International Scholarly Research Network
ISRN Applied Mathematics
Volume 2012, Article ID 727398, 15 pages
doi:10.5402/2012/727398

Research Article

Existence Results for the $p(x)$-Laplacian with Nonlinear Boundary Condition

Zhiqiang Wei and Zigao Chen

Department of Mathematics and Information Science, North China University of Water Resources and Electric Power, Zhengzhou 450000, China

Correspondence should be addressed to Zhiqiang Wei, weizhiqiang@ncwu.edu.cn

Received 28 March 2012; Accepted 8 May 2012

Academic Editors: F. Jauberteau and Y. Tsompanakis

By using the variational method, under appropriate assumptions on the perturbation terms $f(x,u), g(x,u)$ such that the associated functional satisfies the global minimizer condition and the fountain theorem, respectively, the existence and multiple results for the $p(x)$-Laplacian with nonlinear boundary condition in bounded domain Ω were studied. The discussion is based on variable exponent Lebesgue and Sobolev spaces.

1. Introduction

In recent years, increasing attention has been paid to the study of differential and partial differential equations involving variable exponent conditions. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics, or calculus of variations. For more information on modeling physical phenomena by equations involving $p(x)$-growth condition we refer to [1–3]. The appearance of such physical models was facilitated by the development of variable exponent Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where $p(x)$ is a real-valued function. Variable exponent Lebesgue spaces appeared for the first time in the literature as early as 1931 in an article by Orlicz [4]. The spaces $L^{p(x)}$ are special cases of Orlicz spaces L^{φ} originated by Nakano [5] and developed by Musielak and Orlicz [6, 7], where $f \in L^{\varphi}$ if and only if $\int \varphi(x, |f(x)|)dx < \infty$ for a suitable φ. Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. In that context we refer to the studies of Tsenov [8], Sharapudinov [9], and Zhikov [10, 11].

In this paper, we consider the following nonlinear elliptic boundary value problem:

$$\begin{aligned} -\operatorname{div}\left(a(x)|\nabla u|^{p(x)-2}\nabla u\right)+b(x)|u|^{p(x)-2}u&=\lambda f(x,u), \quad x\in\Omega,\\ a(x)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu}&=c(x)|u|^{q(x)-2}u+\mu g(x,u), \quad x\in\partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $\partial/\partial\nu$ is outer unit normal derivative, $p(x)\in C(\overline{\Omega})$, $q(x)\in C(\partial\Omega)$, $p(x)$, $q(x)>1$, and $p(x)\neq q(y)$ for any $x\in\Omega$, $y\in\partial\Omega$; λ, $\mu\in\mathbb{R}$; $f:\Omega\times\mathbb{R}\to\mathbb{R}$, and $g:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathédory functions. Throughout this paper, we assume that $a(x), b(x)$, and $c(x)$ satisfy $0<a_1\leq a(x)\leq a_2$, $0<b_1\leq b(x)\leq b_2$, and $0\leq c_1\leq c(x)\leq c_2$.

The operator $-\Delta_{p(x)}u := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, which is a natural extension of the p-Laplace operator, with p being a positive constant. However, such generalizations are not trivial since the $p(x)$-Laplace operator possesses a more complicated structure than the p-Laplace operator, for example, it is inhomogeneous. For related results involving the Laplace operator, see [12, 13].

In the past decade, many people have studied the nonlinear boundary value problems involving p-Laplacian. For example, if $\lambda=\mu=1$, $a(x)=b(x)=c(x)\equiv 1$, $p(x)\equiv p$, and $q(x)\equiv q$ (a constant), then problem (1.1) becomes

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)+|u|^{p-2}u&=f(x,u), \quad x\in\Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu}&=|u|^{q-2}u+g(x,u), \quad x\in\partial\Omega. \end{aligned} \tag{1.2}$$

Bonder and Rossi [14] considered the existence of nontrivial solutions of problem (1.2) when $f(x,u)\equiv 0$ and discussed different cases when $g(x,u)$ is subcritical, critical, and supercritical with respect to u. We also mention that Martínez and Rossi [15] studied the existence of solutions when $p=q$ and the perturbation terms $f(x,u)$ and $g(x,u)$ satisfy the Landesman-Lazer-type conditions. Recently, J.-H. Zhao and P.-H. Zhao [16] studied the nonlinear boundary value problem, assumed that $f(x,u)$ and $g(x,u)$ satisfy the Ambrosetti-Rabinowitz-type condition, and got the multiple results.

If $\lambda=\mu=1$, $p(x)\equiv p$, and $q(x)\equiv q$ (a constant), then problem (1.1) becomes

$$\begin{aligned} -\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right)+b(x)|u|^{p-2}u&=f(x,u), \quad x\in\Omega,\\ a(x)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu}&=c(x)|u|^{q-2}u+g(x,u), \quad x\in\partial\Omega. \end{aligned} \tag{1.3}$$

There are also many people who studied the p-Laplacian nonlinear boundary value problems involving (1.3). For example, Cîrstea and Rădulescu [17] used the weighted Sobolev space to discuss the existence and nonexistence results and assumed that $f(x,u)$ is a special case in the problem (1.3), where Ω is an unbounded domain. Pflüger [18], by using the same technique, considered the existence and multiplicity of solutions when $b(x)\equiv 0$. The author showed the existence result when $f(x,u)$ and $g(x,u)$ are superlinear and satisfy the Ambrosetti-Rabinowitz-type condition and got the multiplicity of solutions when one of $f(x,u)$ and $g(x,u)$ is sublinear and the other one is superlinear.

