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Research Article

Darboux Transformation and Explicit Solutions for a Generalized Sawada-Kotera Equation

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A generalized Sawada-Kotera equation and its Lax pairs are proposed. With the help of the gauge transformation between spectral problems, a Darboux transformation for the generalized SK equation is constructed. As an application of the Darboux transformation, we give some explicit solutions of the generalized SK equation such as the rational solutions, soliton solutions, and periodic solutions.

1. Introduction

The Sawada-Kotera (SK) equation

$$u_t = -u_{xxxxx} + 15\left(uu_{xx} - u^3\right)_x \tag{1}$$

was first proposed by Sawada and Kotera when they gave a method for finding N-soliton solutions of the KdV equation and the KdV-like equation [1]. In [2], Caudrey et al. showed that (1) was a member of a new hierarchy of KdV equations. The SK equation's physical importance was illustrated by Aiyer et al. in [3]. Then, the equation has been investigated by many authors [4–8]. The aim of the present paper is using the Darboux transformation [9–12] to study a generalized SK equation:

$$\begin{aligned} u_t &= -u_{xxxxx} + 15\left(uu_{xx} - u^3\right)_x - 15\left(vu_x\right)_x - 10vv_x, \\ v_t &= -v_{xxxxx} + 15\left(uv_{xx} + vv_x - 3vu^2\right)_x + 30\left(vu_x\right)_{xx}. \end{aligned} \tag{2}$$

The present paper is organized as follows. In Section 2, with the aid of the Lax pairs of the SK equation [13, 14] and extending them by adding one potential function, we propose a generalized SK equation and its Lax pairs. Based on the gauge transformation between spectral problems, we derive a Darboux transformation of the generalized SK equation. In Section 3, the Darboux transformation is applied to the generalized SK equation, by which explicit solutions (we have verified the correctness of the solutions by using the Mathematic 5.0.) of the generalized SK equation are derived, including rational solutions, soliton solutions, and periodic solutions.

2. Darboux Transformation of the Generalized Sawada-Kotera Equation

In this section, we will derive a generalized SK equation and its Darboux transformation. To this end, we first introduce the Lax pairs:

$$\mathscr{L}\psi = \lambda\psi, \qquad \psi_t = \mathscr{B}\psi, \tag{3}$$

where operators $\mathscr{L}$ and $\mathscr{B}$ are defined as follows:

$$\begin{aligned} \mathscr{L} &= \partial^3 - 3u\partial + v, \\ \mathscr{B} &= 9\partial^5 - 45u\partial^3 + 15\left(v - 3u_x\right)\partial^2 \\ &\quad + 15\left(3u^2 - 2u_{xx} + v_x\right)\partial + 10\left(v_{xx} - 3uv\right). \end{aligned} \tag{4}$$

Then the compatibility condition between the two equations of (3) yields the Lax equation, $\mathscr{L}_t = [\mathscr{B}, \mathscr{L}]$, which is equivalent to the generalized SK equation:

$$\begin{aligned} u_t &= -u_{xxxxx} + 15\left(uu_{xx} - u^3\right)_x - 15\left(vu_x\right)_x - 10vv_x, \\ v_t &= -v_{xxxxx} + 15\left(uv_{xx} + vv_x - 3u^2v\right)_x + 30\left(u_xv\right)_{xx}. \end{aligned} \tag{5}$$

If we choose $v = 0$ and $v = -(3/2)u_x$, (5) can be, respectively, reduced to the SK equation:

$$u_t = -u_{xxxxx} - 45u^2u_x + 15uu_{xxx} + 15u_xu_{xx} \tag{6}$$

and the Kaup-Kupershmidt equation:

$$u_t = -u_{xxxxx} - 45u^2u_x + 15uu_{xxx} + \frac{75}{2}u_xu_{xx}. \tag{7}$$

Theorem 1. *Let f satisfy (3) with $\lambda = \lambda_0$ and $A = -(\ln f)_x$. Then the following Darboux transformation gives the relation about the original solutions u, v of (5) and its new ones $\overline{u}$, $\overline{v}$:*

$$\begin{aligned}\overline{u} &= u + A_x,\\ \overline{v} &= v - 3u_x + 3AA_x - 3A_{xx}.\end{aligned} \tag{8}$$

