

Definition 2.1. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be ICS if T is injective, continuous, and it has the property: for every sequence $\{x_n\}$ in X, if $\{Tx_n\}$ is convergent then, $\{x_n\}$ is also convergent.

Let Φ be the set of all functions $\phi : [0, \infty) \to [0, \infty)$ such that

(1) $\phi(t) < t$ for all $t \in (0, +\infty)$,

(2) $\lim_{r \to t^+} \phi(r) < t$ for all $t \in (0, +\infty)$.

Our first result is given by the following theorem.

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Suppose also that $T : X \to X$ is an ICS mapping and $F : X^4 \to X$ is such that F has the mixed monotone property. Assume that there exists $\phi \in \Phi$ such that*

$$d(TF(x, y, z, w), TF(u, v, r, s)) \leq \phi(\max\{d(Tx, Tu), d(Ty, Tv), d(Tz, Tr), d(Tw, Ts)\}) \tag{2.1}$$

for any $x, y, z, w, u, v, r, s \in X$ for which $x \leq u$, $v \leq y$, $z \leq r$, and $s \leq w$. Additionally assume that either

(a) *F is continuous, or*

(b) *X has the following properties:*

  (i) *if nondecreasing sequence $x_n \to x$ (respectively, $z_n \to z$), then $x_n \leq x$ (respectively, $z_n \leq z$) for all n,*

  (ii) *if nonincreasing sequence $y_n \to y$ (respectively, $w_n \to w$), then $y_n \geq y$ (respectively, $w_n \geq w$) for all n.*

If there exist $x_0, y_0, z_0, w_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0, w_0)$, $y_0 \geq F(y_0, z_0, w_0, x_0)$, $z_0 \leq F(z_0, w_0, x_0, y_0)$, and $w_0 \geq F(w_0, x_0, y_0, z_0)$, then there exist $x, y, z, w \in X$ such that

$$F(x, y, z, w) = x, \qquad F(y, z, w, x) = y, \qquad F(z, w, x, y) = z, \qquad F(w, x, y, z) = w. \tag{2.2}$$

that is, F has a quadruple fixed point.

Proof. Let $x_0, y_0, z_0, w_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0, w_0), \qquad y_0 \geq F(y_0, z_0, w_0, x_0), \tag{2.3}$$

$$z_0 \leq F(z_0, w_0, x_0, y_0), \qquad w_0 \geq F(w_0, x_0, y_0, z_0). \tag{2.4}$$

Set

$$\begin{gathered} x_1 = F(x_0, y_0, z_0, w_0), \qquad y_1 = F(y_0, z_0, w_0, x_0), \\ z_1 = F(z_0, w_0, x_0, y_0), \qquad w_1 = F(w_0, x_0, y_0, z_0). \end{gathered} \tag{2.5}$$

Then, $x_0 \leq x_1$, $y_0 \geq y_1$, $z_0 \leq z_1$, and $w_0 \geq w_1$. Again, define $x_2 = F(x_1, y_1, z_1, w_1)$, $y_2 = F(y_1, z_1, w_1, x_1)$, $z_2 = F(z_1, w_1, x_1, y_1)$, and $w_2 = F(w_1, x_1, y_1, z_1)$. Since F has the mixed monotone property, we have $x_0 \leq x_1 \leq x_2$, $y_2 \leq y_1 \leq y_0$, $z_0 \leq z_1 \leq z_2$, and $w_2 \leq w_1 \leq w_0$. By continuing this process, we can construct four sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ in X such that

$$\begin{gathered} x_{n+1} = F(x_n, y_n, z_n, w_n), \qquad y_{n+1} = F(y_n, z_n, w_n, x_n), \\ z_{n+1} = F(z_1, w_1, x_1, y_1), \qquad w_{n+1} = F(w_n, x_n, y_n, z_n). \end{gathered} \tag{2.6}$$

Since F has the mixed monotone property, by using a mathematical induction it is easy to see that

$$x_n \leq x_{n+1}, \quad y_{n+1} \leq y_n, \quad z_n \leq z_{n+1}, \quad w_{n+1} \leq w_n, \quad \text{for } n = 0, 1, 2, \ldots, \tag{2.7}$$

Assume that, for some $n \in \mathbb{N}$,

$$x_n = x_{n+1}, \qquad y_n = y_{n+1}, \qquad z_n = z_{n+1}, \qquad w_n = w_{n+1}. \tag{2.8}$$

Then, by (2.6), (x_n, y_n, z_n, w_n) is a quadruple fixed point of F. Therefore, in the rest of the proof, for any $n \in \mathbb{N}$ we will assume that

$$x_n \neq x_{n+1} \quad \text{or} \quad y_n \neq y_{n+1} \quad \text{or} \quad z_n \neq z_{n+1} \quad \text{or} \quad w_n \neq w_{n+1}. \tag{2.9}$$

Since T is injective, for any $n \in \mathbb{N}$,

$$0 < \max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}), d(Tz_n, Tz_{n+1}), d(Tw_n, Tw_{n+1})\}. \tag{2.10}$$