More recently, the study on the nonlinear boundary value problems with variable exponent has received considerable attention. For example, Deng [19] studied the eigenvalue of $p(x)$-Laplacian Steklov problem, and discussed the properties of the eigenvalue sequence under different conditions. Fan [20] discussed the boundary trace embedding theorems for variable exponent Sobolev spaces and some applications. Yao [21] constrained the two nonlinear perturbation terms $f(x,u)$ and $g(x,u)$ in appropriate conditions and got a number of results for the existence and multiplicity of solutions. Motivated by Yao and problem (1.3), we consider the more general form of the variable exponent boundary value problem (1.1). Under appropriate assumptions on the perturbation terms $f(x,u)$ and $g(x,u)$, by using the global minimizer method and fountain theorem, respectively, the existence and multiplicity of solutions of (1.1) were obtained. These results extend some of the results in [21] and the classical results for the p-Laplacian in [14, 16, 22–24].

2. Preliminaries

In order to discuss problem (1.1), we need some results for the spaces $W^{1,p(x)}(\Omega)$, which we call variable exponent Sobolev spaces. We state some basic properties of the spaces $W^{1,p(x)}(\Omega)$, which will be used later (for more details, see [25, 26]). Let Ω be a bounded domain of $\mathbb{R}^n$, and denote

$$C_+\left(\overline{\Omega}\right) = \left\{p(x) \mid p(x) \in C\left(\overline{\Omega}\right);\ p(x) > 1,\ \forall x \in \overline{\Omega}\right\}. \tag{2.1}$$

For $p(x) \in C_+(\overline{\Omega})$ write

$$p^+ = \max_{x\in\overline{\Omega}} p(x), \qquad p^- = \min_{x\in\overline{\Omega}} p(x). \tag{2.2}$$

We can also denote $C_+(\partial\Omega)$ and q^+, q^- for any $q(x) \in C(\partial\Omega)$, and define

$$\begin{aligned} L^{p(x)}(\Omega) &= \left\{u \mid u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)}\, dx < \infty\right\}, \\ L^{p(x)}(\partial\Omega) &= \Big\{u \mid u : \partial\Omega \longrightarrow \mathbb{R} \text{ is a measurable real-valued function}, \\ &\qquad \int_{\partial\Omega} |u(x)|^{p(x)}\, d\sigma < \infty\Big\}, \end{aligned} \tag{2.3}$$

with norms on $L^{p(x)}(\Omega)$ and $L^{p(x)}(\partial\Omega)$ defined by

$$\begin{aligned} |u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} &= \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}, \\ |u|_{L^{p(x)}(\partial\Omega)} &= \inf\left\{\tau > 0 : \int_{\partial\Omega} \left|\frac{u(x)}{\tau}\right|^{p(x)} d\sigma \le 1\right\}, \end{aligned} \tag{2.4}$$

where $d\sigma$ is the surface measure on $\partial\Omega$. Then, $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ and $(L^{p(x)}(\partial\Omega), |\cdot|_{L^{p(x)}(\partial\Omega)})$ become Banach spaces, which we call variable exponent Lebesgue spaces. Let us define the space

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}, \tag{2.5}$$

equipped with the norm

$$\|u\| = \inf\left\{\lambda > 0 : \int_\Omega \left(\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} + \left|\frac{u(x)}{\lambda}\right|^{p(x)}\right) dx \le 1\right\}. \tag{2.6}$$

For $u \in W^{1,p(x)}(\Omega)$, if we define

$$\|u\|' = \inf\left\{\lambda > 0 : \int_\Omega \left(a(x)\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} + b(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)}\right) dx \le 1\right\}, \tag{2.7}$$

then, from the assumptions of $a(x)$ and $b(x)$, it is easy to check that $\|u\|'$ is an equivalent norm on $W^{1,p(x)}(\Omega)$. For simplicity, we denote

$$\Gamma(u) = \int_\Omega \left(a(x)|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx. \tag{2.8}$$

Hence, we have (see [27])

(i) if $\Gamma(u) \ge 1$, then $\xi_1\|u\|^{p^-} \le \Gamma(u) \le \xi_2\|u\|^{p^+}$,

(ii) if $\Gamma(u) \le 1$, then $\zeta_1\|u\|^{p^+} \le \Gamma(u) \le \zeta_2\|u\|^{p^-}$,

where ξ_1, ξ_2 and ζ_1, ζ_2 are positive constants independent of u.

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.1 (see [21, 28]). (1) *The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, uniformly convex Banach space, and its conjugate space is $L^{q(x)}(\Omega)$, where $1/q(x) + 1/p(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, one has*

$$\left|\int_\Omega uv\ dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)}. \tag{2.9}$$

(2) If $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \le p_2(x)$, for any $x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the imbedding is continuous.