Proof. Assume that ψ satisfies (3) and $A = -(\ln f)_x$. Let

$$\overline{\psi} = \psi_x + A\psi. \tag{9}$$

Using the first expression of (3), a direct calculation gives the following equations:

$$\begin{aligned}\overline{\psi}_x &= \psi_{xx} + A\psi_x + A_x\psi,\\ \overline{\psi}_{xx} &= A\psi_{xx} + (3u + 2A_x)\psi_x + (A_{xx} - v + \lambda)\psi,\\ \overline{\psi}_{xxx} &= 3(u + A_x)\psi_{xx} + (3u_x - v + 3A_{xx} + 3uA + \lambda)\psi_x\\ &\quad + (A_{xxx} - v_x - vA + \lambda A)\psi.\end{aligned} \tag{10}$$

Substituting (9) and (10) into the following equation:

$$\overline{\mathscr{L}}\overline{\psi} = \lambda\overline{\psi}, \tag{11}$$

where

$$\overline{\mathscr{L}} = \partial^3 - 3\overline{u}\partial + \overline{v} \tag{12}$$

and comparing the coefficients of ψ, ψ_x, and ψ_{xx}, we obtain the following:

$$\begin{aligned}\overline{u} - u - A_x &= 0,\\ 3\overline{u}A - \overline{v} - 3u_x + v - 3A_{xx} - 3uA &= 0,\end{aligned} \tag{13}$$

$$3\overline{u}A_x - \overline{v}A - A_{xxx} + v_x + vA = 0. \tag{14}$$

Equation (13) implies the following:

$$\begin{aligned}\overline{u} &= u + A_x,\\ \overline{v} &= v - 3u_x + 3AA_x - 3A_{xx}.\end{aligned} \tag{15}$$

Substituting (15) into (14) and integrating it once, we have the following:

$$3uA + 3AA_x - A^3 - A_{xx} + v = \lambda_0, \tag{16}$$

where λ_0 is a constant of integration. Through direct calculations, we arrive at the following:

$$\begin{aligned}-\frac{f_{xx}}{f} &= A_x - A^2,\\ -\frac{f_{xxx}}{f} &= A_{xx} - 3AA_x + A^3,\\ -\frac{f_{4x}}{f} &= A_{xxx} - 4AA_{xx} - 3A_x^2 + 6A^2A_x - A^4,\\ -\frac{f_{5x}}{f} &= A_{xxxx} - 5AA_{xxx} - 10A_xA_{xx}\\ &\quad + 15AA_x^2 + 10A^2A_{xx} - 10A^3A_x + A^5.\end{aligned} \tag{17}$$

Using (17) and $A = -(\ln f)_x$, a simple reduction shows that (16) gives rise to the following:

$$\mathscr{L}f = \lambda_0 f. \tag{18}$$

Similarly, we consider the following equation:

$$\overline{\psi}_t = \overline{\mathscr{B}}\overline{\psi}, \tag{19}$$

where

$$\begin{aligned}\overline{\mathscr{B}} &= 9\partial^5 - 45\overline{u}\partial^3 + 15(\overline{v} - 3\overline{u}_x)\partial^2\\ &\quad + 15(3\overline{u}^2 - 2\overline{u}_{xx} + \overline{v}_x)\partial + 10(\overline{v}_{xx} - 3\overline{u}\overline{v}).\end{aligned} \tag{20}$$

Seeing (3), (8), and (9), a direct calculation shows that (19) gives the following:

$$\begin{aligned}A_t &= 9A_{xxxxx} - 10v_{xxx} - 45uA_{xxx} + 30uv_x\\ &\quad + 15vA_{xx} - 90A_{xx}^2 + 30u_xv - 90u_xA_{xx}\\ &\quad + 45u^2A_x + 90uA_x^2 + 90A_x^3 - 75u_{xx}A_x\\ &\quad - 120A_xA_{xxx} + 15v_xA_x + 270AA_xA_{xx}\\ &\quad + 30A^2A_{xxx} - 30AA_{xxxx} - 30u_{xxx}A\\ &\quad - 90uA^2A_x + 90uAA_{xx} + 90uu_xA\\ &\quad - 30vAA_x - 90A^2A_x^2 + 90u_xAA_x,\end{aligned} \tag{21}$$

which together with (17) implies the following:

$$f_t = \mathscr{B}f. \tag{22}$$

This means that both of the Lax pairs (3) and (11) and (19) have the same form; that is, they lead to the same equation (5). Therefore, original solutions u, v of the generalized SK equation (5) are mapped into its new ones $\overline{u}$, $\overline{v}$ by the Darboux transformation (8). □