Due to (2.1), (2.6), and (2.7), we have

$$\begin{aligned}
d(Tx_n, Tx_{n+1}) &= d(TF(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}), TF(x_n, y_n, z_n, w_n)) \\
&\leq \phi(\max\{d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n), d(Tz_{n-1}, Tz_n), d(Tw_{n-1}, Tw_n)\}), \\
d(Ty_{n+1}, Ty_n) &= d(TF(y_n, z_n, w_n, x_n), TF(y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1})) \\
&\leq \phi(\max\{d(Ty_{n-1}, Ty_n), d(Tz_{n-1}, Tz_n), d(Tw_{n-1}, Tw_n), d(Tx_{n-1}, Tx_n)\}), \\
d(Tz_n, Tz_{n+1}) &= d(TF(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}), TF(z_n, w_n, x_n, y_n)) \\
&\leq \phi(\max\{d(Tz_{n-1}, Tz_n), d(Tw_{n-1}, Tw_n), d(Tx_{n-1}, Tz_n), d(Ty_{n-1}, Ty_n)\}), \\
d(Tw_{n+1}, Tw_n) &= d(TF(w_n, x_n, y_n, z_n), TF(w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1})) \\
&\leq \phi(\max\{d(Tw_{n-1}, Tw_n), d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n), d(Tz_{n-1}, Tz_n)\}).
\end{aligned} \tag{2.11}$$

Using the fact that $\phi(t) < t$ for all $t > 0$ together with (2.11), we obtain that

$$\begin{aligned}
0 &< \max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}), d(Tz_n, Tz_{n+1}), d(Tw_n, Tw_{n+1})\} \\
&\leq \phi(\max\{d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n), d(Tz_{n-1}, Tz_n), d(Tw_{n-1}, Tw_n)\}) \\
&< \max\{d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n), d(Tz_{n-1}, Tz_n), d(Tw_{n-1}, Tw_n)\}.
\end{aligned} \tag{2.12}$$

It follows that

$$\begin{aligned}
&\max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}), d(Tz_n, Tz_{n+1}), d(Tw_n, Tw_{n+1})\} \\
&\quad < \max\{d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n), d(Tz_{n-1}, Tz_n), d(Tw_{n-1}, Tw_n)\}.
\end{aligned} \tag{2.13}$$

Thus, $\max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}), d(Tz_n, Tz_{n+1}), d(Tw_n, Tw_{n+1})\}$ is a positive decreasing sequence. Hence, there exists $r \geq 0$ such that

$$\lim_{n \to +\infty} \max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}), d(Tz_n, Tz_{n+1}), d(Tw_n, Tw_{n+1})\} = r. \tag{2.14}$$

Suppose that $r > 0$. Letting $n \to +\infty$ in (2.12), we obtain that

$$\begin{aligned} 0 < r &\leq \lim_{n\to+\infty} \phi\left(\max\left\{d(Tx_n, Tx_{n+1}), d\left(Ty_n, Ty_{n+1}\right), d(Tz_n, Tz_{n+1}), d(Tw_n, Tw_{n+1})\right\}\right) \\ &= \lim_{t\to r^+} \phi(t) < r, \end{aligned} \tag{2.15}$$

which is a contradiction. Therefore, we deduce that

$$\lim_{n\to+\infty} \max\left\{d(Tx_n, Tx_{n+1}), d\left(Ty_n, Ty_{n+1}\right), d(Tz_n, Tz_{n+1}), d(Tw_n, Tw_{n+1})\right\} = 0. \tag{2.16}$$

We will show that $\{Tx_n\}$, $\{Ty_n\}$, $\{Tz_n\}$, and $\{Tw_n\}$ are Cauchy sequences. Assume the contrary, that is, either $\{Tx_n\}$ or $\{Ty_n\}$ or $\{Tz_n\}$ or $\{Tw_n\}$ is not a Cauchy sequence, consequently,

$$\lim_{n,m\to+\infty} d(Tx_m, Tx_n) \neq 0 \quad \text{or} \quad \lim_{n,m\to+\infty} d\left(Ty_m, Ty_n\right) \neq 0, \tag{2.17}$$

or $\lim_{n,m\to+\infty} d(Tz_m, Tz_n) \neq 0$ or $\lim_{n,m\to+\infty} d(Tw_m, Tw_n) \neq 0$. This means that there exists $\varepsilon > 0$ for which we can find subsequences of integers (m_k) and (n_k) with $n_k > m_k > k$ such that

$$\max\left\{d(Tx_{m_k}, Tx_{n_k}), d\left(Ty_{m_k}, Ty_{n_k}\right), d(Tz_{m_k}, Tz_{n_k}), d(Tw_{m_k}, Tw_{n_k})\right\} \geq \varepsilon. \tag{2.18}$$

Furthermore, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (2.18). Then,

$$\max\left\{d(Tx_{m_k}, Tx_{n_k-1}), d\left(Ty_{m_k}, Ty_{n_k-1}\right), d(Tz_{m_k}, Tz_{n_k-1}), d(Tw_{m_k}, Tw_{n_k-1})\right\} < \varepsilon. \tag{2.19}$$

By the triangle inequality and (2.19), we have

$$\begin{aligned} d(Tx_{m_k}, Tx_{n_k}) &\leq d(Tx_{m_k}, Tx_{n_k-1}) + d(Tx_{n_k-1}, Tx_{n_k}) \\ &< \epsilon + d(Tx_{n_k-1}, Tx_{n_k}). \end{aligned} \tag{2.20}$$