Proposition 2.2 (see [20, 21, 28]). (1) *$W^{1,p(x)}(\Omega)$, $W_0^{1,p(x)}(\Omega)$ are separable reflexive Banach spaces.*

(2) *If* $q(x) \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$, *then the embedding from* $W^{1,p(x)}(\Omega)$ *into* $L^{q(x)}(\Omega)$ *is compact and continuous, where*

$$p^*(x) = \begin{cases} \dfrac{np(x)}{n-p(x)}, & \text{if } p(x) < n, \\ \infty, & \text{if } p(x) \geq n. \end{cases} \tag{2.10}$$

(3) *If* $q(x) \in C_+(\partial\Omega)$ *and* $q(x) < p_*(x)$ *for any* $x \in \partial\Omega$, *then the trace imbedding from* $W^{1,p(x)}(\Omega)$ *into* $L^{q(x)}(\partial\Omega)$ *is compact and continuous, where*

$$p_*(x) = \begin{cases} \dfrac{(n-1)p(x)}{n-p(x)}, & \text{if } p(x) < n, \\ \infty, & \text{if } p(x) \geq n. \end{cases} \tag{2.11}$$

(4) *(Poincaré inequality) There is a constant* $C > 0$, *such that*

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega). \tag{2.12}$$

Proposition 2.3 (see [21, 28, 29]). *If* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function and satisfies*

$$\left|f(x,s)\right| \leq a(x) + b|s|^{p_1(x)/p_2(x)}, \quad \text{for any } x \in \overline{\Omega},\ s \in \mathbb{R}, \tag{2.13}$$

where $p_1(x)$, $p_2(x) \in C_+(\overline{\Omega})$, $a(x) \in L^{p_2(x)}(\Omega)$, $a(x) \geq 0$, *and* $b \geq 0$ *is a constant, then the Nemytsky operator from* $L^{p_1(x)}(\Omega)$ *to* $L^{p_2(x)}(\Omega)$ *defined by* $(N_f(u))(x) = f(x,u(x))$ *is a continuous and bounded operator.*

Proposition 2.4 (see [21, 28, 30]). *Denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\, dx, \quad \forall u \in L^{p(x)}(\Omega). \tag{2.14}$$

Then,

(1) $|u|_{p(x)} < 1(=1;>1)$ *if and only if* $\rho(u) < 1(=1;>1)$,

(2) $|u|_{p(x)} > 1$ *implies* $|u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$ *and* $|u|_{p(x)} < 1$ *implies* $|u|_{p(x)}^{p^-} \geq \rho(u) \geq |u|_{p(x)}^{p^+}$,

(3) $|u|_{p(x)} \to 0$ *if and only if* $\rho(u) \to 0$ *and* $|u(x)|_{p(x)} \to \infty$ *if and only if* $\rho(u) \to \infty$.

Proposition 2.5 (see [19]). *Denote*

$$\rho(u) = \int_{\partial\Omega} |u|^{p(x)}\, d\sigma, \quad \forall u \in L^{p(x)}(\partial\Omega). \tag{2.15}$$

Then,

(1) $|u|_{L^{p(x)}(\partial\Omega)} > 1$ *implies* $|u|_{L^{p(x)}(\partial\Omega)}^{p^-} \leq \rho(u) \leq |u|_{L^{p(x)}(\partial\Omega)}^{p^+}$,

(2) $|u|_{L^{p(x)}(\partial\Omega)} < 1$ *implies* $|u|_{L^{p(x)}(\partial\Omega)}^{p^-} \geq \rho(u) \geq |u|_{L^{p(x)}(\partial\Omega)}^{p^+}$.

3. Assumptions and Statement of Main Results

In the following, let X denote the generalized Sobolev space $W^{1,p(x)}(\Omega)$, X^* denote the dual space of $W^{1,p(x)}(\Omega)$, $\langle\cdot\rangle$ denote the dual pair, and let $\to$ represent strong convergence, $\rightharpoonup$ represent weak convergence, C, C_i represent the generic positive constants.

Now we state the assumptions on perturbation terms $f(x,u)$ and $g(x,u)$ for problem (1.1) as follows:

(f_0) $f : \Omega\times\mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition and there exist two constants $c_1 \geq 0$, $c_2 > 0$ such that

$$\left|f(x,u)\right| \leq c_1 + c_2|u|^{\alpha(x)-1}, \quad \forall(x,u)\in\Omega\times\mathbb{R}, \tag{3.1}$$

where $\alpha(x)\in C_+(\overline{\Omega})$ and $\alpha(x) < p^*(x)$ for any $x\in\overline{\Omega}$.

(f_1) There exist $M_1 > 0$, $\theta_1 > p^+$ such that

$$0 < \theta_1 F(x,u) \leq f(x,u)u, \quad |u| \geq M_1,\ \forall x\in\Omega. \tag{3.2}$$

(f_2) $f(x,-u) = -f(x,u)$, for all $x\in\Omega$, $u\in\mathbb{R}$.