3. Explicit Solutions of the Generalized Sawada-Kotera Equation

In this section, we will construct explicit solutions of the generalized SK equation (5) by using the Darboux transformation (8).

(I) We choose a trivial solution $u = 0$, $v = 0$ of (5). Then (3) with $\lambda = \lambda_0$ is reduced to the following:

$$\psi_{xxx} = \lambda_0\psi, \qquad \psi_t = 9\psi_{xxxxx}. \tag{23}$$

Let $\lambda_0 = -k^3$ $(k \neq 0)$. We can see that (23) has a general solution:

$$f = c_1 \exp(\Delta_1) + c_2 \exp\left(-\frac{1}{2}\Delta_1\right)\cos(\Delta_2) + c_3 \exp\left(-\frac{1}{2}\Delta_1\right)\sin(\Delta_2), \tag{24}$$

where c_j, $(j = 1, 2, 3)$ are constants and

$$\Delta_1 = -kx - 9k^5 t, \qquad \Delta_2 = \frac{\sqrt{3}}{2}\left(kx - 9k^5 t\right). \tag{25}$$

Using the Darboux transformation (8), we get an explicit solution of (5)

$$\overline{u} = \frac{3\left(c_2^2 + c_3^2\right)k^2 \exp(-3\Delta_1) - 6c_1 k^2 \exp(-(3/2)\Delta_1)\left[\left(c_2 + \sqrt{3}c_3\right)\cos(\Delta_2) - \left(\sqrt{3}c_2 - c_3\right)\sin(\Delta_2)\right]}{4[c_1 + c_2\exp(-(3/2)\Delta_1)\cos(\Delta_2) + c_3\exp(-(3/2)\Delta_1)\sin(\Delta_2)]^2},$$

$$\begin{aligned}\overline{v} = \Bigg(-\Bigg\{&9\exp\left(-\frac{3}{2}\Delta_1\right)k^3\Bigg(4c_1^2\left[\left(c_2 - \sqrt{3}c_3\right)\cos(\Delta_2) + \left(\sqrt{3}c_2 + c_3\right)\sin(\Delta_2)\right]\\
&+ \exp(-3\Delta_1)\left(\left[\left(c_2 - \sqrt{3}c_3\right)\cos(\Delta_2) + \left(\sqrt{3}c_2 + c_3\right)\sin(\Delta_2)\right]\left(c_2^2 + c_3^2\right)\right)\\
&+ 2\exp\left(-\frac{3}{2}\Delta_1\right)c_1\Big(\left(6 + \cos(2\Delta_2) - \sqrt{3}\sin(2\Delta_2)\right)c_2^2\\
&+ 2\left(\sin(2\Delta_2) + \sqrt{3}\cos(2\Delta_2)\right)c_2c_3\\
&+ \left(6 - \cos(2\Delta_2) + \sqrt{3}\sin(2\Delta_2)\right)c_3^2\Big)\Bigg)\Bigg\}\Bigg)\\
&\times\left(8\left[c_1 + c_2\exp\left(-\frac{3}{2}\Delta_1\right)\cos(\Delta_2) + c_3\exp\left(-\frac{3}{2}\Delta_1\right)\sin(\Delta_2)\right]^3\right)^{-1}.\end{aligned} \tag{26}$$

Particularly, when we choose $c_1 = 0$, $c_2 = c_3 = 1$, we can get a periodic solution of (5):

$$\begin{aligned}\overline{u} &= \frac{3k^2}{2[\cos(\Delta_2) + \sin(\Delta_2)]^2},\\
\overline{v} &= -3k^3 + 3k^3\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\cos(\Delta_2) - \sin(\Delta_2)}{\cos(\Delta_2) + \sin(\Delta_2)}\right)^3\\
&\quad + 3k^3\frac{2\sin(2\Delta_2) - 1}{[\cos(\Delta_2) + \sin(\Delta_2)]^2}.\end{aligned} \tag{27}$$

Plots of the solutions are given in Figures 1 and 2.