Thus, by (2.16), we obtain

$$\lim_{k\to+\infty} d(Tx_{m_k}, Tx_{n_k}) \leq \lim_{k\to+\infty} d(Tx_{m_k}, Tx_{n_k-1}) \leq \varepsilon. \tag{2.21}$$

Similarly, we have

$$\begin{aligned} \lim_{k\to+\infty} d\left(Ty_{m_k}, Ty_{n_k}\right) &\leq \lim_{k\to+\infty} d\left(Ty_{m_k}, Ty_{n_k-1}\right) \leq \varepsilon, \\ \lim_{k\to+\infty} d(Tz_{m_k}, Tz_{n_k}) &\leq \lim_{k\to+\infty} d(Tz_{m_k}, Tz_{n_k-1}) \leq \varepsilon, \\ \lim_{k\to+\infty} d(Tw_{m_k}, Tw_{n_k}) &\leq \lim_{k\to+\infty} d(Tw_{m_k}, Tw_{n_k-1}) \leq \varepsilon. \end{aligned} \tag{2.22}$$

Again, by (2.19), we have

$$\begin{aligned} d(Tx_{m_k}, Tx_{n_k}) &\le d(Tx_{m_k}, Tx_{m_k-1}) + d(Tx_{m_k-1}, Tx_{n_k-1}) + d(Tx_{n_k-1}, Tx_{n_k}) \\ &\le d(Tx_{m_k}, Tx_{m_k-1}) + d(Tx_{m_k-1}, Tx_{m_k}) \\ &\quad + d(Tx_{m_k}, Tx_{n_k-1}) + d(Tx_{n_k-1}, Tx_{n_k}) \\ &< d(Tx_{m_k}, Tx_{m_k-1}) + d(Tx_{m_k-1}, Tx_{m_k}) + \varepsilon + d(Tx_{n_k-1}, Tx_{n_k}). \end{aligned} \tag{2.23}$$

Letting $k \to +\infty$ and using (2.16), we get

$$\begin{aligned} \lim_{k\to+\infty} d(Tx_{m_k}, Tx_{n_k}) &\le \lim_{k\to+\infty} d(Tx_{m_k-1}, Tx_{n_k-1}) \le \varepsilon, \\ \lim_{k\to+\infty} d(Ty_{m_k}, Ty_{n_k}) &\le \lim_{k\to+\infty} d(Ty_{m_k-1}, Ty_{n_k-1}) \le \varepsilon, \\ \lim_{k\to+\infty} d(Tz_{m_k}, Tz_{n_k}) &\le \lim_{k\to+\infty} d(Tz_{m_k-1}, Tz_{n_k-1}) \le \varepsilon, \\ \lim_{k\to+\infty} d(Tw_{m_k}, Tw_{n_k}) &\le \lim_{k\to+\infty} d(Tw_{m_k-1}, Tw_{n_k-1}) \le \varepsilon. \end{aligned} \tag{2.24}$$

Using (2.18) and (2.24), we have

$$\begin{aligned} &\lim_{k\to+\infty} \max\{d(Tx_{m_k}, Tx_{n_k}), d(Ty_{m_k}, Ty_{n_k}), d(Tz_{m_k}, Tz_{n_k}), d(Tw_{m_k}, Tw_{n_k})\} \\ &\quad = \lim_{k\to+\infty} \max\{d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1}), d(Tw_{m_k-1}, Tw_{n_k-1})\} \\ &\quad = \varepsilon. \end{aligned} \tag{2.25}$$

Now, using inequality (2.1), we obtain

$$\begin{aligned} &d(Tx_{m_k}, Tx_{n_k}) \\ &\quad = d(TF(x_{m_k-1}, y_{m_k-1}, z_{m_k-1}, w_{m_k-1}), TF(x_{n_k-1}, y_{n_k-1}, z_{n_k-1}, w_{n_k-1})) \\ &\quad \le \phi(\max\{d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1}), d(Tw_{m_k-1}, Tw_{n_k-1})\}) \\ &d(Ty_{m_k}, Ty_{n_k}) \\ &\quad = d(TF(y_{m_k-1}, z_{m_k-1}, w_{m_k-1}, x_{m_k-1}), TF(y_{n_k-1}, z_{n_k-1}, w_{n_k-1}, x_{n_k-1})) \\ &\quad \le \phi(\max\{d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1}), d(Tw_{m_k-1}, Tw_{n_k-1}), d(Tx_{m_k-1}, Tx_{n_k-1})\}), \\ &d(Ty_{m_k}, Ty_{n_k}) \\ &\quad = d(TF(z_{m_k-1}, w_{m_k-1}, x_{m_k-1}, y_{m_k-1}), TF(z_{n_k-1}, w_{n_k-1}, x_{n_k-1}, y_{n_k-1})) \\ &\quad \le \phi(\max\{d(Tz_{m_k-1}, Tz_{n_k-1}), d(Tw_{m_k-1}, Tw_{n_k-1}), d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1})\}), \\ &d(Ty_{m_k}, Ty_{n_k}) \\ &\quad = d(TF(w_{m_k-1}, x_{m_k-1}, y_{m_k-1}, z_{m_k-1}), TF(w_{n_k-1}, x_{n_k-1}, y_{n_k-1}, z_{n_k-1})) \\ &\quad \le \phi(\max\{d(Tw_{m_k-1}, Tw_{n_k-1}), d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1})\}) \end{aligned} \tag{2.26}$$