(g_0) $g : \partial\Omega\times\mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition and there exist two constants $c_1' \geq 0$, $c_2' > 0$ such that

$$\left|g(x,u)\right| \leq c_1' + c_2'|u|^{\beta(x)-1}, \quad \forall(x,u)\in\partial\Omega\times\mathbb{R}, \tag{3.3}$$

where $\beta(x)\in C_+(\partial\Omega)$ and $\beta(x) < p_*(x)$ for any $x\in\partial\Omega$.

(g_1) There exist $M_2 > 0$, $\theta_2 > p^+$ such that

$$0 < \theta_2 G(x,u) \leq g(x,u)u, \quad |u| \geq M_2,\ \forall x\in\partial\Omega. \tag{3.4}$$

(g_2) $g(x,-u) = -g(x,u)$, for all $x\in\partial\Omega, u\in\mathbb{R}$.

The functional associated with problem (1.1) is

$$\begin{aligned}\varphi(u) = \int_\Omega \frac{a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)}}{p(x)}\,\mathrm{d}x - \lambda\int_\Omega F(x,u)\,dx \\ - \int_{\partial\Omega}\frac{c(x)}{q(x)}|u|^{q(x)}\,\mathrm{d}\sigma - \mu\int_{\partial\Omega} G(x,u)\,d\sigma,\end{aligned} \tag{3.5}$$

where $F(x,u)$ and $G(x,u)$ are denoted by

$$F(x,u) = \int_0^u f(x,s)\,\mathrm{d}s, \qquad G(x,u) = \int_0^u g(x,s)\,ds. \tag{3.6}$$

By Propositions 3.1 and 3.2, and assumptions (f_0), (g_0), it is easy to see that the functional $\varphi \in C^1(X,\mathbb{R})$; moreover, φ is even if (f_2) and (g_3) hold. Then,

$$\begin{aligned}\langle \varphi'(u), v\rangle = &\int_\Omega \left(a(x)|\nabla u|^{p(x)-2}\nabla u \nabla v + b(x)|u|^{p(x)-2}uv\right) dx - \lambda \int_\Omega f(x,u)v\,dx \\ &- \int_{\partial\Omega} c(x)|u|^{q(x)-2}uv\,d\sigma - \mu \int_{\partial\Omega} g(x,u)v\,d\sigma,\end{aligned} \tag{3.7}$$

so the weak solution of (1.1) corresponds to the critical point of the functional φ.

Before giving our main results, we first give several propositions that will be used later.

Proposition 3.1 (see [31]). *If one denotes*

$$I(u) = \int_\Omega \frac{a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)}}{p(x)}\,dx, \quad \forall u \in X, \tag{3.8}$$

then $I \in C^1(X,\mathbb{R})$ and the derivative operator of I, denoted by I', is

$$\langle I'(u), v\rangle = \int_\Omega \left(a(x)|\nabla u|^{p(x)-2}\nabla u \nabla v + b(x)|u|^{p(x)-2}uv\right) dx, \quad \forall u,v \in X, \tag{3.9}$$

and one has:

(i) *$I' : X \to X^*$ is a continuous, bounded, and strictly monotone operator,*

(ii) *I' is a mapping of (S+) type, that is, if $u_n \rightharpoonup u$ in X and* $\limsup_{n\to\infty}\langle I'(u_n) - I'(u), u_n - u\rangle \le 0$, *then $u_n \to u$ in X,*

(iii) *$I' : X \to X^*$ is a homeomorphism.*

Proposition 3.2 (see [19]). *If one denotes*

$$J(u) = \int_{\partial\Omega} \frac{c(x)}{q(x)}|u|^{q(x)}\,d\sigma, \quad \forall u \in X, \tag{3.10}$$

where $q(x) \in C_+(\partial\Omega)$ and $q(x) < p_*(x)$ for any $x \in \partial\Omega$, then $J \in C^1(X,\mathbb{R})$ and the derivative operator J' of J is

$$\langle J'(u), v\rangle = \int_{\partial\Omega} c(x)|u|^{q(x)-2}uv\,d\sigma, \quad \forall u,v \in X, \tag{3.11}$$

and one has that $J : X \to \mathbb{R}$ and $J' : X \to X^*$ are sequentially weakly-strongly continuous, namely, $u_n \rightharpoonup u$ in X implies $J'(u_n) \to J'(u)$.

Let X be a reflexive and separable Banach space. There exist $e_i \in X$ and $e_j^* \in X^*$ such that

$$X = \overline{\operatorname{span}\{e_i : i = 1, 2, \ldots\}}, \qquad X^* = \overline{\operatorname{span}\left\{e_j^* : j = 1, 2, \ldots\right\}},$$
$$\left\langle e_i, e_j^* \right\rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{3.12}$$

$k = 1, 2, \ldots$

$$X_k = \operatorname{span}\{e_k\}, \qquad Y_k = \bigoplus_{i=1}^{k} X_i, \qquad Z_k = \overline{\bigoplus_{i \geq k} X_i}. \tag{3.13}$$

One important aspect of applying the standard methods of variational theory is to show that the functional φ satisfies the *Palais-Smale* condition, which is introduced by the following definition.