(II) We consider the trivial solution $u = 0$, $v = 1$ of (5). Then (3) with $\lambda = \lambda_0$ is reduced to the following:

$$\psi_{xxx} = (\lambda_0 - 1)\psi, \qquad \psi_t = 9\psi_{xxxxx} + 15\psi_{xx}. \tag{28}$$

Case 1. When $\lambda_0 = 1$, it is easy to see that (28) has a general solution:

$$f = \alpha x^2 + \beta x + \gamma + 30\alpha t, \tag{29}$$

where α, β, and γ are constants. Using the Darboux transformation (8), we get a rational solution of the generalized SK equation (5):

$$\begin{aligned}\overline{u} &= \frac{2\alpha^2x^2 + 2\alpha\beta x + \beta^2 - 2\alpha\gamma - 60\alpha^2 t}{[\alpha x^2 + \beta x + \gamma + 30\alpha t]^2},\\
\overline{v} &= 1 + \frac{3\left(\beta^2 - 4\alpha(\gamma + 30\alpha t)\right)(\beta + 2\alpha x)}{[\alpha x^2 + \beta x + \gamma + 30\alpha t]^3}.\end{aligned} \tag{30}$$

Case 2. When $\lambda_0 = 1 - k^3$ $(k \neq 0)$, (28) has a general solution:

$$f = c_1\exp(\Delta_1) + c_2\exp\left(-\frac{1}{2}\Delta_1\right)\cos(\Delta_2) + c_3\exp\left(-\frac{1}{2}\Delta_1\right)\sin(\Delta_2), \tag{31}$$

where c_j, $(j = 1, 2, 3)$ are constants and

$$\begin{aligned}\Delta_1 &= -kx - \left(9k^5 - 15k^2\right)t,\\
\Delta_2 &= \frac{\sqrt{3}}{2}\left(kx - 9k^5 t + 15k^2 t\right).\end{aligned} \tag{32}$$

Using the Darboux transformation (8), we get an explicit solution of (5):

$$\overline{u} = \frac{3\left(c_2^2+c_3^2\right)k^2\exp\left(-3\Delta_1\right) - 6c_1k^2\exp\left(-(3/2)\Delta_1\right)\left[\left(c_2+\sqrt{3}c_3\right)\cos\left(\Delta_2\right) - \left(\sqrt{3}c_2 - c_3\right)\sin\left(\Delta_2\right)\right]}{4\left[c_1 + c_2\exp\left(-(3/2)\Delta_1\right)\cos\left(\Delta_2\right) + c_3\exp\left(-(3/2)\Delta_1\right)\sin\left(\Delta_2\right)\right]^2},$$

$$\begin{aligned}\overline{v} = {} & \left(1 - \left\{9\exp\left(-\frac{3}{2}\Delta_1\right)k^3\left(4c_1^2\left[\left(c_2-\sqrt{3}c_3\right)\cos\left(\Delta_2\right) + \left(\sqrt{3}c_2 + c_3\right)\sin\left(\Delta_2\right)\right]\right.\right.\right. \\ & + \exp\left(-3\Delta_1\right)\left(\left[\left(c_2-\sqrt{3}c_3\right)\cos\left(\Delta_2\right) + \left(\sqrt{3}c_2+c_3\right)\sin\left(\Delta_2\right)\right]\left(c_2^2+c_3^2\right)\right) \\ & + 2\exp\left(-\frac{3}{2}\Delta_1\right)c_1\left(\left(6+\cos\left(2\Delta_2\right) - \sqrt{3}\sin\left(2\Delta_2\right)\right)c_2^2\right. \\ & + 2\left(\sin\left(2\Delta_2\right) + \sqrt{3}\cos\left(2\Delta_2\right)\right)c_2c_3 \\ & \left.\left.\left.\left. + \left(6 - \cos\left(2\Delta_2\right) + \sqrt{3}\sin\left(2\Delta_2\right)\right)c_3^2\right)\right)\right\}\right) \\ & \times\left(8\left[c_1 + c_2\exp\left(-\frac{3}{2}\Delta_1\right)\cos\left(\Delta_2\right)c_3\exp\left(-\frac{3}{2}\Delta_1\right)\sin\left(\Delta_2\right)\right]^3\right)^{-1}.\end{aligned} \tag{33}$$