From (2.26), we deduce that

$$\max\{d(Tx_{m_k}, Tx_{n_k}), d(Ty_{m_k}, Ty_{n_k}), d(Tz_{m_k}, Tz_{n_k}), d(Tw_{m_k}, Tw_{n_k})\}$$
$$\leq \phi(\max\{d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1}), d(Tw_{m_k-1}, Tw_{n_k-1})\}). \tag{2.27}$$

Letting $k \to +\infty$ in (2.27) and by using (2.25), we get that

$$0 < \varepsilon \leq \lim_{t \to \varepsilon^+} \phi(t) < \varepsilon, \tag{2.28}$$

which is a contradiction. Thus, $\{Tx_n\}$, $\{Ty_n\}$, $\{Tz_n\}$, and $\{Tw_n\}$ are Cauchy sequences in (X, d). Since X is a complete metric space, $\{Tx_n\}$, $\{Ty_n\}$, $\{Tz_n\}$, and $\{Tw_n\}$ are convergent sequences.

Since T is an ICS mapping, there exist $x, y, z, w \in X$ such that

$$\lim_{n \to +\infty} x_n = x, \qquad \lim_{n \to +\infty} y_n = y, \qquad \lim_{n \to +\infty} z_n = z, \qquad \lim_{n \to +\infty} w_n = w. \tag{2.29}$$

Since T is continuous, we have

$$\lim_{n \to +\infty} Tx_n = Tx, \qquad \lim_{n \to +\infty} Ty_n = Ty, \qquad \lim_{n \to +\infty} Tz_n = Tz, \qquad \lim_{n \to +\infty} Tw_n = Tw. \tag{2.30}$$

Suppose now the assumption (a) holds, that is, F is continuous. By (2.6), (2.29), and (2.30) we obtain

$$\begin{aligned}
x &= \lim_{n \to +\infty} x_{n+1} = \lim_{n \to +\infty} F(x_n, y_n, z_n, w_n) \\
&= F\left(\lim_{n \to +\infty} x_n, \lim_{n \to +\infty} y_n, \lim_{n \to +\infty} z_n, \lim_{n \to +\infty} w_n\right) = F(x, y, z, w), \\
y &= \lim_{n \to +\infty} y_{n+1} = \lim_{n \to +\infty} F(y_n, z_n, w_n, x_n) \\
&= F\left(\lim_{n \to +\infty} y_n, \lim_{n \to +\infty} z_n, \lim_{n \to +\infty} w_n, \lim_{n \to +\infty} x_n\right) = F(y, z, w, x), \\
z &= \lim_{n \to +\infty} z_{n+1} = \lim_{n \to +\infty} F(z_n, w_n, x_n, y_n) \\
&= F\left(\lim_{n \to +\infty} z_n, \lim_{n \to +\infty} w_n, \lim_{n \to +\infty} x_n, \lim_{n \to +\infty} y_n\right) = F(z, w, x, y), \\
w &= \lim_{n \to +\infty} w_{n+1} = \lim_{n \to +\infty} F(w_n, x_n, y_n, z_n) \\
&= F\left(\lim_{n \to +\infty} w_n, \lim_{n \to +\infty} x_n, \lim_{n \to +\infty} y_n, \lim_{n \to +\infty} z_n\right) = F(w, x, y, z).
\end{aligned} \tag{2.31}$$

We have proved that F has a quadruple fixed point.

Suppose now the assumption (b) holds. Since $\{x_n\}$ and $\{z_n\}$ are nondecreasing with $x_n \to x$ and $z_n \to z$ and also $\{y_n\}$ and $\{w_n\}$ are nonincreasing, with $y_n \to y$ and $w_n \to w$, we have

$$x_n \leq x, \qquad y_n \geq y, \qquad z_n \leq z, \qquad w_n \geq w \tag{2.32}$$

for all n. Consider now

$$\begin{aligned} d(Tx, TF(x,y,z,w)) &\leq d(Tx, Tx_{n+1}) + d(Tx_{n+1}, TF(x,y,z,w)) \\ &= d(Tx, Tx_{n+1}) + d(TF(x_n,y_n,z_n,w_n), TF(x,y,z,w)) \\ &\leq d(Tx, Tx_{n+1}) + \phi(\max\{d(Tx_n,Tx), d(Ty_n,Ty), d(Tz_n,Tz), \\ &\qquad d(Tw_n,Tw)\}). \end{aligned} \tag{2.33}$$

Taking $n \to \infty$ and using (2.30), the right-hand side of (2.33) tends to 0, so we get that $d(Tx, TF(x,y,z,w)) = 0$. Thus, $Tx = TF(x,y,z,w)$, and since T is injective, we get that $x = F(x,y,z,w)$. Analogously, one finds that

$$F(y,z,w,x) = y, \qquad F(z,w,x,y) = z, \qquad F(w,x,y,z) = w. \tag{2.34}$$

Thus, we proved that F has a quadrupled fixed point. This completes the proof of Theorem 2.2. □

Repeating the same proof of Theorem 2.2, we may state the following corollary.