Definition 3.3. Let $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. Then, functional φ satisfies the $(\mathrm{PS})_c$ condition if any sequence $\{u_n\} \subset X$ such that

$$\varphi(u_n) \longrightarrow c, \quad \varphi'(u_n) \longrightarrow 0 \text{ in } X^*, \quad \text{as } n \longrightarrow \infty \tag{3.14}$$

contains a subsequence converging to a critical point of φ.

In what follows we write the $(\mathrm{PS})_c$ condition simply as the (PS) condition if it holds for every level $c \in \mathbb{R}$ for the *Palais-Smale* condition at level c.

Proposition 3.4 (Fountain theorem, see [23, 32]). *Assume that*

(A1) *X is a Banach space, $\varphi \in C^1(X, \mathbb{R})$ is an even functional, the subspaces X_k, Y_k and Z_k are defined by* (3.13).*Suppose that, for every $k \in \mathbf{N}$, there exist $\rho_k > \gamma_k > 0$ such that*

(A2) $\inf_{u \in Z_k, \|u\| = \gamma_k} \varphi(u) \to \infty$ *as* $k \to \infty$,

(A3) $\max_{u \in Y_k, \|u\| = \rho_k} \varphi(u) \leq 0$,

(A4) *φ satisfies $(PS)_c$ condition for every $c > 0$.*

Then, φ has a sequence of critical values tending to $+\infty$.

Proposition 3.5 (see [21]). *Suppose that hypotheses $\alpha(x) \in C_+(\overline{\Omega})$, $\alpha(x) < p^*(x)$, for all $x \in \overline{\Omega}$, and if $q(x) \in C_+(\partial\Omega)$, $q(x) < p_*(x)$, for all $x \in \partial\Omega$, denote*

$$\alpha_k = \sup\left\{|u|_{L^{\alpha(x)}(\Omega)} : \|u\| = 1, u \in Z_k\right\};$$
$$q_k = \sup\left\{|u|_{L^{q(x)}(\partial\Omega)} : \|u\| = 1, u \in Z_k\right\}, \tag{3.15}$$

then $\lim_{k \to \infty} \alpha_k = 0$, $\lim_{k \to \infty} q_k = 0$.

Let us introduce the following lemma that will be useful in the proof of our main result.

Lemma 3.6. *Let $\lambda, \mu \geq 0$, $q^- > \theta_1$, θ_2, and assume that $(f_0), (f_1)$, (g_0), and (g_1) are satisfied, then φ satisfies (PS) condition.*

Proof. By Propositions 2.2 and 2.3, we know that if we denote

$$\Phi(u) = \lambda \int_\Omega F(x,u)\ dx + \mu \int_{\partial\Omega} G(x,u)\ d\sigma, \tag{3.16}$$

then Φ is weakly continuous and its derivative operator, denoted by Φ', is compact. By Propositions 3.1 and 3.2, we deduce that $\varphi' = I' - J' - \Phi'$ is also of (S+) type. To verify that φ satisfies (PS) condition on X, it is enough to verify that any (PS) sequence is bounded. Suppose that $\{u_n\} \subset X$ such that

$$\varphi(u_n) \longrightarrow c, \quad \varphi'(u_n) \longrightarrow 0, \text{ in } \ X^*, \text{ as } n \longrightarrow \infty. \tag{3.17}$$

Then, for n large enough, we can find $M_3 > 0$ such that

$$|\varphi(u_n)| \leq M_3. \tag{3.18}$$

Since $\varphi'(u_n) \to 0$, we have $\langle \varphi'(u_n), u_n\rangle \to 0$. In particular, $\{\langle \varphi'(u_n), u_n\rangle\}$ is bounded. Thus, there exists $M_4 > 0$ such that

$$|\langle \varphi'(u_n), u_n\rangle| \leq M_4. \tag{3.19}$$

We claim that the sequence $\{u_n\}$ is bounded. If it is not true, by passing a subsequence if necessary, we may assume that $\|u_n\| \to +\infty$. Without loss of generality, we assume that $\|u_n\| \geq 1$ appropriately large such that $\xi_1 \|u\|^{p^-} < \zeta_1 \|u\|^{p^+}$ for any $x \in \Omega$. From (3.18) and (3.19) and letting $\theta = \min\{\theta_1, \theta_2\}$, then $\theta < q^-$, we have

$$\begin{aligned} M_3 \geq \varphi(u_n) &= I(u_n) - J(u_n) - \Phi(u_n) \\ &\geq \frac{1}{p^+}\Gamma(u_n) - \frac{1}{q^-}\int_{\partial\Omega} c(x)|u_n|^{q(x)}\ d\sigma - \Phi(u_n), \\ &\geq \frac{1}{p^+}\Gamma(u_n) - \frac{1}{\theta}\int_{\partial\Omega} c(x)|u_n|^{q(x)}\ d\sigma - \Phi(u_n), \end{aligned} \tag{3.20}$$

$$M_4 \geq -\langle \varphi'(u_n), u_n\rangle = -\Gamma(u_n) + \int_{\partial\Omega} c(x)|u_n|^{q(x)}\ d\sigma + \langle \Phi'(u_n), u_n\rangle. \tag{3.21}$$