(III) We choose another trivial solution $u = 1$, $v = 0$ of (5). Then (3) with $\lambda = \lambda_0$ is reduced to the following:

$$\psi_{xxx} = 3\psi_x + \lambda_0\psi, \qquad \psi_t = 9\psi_{xxxxx} - 45\psi_{xxx} + 45\psi_x. \tag{34}$$

Case 1. For $\lambda_0 = 0$, a direct calculation gives a general solution of (34):

$$f = c_1\exp\left(\Delta\right) + c_2\exp\left(-\Delta\right), \tag{35}$$

where c_1, c_2 are constants and $\Delta = \sqrt{3}x - 9\sqrt{3}t$. Using the Darboux transformation (8), we get a soliton solution of the generalized SK equation (5) ($c_1 = c_2 = 1$):

$$\begin{aligned}\overline{u} &= 3[\tanh(\Delta)]^2 - 2, \\ \overline{v} &= 9\sqrt{3}[\tanh(\Delta)]^3 - 9\sqrt{3}\tanh\left(\Delta\right).\end{aligned} \tag{36}$$

Plots of the solutions are given in Figures 3 and 4.

Case 2. For $\lambda_0 = k^3 - 3k$ $(k \neq 0)$, (34) has a general solution:

$$f = c_1\exp\left(\Delta_1\right) + c_2\exp\left(\Delta_2\right) + c_3\exp\left(\Delta_3\right), \tag{37}$$

where c_j, $(j = 1, 2, 3)$ are constants and

$$\begin{aligned}\Delta_1 &= kx + \left(9k^5 - 45k^3 + 45k\right)t, \\ \Delta_2 &= \frac{-k+\sqrt{12-3k^2}}{2}x \\ &\quad - \frac{9}{2}\left(5k - 5k^3 + k^5 + \sqrt{12-3k^2}\right. \\ &\qquad \left. -3k^2\sqrt{12-3k^2} + k^4\sqrt{12-3k^2}\right)t, \\ \Delta_3 &= \frac{-k-\sqrt{12-3k^2}}{2}x \\ &\quad + \frac{9}{2}\left(-5k + 5k^3 - k^5 + \sqrt{12-3k^2}\right. \\ &\qquad \left. -3k^2\sqrt{12-3k^2} + k^4\sqrt{12-3k^2}\right)t.\end{aligned} \tag{38}$$

Using the Darboux transformation (8), we get an explicit solution of (5):

$$\begin{aligned}\overline{u} &= 1 + \frac{3}{2}\frac{2c_2c_3\left(k^2-4\right)\exp\left(\Delta_2+\Delta_3\right) - c_1c_2\left(2+k^2-k\sqrt{12-3k^2}\right)\exp\left(\Delta_1+\Delta_2\right) - c_1c_3\left(2+k^2+k\sqrt{12-3k^2}\right)\exp\left(\Delta_1+\Delta_3\right)}{\left[c_1\exp\left(\Delta_1\right)+c_2\exp\left(\Delta_2\right)+c_3\exp\left(\Delta_3\right)\right]^2}, \\ \overline{v} &= 3k^3 - 9k + 3\frac{\Delta}{\left[c_1\exp\left(\Delta_1\right)+c_2\exp\left(\Delta_2\right)+c_3\exp\left(\Delta_3\right)\right]^3},\end{aligned} \tag{39}$$

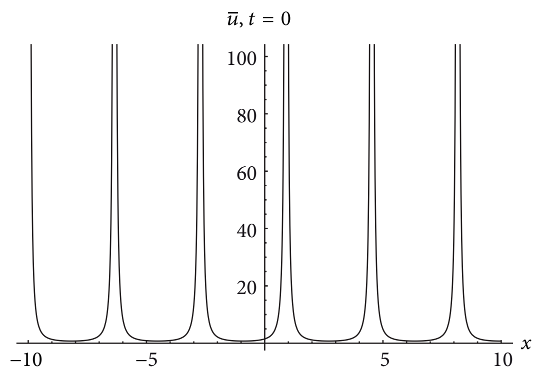

Figure 1: $\overline{u}$.