Corollary 2.3. *Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X,d) is a complete metric space. Suppose also $T : X \to X$ is an ICS mapping and $F : X^4 \to X$ is such that F has the mixed monotone property. Assume that there exists $\phi \in \Phi$ such that*

$$d(TF(x,y,z,w), TF(u,v,t,s)) \leq \phi\left(\frac{d(Tx,Tu) + d(Ty,Tv) + d(Tz,Tt) + d(Tw,Ts)}{4}\right) \tag{2.35}$$

for any $x,y,z,w,u,v,t,s \in X$ for which $x \leq u$, $v \leq y$, $z \leq t$, and $s \leq w$. Additionally suppose that either

(a) *F is continuous, or*

(b) *X has the following property:*

 (i) *if non-decreasing sequence $x_n \to x$ (resp., $z_n \to z$), then $x_n \leq x$ (resp., $z_n \leq z$) for all n,*

 (ii) *if non-increasing sequence $y_n \to y$ (resp., $w_n \to w$), then $y_n \geq y$ (resp., $w_n \geq w$) for all n.*

If there exist $x_0, y_0, z_0, w_0 \in X$ *such that* $x_0 \le F(x_0, y_0, z_0, w_0)$, $y_0 \ge F(y_0, z_0, w_0, x_0)$, $z_0 \le F(z_0, w_0, x_0, y_0)$ *and* $w_0 \ge F(w_0, w_0, y_0, z_0)$*, then there exist* $x, y, z, w \in X$ *such that*

$$F(x,y,z,w) = x, \qquad F(y,z,w,x) = y, \qquad F(z,w,x,y) = z\ , \qquad F(w,x,y,z) = w, \tag{2.36}$$

that is, F has a quadruple fixed point.

Corollary 2.4. *Let* $(X, \le)$ *be a partially ordered set and suppose that there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Suppose also that* $T : X \to X$ *is an ICS mapping and* $F : X^4 \to X$ *is such that* F *has the mixed monotone property. Assume that there exists* $k \in [0, 1)$ *such that*

$$d(TF(x,y,z,w), TF(u,v,t,s)) \le k \max\{d(Tx,Tu), d(Ty,Tv), d(Tz,Tt), d(Tw,Ts)\} \tag{2.37}$$

for any $x, y, z, w, u, v, t, s \in X$ *for which* $x \le u$, $v \le y$, $z \le t$, *and* $s \le w$. *Suppose that either*

(a) *F is continuous, or*

(b) *X has the following property:*

(i) *if nondecreasing sequence* $x_n \to x$ *(resp.,* $z_n \to z$*), then* $x_n \le x$ *(resp.,* $z_n \le z$*) for all* n,

(ii) *if nonincreasing sequence* $y_n \to y$ *(resp.,* $w_n \to w$*), then* $y_n \ge y$ *(resp.,* $w_n \ge w$*) for all* n.

If there exist $x_0, y_0, z_0, w_0 \in X$ *such that* $x_0 \le F(x_0, y_0, w_0, z_0)$, $y_0 \ge F(y_0, z_0, w_0, x_0)$, $z_0 \le F(z_0, w_0, x_0, y_0)$, *and* $w_0 \ge F(w_0, w_0, y_0, z_0)$ *then, there exist* $x, y, z, w \in X$ *such that*

$$F(x,y,z,w) = x, \qquad F(y,z,w,x) = y, \qquad F(z,w,x,y) = z, \qquad F(w,x,y,z) = w, \tag{2.38}$$

that is, F has a quadruple fixed point.

Proof. It suffices to remark that $\phi(t) = kt$ in Theorem 2.2. □

Corollary 2.5. *Let* $(X, \le)$ *be a partially ordered set and suppose that there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Suppose also that* $T : X \to X$ *is an ICS mapping and* $F : X^4 \to X$ *is such that* F *has the mixed monotone property. Assume that there exists* $k \in [0, 1)$ *such that*

$$d(TF(x,y,z,w), TF(u,v,t,s)) \le \frac{k}{4}(d(Tx,Tu), d(Ty,Tv), d(Tz,Tt), d(Tw,Ts)) \tag{2.39}$$

for any $x, y, z, w, u, v, t, s \in X$ *for which* $x \le u$, $v \le y$, $z \le t$ *and* $s \le w$. *Suppose that either*

(a) *F is continuous, or*

(b) *X has the following property:*

(i) *if nondecreasing sequence* $x_n \to x$ *(resp.,* $z_n \to z$*), then* $x_n \le x$ *(resp.,* $z_n \le z$*) for all* n,

(ii) *if nonincreasing sequence* $y_n \to y$ *(resp.,* $w_n \to w$*), then* $y_n \geq y$ *(resp.,* $w_n \geq w$*) for all* n.

If there exist $x_0, y_0, z_0, w_0 \in X$ *such that* $x_0 \leq F(x_0, y_0, w_0, z_0)$, $y_0 \geq F(y_0, z_0, w_0, x_0)$, $z_0 \leq F(z_0, w_0, x_0, y_0)$ *and* $w_0 \geq F(w_0, w_0, y_0, z_0)$ *then there exist* $x, y, z, w \in X$ *such that*

$$F(x,y,z,w) = x, \qquad F(y,z,w,x) = y, \qquad F(z,w,x,y) = z, \qquad F(w,x,y,z) = w \tag{2.40}$$

that is, F has a quadruple fixed point.