By virtue of assumptions (f_1) and (g_1) and combining (3.20) and (3.21), we have

$$\begin{aligned}\theta M_3 + M_4 &\geq \left(\frac{\theta}{p^+} - 1\right)\Gamma(u_n) - \theta\Phi(u_n) + \langle \Phi'(u_n), u_n\rangle \\ &\geq \left(\frac{\theta}{p^+} - 1\right)\xi_1\|u_n\|^{p^-} + \lambda\int_\Omega \left(f(x,u_n)u_n - \theta F(x,u_n)\right)\, dx \\ &\quad + \mu\int_{\partial\Omega}\left(g(x,u_n)u_n - \theta G(x,u_n)\right)\, d\sigma \\ &\geq \left(\frac{\theta}{p^+} - 1\right)\xi_1\|u_n\|^{p^-} - C.\end{aligned} \tag{3.22}$$

Note that $\theta = \min\{\theta_1, \theta_2\} > p^+$, let $n \to \infty$ we obtian a contradiction. It follows that the sequence $\{u_n\}$ is bounded in X. Therefore, φ satisfies (PS) condition. □

Under appropriate assumptions on the perturbation terms $f(x,u), g(x,u)$, a sequence of weak solutions with energy values tending to $+\infty$ was obtained. The main result of the paper reads as follows.

Theorem 3.7. *Let $\alpha^-, \beta^- > p^+$, $q^- > \theta_1$, θ_2, and λ, $\mu \geq 0$, and assumed that $(f_0) - -(f_2), (g_0) - -(g_2)$ are satisfied; then φ has a sequence of critical points $\{\pm u_n\}$ such that $\varphi(\pm u_n) \to \infty$ as $n \to \infty$.*

Proof. We will prove that φ satisfies the conditions of Proposition 3.4. Obviously, because of the assumptions of (f_2) and (g_2), φ is an even functional and satisfies (PS) condition (see Lemma 3.6). We will prove that if k is large enough, then there exist $\rho_k > \gamma_k > 0$ such that (A2) and (A3) hold. By virtue of (f_0), (g_0), there exist two positive constants C_1, C_2 such that

$$|F(x,u)| \leq C_1\left(1 + |u|^{\alpha(x)}\right), \quad (x,u) \in \Omega\times\mathbb{R}; \qquad |G(x,u)| \leq C_2\left(1 + |u|^{\beta(x)}\right), \quad (x,u) \in \partial\Omega\times\mathbb{R}. \tag{3.23}$$

Letting $u \in Z_k$ with $\|u\| > 1$ appropriately large such that $\xi_1\|u\|^{p^-} < \zeta_1\|u\|^{p^+}$, we have

$$\begin{aligned}\varphi(u) &= I(u) - J(u) - \Phi(u) \\ &\geq \frac{1}{p^+}\Gamma(u) - \frac{c_2}{q^-}\int_{\partial\Omega}|u|^{q(x)}\, d\sigma - \lambda\int_\Omega C_1\left(1 + |u|^{\alpha(x)}\right)\, dx - \mu\int_{\partial\Omega} C_2\left(1 + |u|^{\beta(x)}\right)\, d\sigma \\ &\geq \frac{1}{p^+}\min\left\{\xi_1\|u\|^{p^-}, \zeta_1\|u\|^{p^+}\right\} - \frac{c_2}{q^-}\max\left\{|u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)}\right\} \\ &\quad - \lambda C_1\max\left\{|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}\right\} - \mu C_2\max\left\{|u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}\right\} - C_3 \\ &\geq \frac{\xi_1}{p^+}\|u\|^{p^-} - C(q^-, \lambda, \mu)\max\Big\{|u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)}, |u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, \\ &\qquad\qquad |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}, |u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}\Big\} - C_3.\end{aligned} \tag{3.24}$$

If $\max\{|u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)}, |u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}, |u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}\} = |u|^{q^+}_{L^{q(x)}(\partial\Omega)}$, then by Proposition 3.5 , we have

$$\varphi(u_n) \geq \frac{\xi_1}{p^+}\|u\|^{p^-} - C(q^-,\lambda,\mu)|u|^{q^+}_{L^{q(x)}(\partial\Omega)} - C_3 \geq \frac{\xi_1}{p^+}\|u\|^{p^-} - C(q^-,\lambda,\mu)q_k^{q^+}\|u\|^{q^+} - C_3. \quad (3.25)$$

Choose $\gamma_k = (q^+C(q^-,\lambda,\mu)(q\_k(q^+))/\xi\_1)^{1/(p^- - q^+)}$. For $u \in Z_k$ with $\|u\| = \gamma_k$, we have

$$\varphi(u) \geq \xi_1\left(\frac{1}{p^+} - \frac{1}{q^+}\right)\gamma_k^{p^-} - C_3. \quad (3.26)$$

Since $q_k \to 0$ as $k \to \infty$ and $1 < p^- \leq p^+ < \theta_1,\ \theta_2 < q^- \leq q^+$, we have $1/p^+ - 1/q^+ > 0$ and $\gamma_k \to \infty$. Thus, for sufficiently large k, we have $\varphi(u) \to \infty$ with $u \in Z_k$ and $\|u\| = \gamma_k$ as $k \to \infty$. In other cases, similarly, we can deduce

$$\varphi(u) \longrightarrow \infty, \qquad \text{since } \alpha_k \longrightarrow 0,\ q_k = 0,\ k \longrightarrow \infty. \quad (3.27)$$

So (A2) holds.