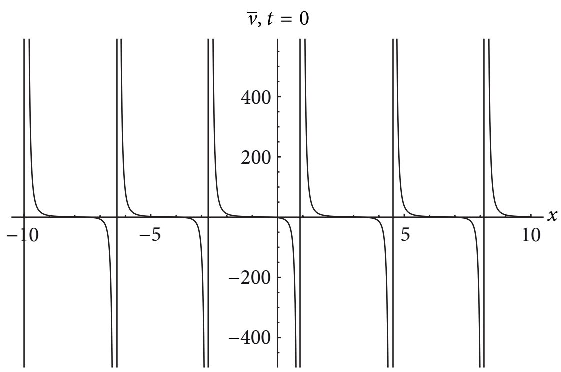

Figure 2: $\overline{v}$.

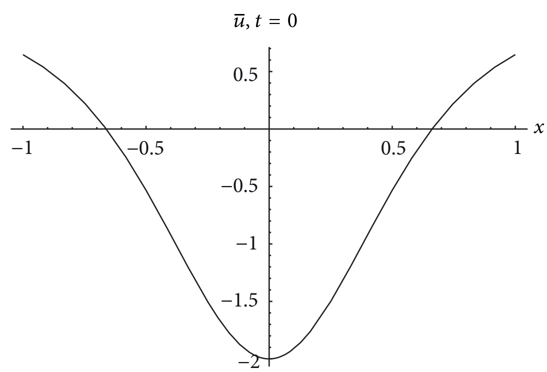

Figure 3: $\overline{u}$.

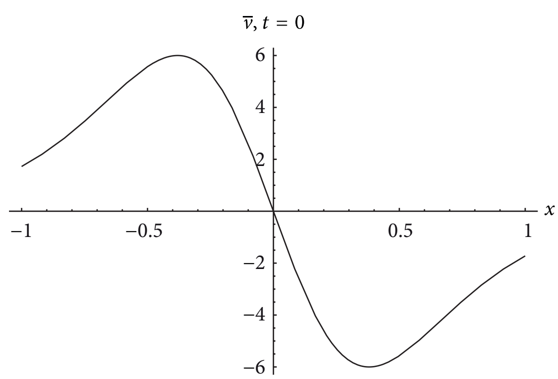

Figure 4: $\overline{v}$.

where

$$\begin{aligned}
\Delta = {} & k\left(3-k^2\right)\left[c_1^3\exp\left(3\Delta_1\right)\right. \\
& \qquad + c_2^3\exp\left(3\Delta_2\right) + c_3^3\exp\left(3\Delta_3\right) \\
& \qquad \left. -12c_1c_2c_3\exp\left(\Delta_1+\Delta_2+\Delta_3\right)\right] \\
& + \frac{3}{2}c_2^2c_3\exp\left(2\Delta_2+\Delta_3\right) \\
& \times\left[2k-k^3-4\sqrt{12-3k^2}+k^2\sqrt{12-3k^2}\right] \\
& + \frac{3}{2}c_2c_3^2\exp\left(\Delta_2+2\Delta_3\right) \\
& \times\left[2k-k^3+4\sqrt{12-3k^2}-k^2\sqrt{12-3k^2}\right] \\
& - \frac{3}{2}c_1^2c_2\exp\left(2\Delta_1+\Delta_2\right)\left(1+k^2\right)\left(k-\sqrt{12-3k^2}\right) \\
& - \frac{3}{2}c_1^2c_3\exp\left(2\Delta_1+\Delta_3\right)\left(1+k^2\right)\left(k+\sqrt{12-3k^2}\right) \\
& + \frac{3}{2}c_1c_3^2\exp\left(\Delta_1+2\Delta_3\right)k\left(8-k^2+k\sqrt{12-3k^2}\right) \\
& + \frac{3}{2}c_1c_2^2\exp\left(\Delta_1+2\Delta_2\right)k\left(8-k^2-k\sqrt{12-3k^2}\right).
\end{aligned} \tag{40}$$

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