Proof. It suffices to take $\phi(t) = kt$ in Corollary 2.3. □

Now, we shall prove the existence and uniqueness of a quadruple fixed point. For a product X^4 of a partial ordered set $(X, \leq)$, we define a partial ordering in the following way: For all $(x, y, z, w), (u, v, t, s) \in X^4$,

$$(x,y,z,w) \leq (u,v,t,s) \Longleftrightarrow x \leq u, \quad y \geq v, \quad z \leq t, \quad w \geq s. \tag{2.41}$$

We say that (x, y, z, w) and (u, v, t, s) are comparable if

$$(x,y,z,w) \leq (u,v,t,s) \quad \text{or} \quad (u,v,s,t) \leq (x,y,z,w). \tag{2.42}$$

Also, we say that (x, y, z, w) is equal to (u, v, t, s) if and only if $x = u, y = v, z = t,\ w = s$.

Theorem 2.6. *In addition to hypotheses of Theorem 2.2, suppose that that for all* (x, y, z, w), $(u, v, t, s) \in X^4$, *there exists* $(a, b, c, d) \in X^4$ *such that* $(F(a, b, c, d), F(b, c, d, a), F(c, d, a, b), F(d, a, b, c))$ *is comparable to* $(F(x, y, z, w), F(y, z, w, x), F(z, w, x, y), F(w, x, y, z))$ *and* $(F(u, v, t, s), F(v, t, s, u), F(t, s, u, v), F(s, u, v, t))$. *Then, F has a unique quadruple fixed point* (x, y, z, w).

Proof. The set of quadruple fixed points of F is not empty due to Theorem 2.2. Assume, now, (x, y, z, w) and (u, v, t, s) are two quadrupled fixed points of F, that is,

$$\begin{aligned} F(x,y,z,w) &= x, & F(u,v,t,s) &= u, \\ F(y,z,w,x) &= y, & F(v,t,s,u) &= v, \\ F(z,w,x,y) &= z, & F(t,s,u,v) &= t, \\ F(w,x,y,z) &= w, & F(s,u,v,t) &= s. \end{aligned} \tag{2.43}$$

We shall show that (x, y, z, w) and (u, v, t, s) are equal. By assumption, there exists $(a, b, c, d) \in X^4$ such that $(F(a, b, c, d), F(b, c, d, a), F(c, d, a, b), F(d, a, b, c))$ is comparable to $(F(x, y, z, w), F(y, z, w, x), F(z, w, x, y), F(w, x, y, z))$ and $(F(u, v, t, s), F(v, t, s, u), F(t, s, u, v), F(s, u, v, t))$.

Define sequences $\{a_n\}, \{b_n\}, \{c_n\}$, and $\{d_n\}$ such that

$$\begin{gathered}
a_0 = a, \quad b_0 = b, \quad c_0 = c, \quad d_0 = d, \quad \text{for any } n \geq 1, \\
\begin{aligned}
a_n &= F(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}), \\
b_n &= F(b_{n-1}, c_{n-1}, d_{n-1}, a_{n-1}), \\
c_n &= F(c_{n-1}, d_{n-1}, a_{n-1}, b_{n-1}), \\
d_n &= F(d_{n-1}, a_{n-1}, b_{n-1}, c_{n-1}),
\end{aligned}
\end{gathered} \tag{2.44}$$

for all n. Further, set $x_0 = x$, $y_0 = y$, $z_0 = z$, $w_0 = w$ and $u_0 = u$, $v_0 = v$, $t_0 = t$, $s_0 = s$ and on the same way define the sequences $\{x_n\}, \{y_n\}, \{z_n\}$, and $\{w_n\}$ and $\{u_n\}, \{v_n\}, \{t_n\}$, and $\{s_n\}$. Then, it is easy that

$$\begin{aligned}
x_n &= F(x, y, z, w), & u_n &= F(u, v, t, s), \\
y_n &= F(y, z, w, x,), & v_n &= F(v, t, s, u), \\
z_n &= F(z, w, x, y), & t_n &= F(t, s, u, v), \\
w_n &= F(w, x, y, z), & s_n &= F(s, u, v, t),
\end{aligned} \tag{2.45}$$

for all $n \geq 1$. Since $(F(x, y, z, w), F(y, z, w, x), F(z, w, x, y), F(w, x, y, z)) = (x_1, y_1, z_1, w_1) = (x, y, z, w)$ is comparable to $(F(a, b, c, d), F(b, c, d, a), F(c, d, a, b)) = (a_1, b_1, c_1, d_1)$, then it is easy to show $(x, y, z, w) \geq (a_1, b_1, c_1, d_1)$. Recursively, we get that

$$(x, y, z, w) \geq (a_n, b_n, c_n, d_n) \quad \forall n. \tag{2.46}$$

By (2.46) and (2.1), we have

$$\begin{aligned}
d(Tx, Ta_{n+1}) &= d(TF(x, y, z, w), TF(a_n, b_n, c_n, d_n)) \\
&\leq \phi(\max\{d(Tx, Ta_n), d(Ty, Tb_n), d(Tz, Tc_n), d(Tw, Td_n)\}), \\
d(Tb_{n+1}, Ty) &= d(TF(b_n, c_n, d_n, a_n), TF(y, z, w, x)) \\
&\leq \phi(\max\{d(Ty, Tb_n), d(Tz, Tc_n), d(Tw, Td_n), d(Tx, Ta_n)\}), \\
d(Tz, Tc_{n+1}) &= d(TF(z, w, x, y), TF(c_n, d_n, a_n, b_n)) \\
&\leq \phi(\max\{d(Tz, Tc_n), d(Tw, Td_n), d(Tx, Ta_n), d(Ty, Tb_n)\}), \\
d(Tw, Td_{n+1}) &= d(TF(w, x, y, z), TF(d_n, a_n, b_n, c_n)) \\
&\leq \phi(\max\{d(Tw, Td_n), d(Tx, Ta_n), d(Ty, Tb_n), d(Tz, Tc_n)\}).
\end{aligned} \tag{2.47}$$