By virtue of (f_1) and (g_1), there exist two positive constants C_4, C_5 such that

$$F(x,u) \geq C_4\left(|u|^{\theta_1} - 1\right), \quad \forall(x,u) \in \Omega \times \mathbb{R}; \qquad G(x,u) \geq C_5\left(|u|^{\theta_2} - 1\right), \quad \forall(x,u) \in \partial\Omega \times \mathbb{R}. \quad (3.28)$$

Letting $u \in Y_k$, we have

$$\begin{aligned}\varphi(u) &\leq \frac{1}{p^-}\Gamma(u) - \frac{c_1}{q^+}\int_{\partial\Omega}|u|^{q(x)}\,d\sigma - \lambda\int_\Omega F(x,u)\,dx - \mu\int_{\partial\Omega}G(x,u)\,d\sigma \\ &\leq \frac{1}{p^-}\max\left\{\xi_2\|u\|^{p^+}, \zeta_2\|u\|^{p^-}\right\} - \frac{c_1}{q^+}\min\left\{|u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)}\right\} - C_4\lambda\int_\Omega|u|^{\theta_1}\,dx \\ &\quad - C_5\mu\int_{\partial\Omega}|u|^{\theta_2}\,d\sigma + C_6.\end{aligned} \quad (3.29)$$

If $\max\{\xi_2\|u\|^{p^+}, \zeta_2\|u\|^{p^-}\} = \xi_2\|u\|^{p^+},\ \min\{|u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)}\} = |u|^{q^-}_{L^{q(x)}(\partial\Omega)}$, then we have

$$\varphi(u) \leq \frac{\xi_2}{p^-}\|u\|^{p^+} - \frac{c_1}{q^+}|u|^{q^-}_{L^{q(x)}(\partial\Omega)} - C_4\lambda\int_\Omega|u|^{\theta_1}\,dx - C_5\mu\int_{\partial\Omega}|u|^{\theta_2}\,d\sigma + C_6. \quad (3.30)$$

Since $\dim Y_k < \infty$, all norms are equivalent in Y_k. So we get

$$\varphi(u) \leq \frac{\xi_2}{p^-}\|u\|^{p^+} - \frac{c_1}{q^+}C_7\|u\|^{q^-} - C_8\lambda\|u\|^{\theta_1} - C_9\mu\|u\|^{\theta_2} + C_6. \quad (3.31)$$

Also, note that $q^- > \theta_1,\ \theta_2 > p^+$, Then, we get $\varphi(u) \to -\infty$ as $\|u\| \to \infty$. For other cases, the proofs are similar and we omit them here. So (A3) holds. From the proof of (A2) and (A3), we can choose $\rho_k > \gamma_k > 0$. Thus, we complete the proof. □

This time our idea is to show that φ possesses a nontrivial global minimum point in X.

Theorem 3.8. *Let $\alpha^+, \beta^+, q^+ < p^-$, and assume (f_0), (g_0) are satisfied; then (1.1) has a weak solution.*

Proof. Firstly, we show that φ is coercive. For sufficiently large norm of u ($\|u\| \geq 1$), and by virtue of (3.23),

$$\begin{aligned}
\varphi(u) &= \int_\Omega \frac{a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)}}{p(x)}\,dx - \lambda\int_\Omega F(x,u)\,dx - \int_{\partial\Omega}\frac{c(x)}{q(x)}|u|^{q(x)}\,d\sigma \\
&\quad - \mu\int_{\partial\Omega} G(x,u)\,d\sigma \\
&\geq \frac{\xi_1}{p^+}\|u\|^{p^-} - |\lambda|\int_\Omega C_1\left(1+|u|^{\alpha(x)}\right)dx - \frac{c_2}{q^-}\int_{\partial\Omega}|u|^{q(x)}\,d\sigma - |\mu|\int_{\partial\Omega} C_2\left(1+|u|^{\beta(x)}\right)d\sigma \\
&\geq \frac{\xi_1}{p^+}\|u\|^{p^-} - |\lambda| C_1 \max\left\{|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}\right\} - \frac{c_2}{q^-}\max\left\{|u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)}\right\} \\
&\quad - |\mu| C_2 \max\left\{|u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}\right\} - C_{10}.
\end{aligned} \tag{3.32}$$