It follows from (2.47) that

$$\begin{aligned}
&\max\{d(Tz, Tc_{n+1}), d(Tw, Td_{n+1}), d(Tx, Ta_{n+1}), d(Ty, Tb_{n+1})\} \\
&\quad \leq \phi(\max\{d(Tz, Tc_n), d(Tw, Td_n), d(Tx, Ta_n), d(Ty, Tb_n)\}).
\end{aligned} \tag{2.48}$$

Therefore, for each $n \geq 1$,

$$\begin{aligned}&\max\{d(Tz,Tc_n),d(Tw,Td_n),d(Tx,Ta_n),d(Ty,Tb_n)\}\\&\quad\leq \phi^n(\max\{d(Tz,Tc_0),d(Tw,Td_0),d(Tx,Ta_0),d(Ty,Tb_0)\}).\end{aligned} \tag{2.49}$$

It is known that $\phi(t) < t$ and $\lim_{r\to t^+}\phi(r) < t$ imply $\lim_{n\to\infty}\phi^n(t) = 0$ for each $t > 0$. Thus, from (2.49),

$$\lim_{n\to\infty}\max\{d(Tz,Tc_n),d(Tw,Td_n),d(Tx,Ta_n),d(Ty,Tb_n)\} = 0. \tag{2.50}$$

This yields that

$$\begin{aligned}&\lim_{n\to\infty} d(Tx,Ta_n) = 0, &&\lim_{n\to\infty} d(Ty,Tb_n) = 0,\\&\lim_{n\to\infty} d(Tz,Tc_n) = 0, &&\lim_{n\to\infty} d(Tw,Td_n) = 0.\end{aligned} \tag{2.51}$$

Analogously, we show that

$$\begin{aligned}&\lim_{n\to\infty} d(Tu,Ta_n) = 0, &&\lim_{n\to\infty} d(Tv,Tb_n) = 0,\\&\lim_{n\to\infty} d(Tt,Tc_n) = 0, &&\lim_{n\to\infty} d(Ts,Td_n) = 0.\end{aligned} \tag{2.52}$$

Combining (2.51) to (2.52) yields that (Tx,Ty,Tz,Tw) and (Tu,Tv,Tt,Ts) are equal. The fact that T is injective gives us $x = u$, $y = v$, $z = t$, and $w = s$. □

We state some examples showing that our results are effective.

Example 2.7. Let $X = [1,64]$ with the metric $d(x,y) = |x-y|$, for all $x,y \in X$ and the usual ordering. Clearly, (X,d) is a complete metric space.

Let $T : X \to X$ and $F : X^4 \to X$ be defined by

$$Tx = \ln(x) + 1, \qquad F(x,y,z,w) = 8\left(\frac{x}{y}\right)^{1/3}, \quad \forall x,y,z,w \in X. \tag{2.53}$$

It is clear that T is an ICS mapping, F has the mixed monotone property and continuous.

Set $\phi(t) = 2t/3$. Taking $x, y, z, w, u, v, s, t \in X$ for which $x \le u$, $v \le y$, $z \le s$, and $t \le w$, we have

$$\begin{aligned} d(TF(x,y,z,w), TF(u,v,s,t)) &= \frac{1}{3}\Big|(\ln x - \ln y) - (\ln u - \ln v)\Big| \\ &\le \frac{1}{3}|\ln x - \ln u| + \frac{1}{3}\Big|\ln y - \ln v\Big| \\ &\le \frac{1}{3}\max\Big\{|\ln x - \ln u|, \Big|\ln y - \ln v\Big|\Big\} \\ &= \phi(\max\{d(Tx,Tu), d(Ty,Tv), d(Tz,Ts), d(Tw,Tt)\}), \end{aligned} \tag{2.54}$$

which is the contractive condition (2.1). Moreover, taking $x_0 = y_0 = z_0 = 8 = w_0$, we have

$$\begin{aligned} &x_0 \le F(x_0, y_0, z_0, w_0), \qquad y_0 \ge F(y_0, z_0, w_0, x_0), \\ &z_0 \le F(z_0, w_0, x_0, y_0)\ , \qquad w_0 \ge F(w_0, x_0, y_0, z_0). \end{aligned} \tag{2.55}$$

Therefore, all the conditions of Theorem 2.2 hold and $(8, 8, 8, 8)$ is the unique quadruple fixed point of F, since also the hypotheses of Theorem 2.6 hold.

On the other hand, we can not apply Corollary 15 of Karapınar [27] to this example. Indeed, for $x = 1 = y = v$, $u = 2$, $1 \le z = s$ and $1 \le t = w$, we have

$$\begin{aligned} d(F(x,y,z,w), F(u,v,s,t)) &= 8\Big|(2)^{1/3} - 1\Big| > \frac{1}{4} > \frac{k}{4}d(x,u) \\ &= \frac{k}{4}\left[d(x,u) + d(y,v) + d(z,s) + d(w,t)\right], \end{aligned} \tag{2.56}$$

for any $k \in [0, 1)$.

Example 2.8. Let $X = \mathbb{R}$ with $d(x, y) = |x - y|$ and natural ordering. Let $T : X \to X$ and $F : X^4 \to X$ be defined by $Tx = x/12$ and $F(x, y, z, w) = 2/5(x - y + z - w)$. It is clear that T is an ICS mapping and F has the monotone property and continuous. Set $\phi(t) = 2t/3 \in \Phi$. It is clear that all conditions of Theorem 2.2 are satisfied and $(0, 0, 0, 0)$ is the desired quadruple point.

Note that Corollary 15 of Karapınar [27] is not applicable. Indeed, for $x = 0$, $u = 1$ and $y = v = z = s = t = w = 0$, we have

$$d(F(x,y,z,w), F(u,v,s,t)) = \frac{2}{5} > \frac{k}{4}d(x,u) = \frac{k}{4}\left[d(x,u) + d(y,v) + d(z,s) + d(w,t)\right], \tag{2.57}$$

for any $k \in [0, 1)$.

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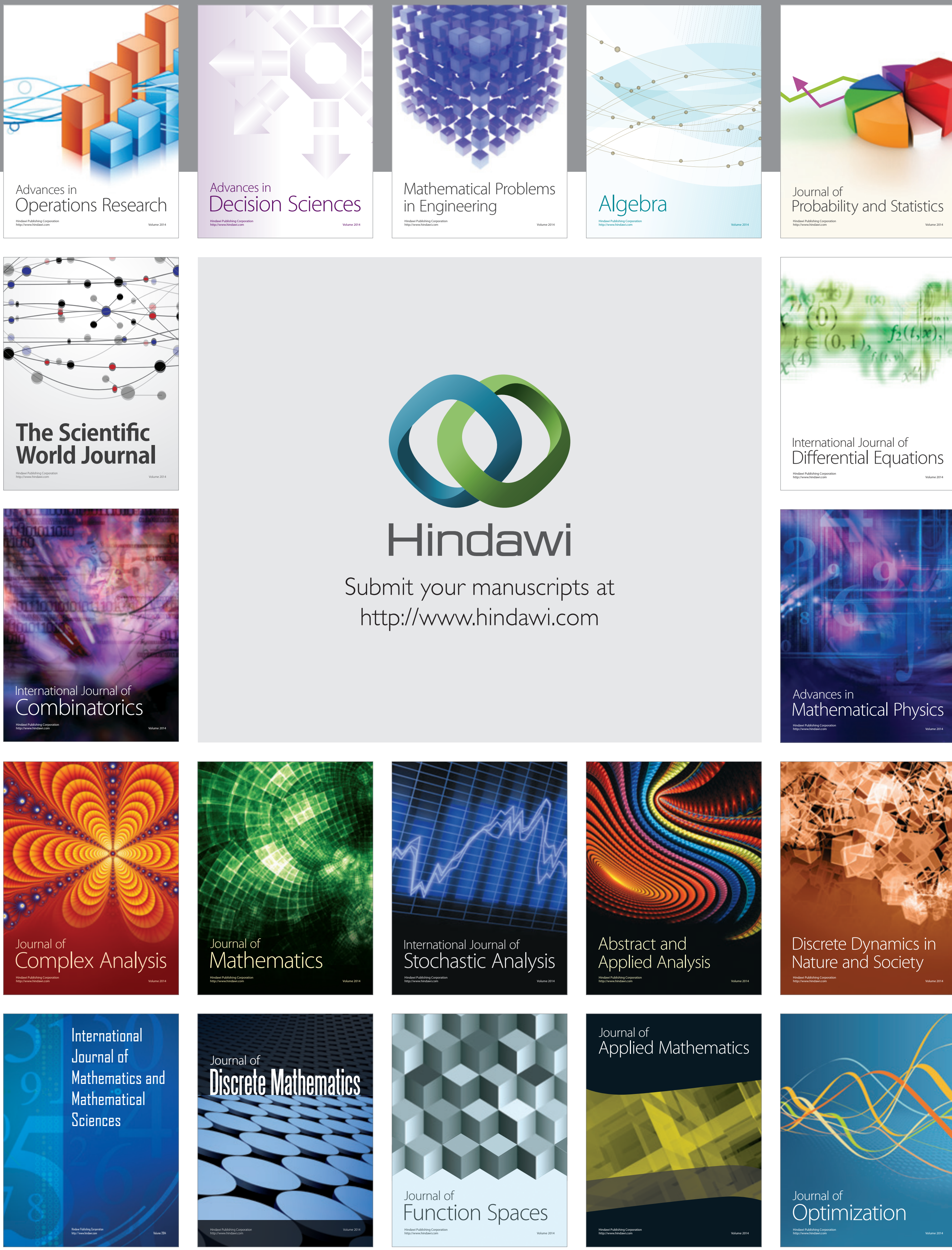

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