If

$$\begin{aligned}
\max\left\{|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}\right\} &= |u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, \\
\max\left\{|u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)}\right\} &= |u|^{q^+}_{L^{q(x)}(\partial\Omega)}, \\
\max\left\{|u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}\right\} &= |u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)},
\end{aligned} \tag{3.33}$$

then

$$\begin{aligned}
\varphi(u) &\geq \frac{\xi_1}{p^+}\|u\|^{p^-} - C_{11}|\lambda|\|u\|^{\alpha^+} - C_{12}\|u\|^{q^+} - C_{13}|\mu|\|u\|^{\beta^+} \\
&\quad - C_{10} \longrightarrow \infty \quad \text{as } \|u\| \longrightarrow \infty.
\end{aligned} \tag{3.34}$$

So φ is coercive since $\alpha^+,\ \beta^+,\ q^+ < p^-$. Secondly, by Proposition 2.2, it is easy to verify that φ is weakly lower semicontinuous. Thus, φ is bounded below and φ attains its infimum in X, that is, $\varphi(u_0) = \inf_{u\in X}\varphi(u)$ and u_0 is a critical point of φ, which is a weak solution of (1.1). □

In the Theorem 3.8, we cannot guarantee that u_0 is nontrivial. In fact, under the assumptions on the above theorem, we can also get a nontrivial weak solution of φ.

Corollary 3.9. *Under the assumptions in Theorem 3.8, if one of the following conditions holds,* (1.1) *has a nontrivial weak solution.*

(1) *If $\lambda, \mu \neq 0$, there exist two positive constants $d_1, d_2 < p^-$ such that*

$$\liminf_{u \to 0} \frac{\operatorname{sgn}(\lambda) F(x,u)}{|u|^{d_1}} > 0, \quad \text{for } x \in \Omega \text{ uniformly},$$
$$\liminf_{u \to 0} \frac{\operatorname{sgn}(\mu) G(x,u)}{|u|^{d_2}} > 0, \quad \text{for } x \in \partial\Omega \text{ uniformly}. \tag{3.35}$$

(2) *If $\lambda = 0$, $\mu \neq 0$, there exist two positive constants $d_2 < p^-$ such that*

$$\liminf_{u \to 0} \frac{\operatorname{sgn}(\mu) G(x,u)}{|u|^{d_2}} > 0, \quad \text{for } x \in \partial\Omega \text{ uniformly}. \tag{3.36}$$

(3) *If $\lambda \neq 0$, $\mu = 0$, there exist two positive constants $d_1 < p^-$ such that*

$$\liminf_{u \to 0} \frac{\operatorname{sgn}(\lambda) F(x,u)}{|u|^{d_1}} > 0, \quad \text{for } x \in \Omega \text{ uniformly}. \tag{3.37}$$

Proof. From Theorem 3.8, we know that φ has a global minimum point u_0. We just need to show that u_0 is nontrivial. We only consider the case $\lambda, \mu \neq 0$ here. From (1), we know that for $0 < u < 1$ small enough, there exists two positive constants $C_{14}, C_{15} > 0$ such that

$$\operatorname{sgn}(\lambda) F(x,u) \geq C_{14}|u|^{d_1}, \qquad \operatorname{sgn}(\mu) G(x,u) \geq C_{15}|u|^{d_2}. \tag{3.38}$$

Choose $\overline{u} \equiv M > 0$; then $\overline{u} \in X$. For $0 < t < 1$ small enough, we have

$$\begin{aligned}
\varphi(t\overline{u}) &\leq \frac{b_2 t^{p^-}}{p^-} \int_\Omega |\overline{u}|^{p(x)}\, dx - |\lambda| \int_\Omega \operatorname{sgn}(\lambda) F(x, t\overline{u})\, dx - \frac{c_1}{q^+} \int_{\partial\Omega} |t\overline{u}|^{q(x)}\, d\sigma \\
&\quad - |\mu| \int_{\partial\Omega} \operatorname{sgn}(\mu) G(x, t\overline{u})\, d\sigma \\
&\leq \frac{b_2 t^{p^-}}{p^-} \int_\Omega |M|^{p(x)}\, \mathrm{d}x - C_{14}|\lambda| t^{d_1} \int_\Omega |M|^{d_1}\, dx - \frac{c_1}{q^+} t^{q^-} \int_{\partial\Omega} |M|^{q(x)}\, d\sigma \\
&\quad - C_{15}|\mu| t^{d_2} \int_{\partial\Omega} |M|^{d_2}\, d\sigma \\
&\leq C_{16} t^{p^-} - C_{17}|\lambda| t^{d_1} - C_{18} c_1 t^{q^-} - C_{19}|\mu| t^{d_2}.
\end{aligned} \tag{3.39}$$

Since $d_1, d_2, < p^-$ and $q^- \leq q^+ < p^-$, there exists $0 < t_0 < 1$ small enough such that $\varphi(t_0\overline{u}) < 0$. So the global minimum point u_0 of φ is nontrivial. □

Remark 3.10. Suppose that $f(x,u) = \text{sgn}(\lambda)|u|^{\alpha(x)-2}u$, $g(x,u) = \text{sgn}(\mu)|u|^{\beta(x)-2}u$ and $p^- > \alpha^+,\ \beta^+,\ q^+$; then the conditions in Corollary 3.9 can be fulfilled.

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