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Research Article

Strong Convergence Theorems for Maximal Monotone Operators, Fixed-Point Problems, and Equilibrium Problems

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We present a new iterative method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions to an equilibrium problem, and the set of zeros of the sum of maximal monotone operators and prove the strong convergence theorems in the Hilbert spaces. We also apply our results to variational inequality and optimization problems.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $S : C \to C$ is nonexpansive if $\|Sx - Sy\| \le \|x - y\|$ for all $x, y \in C$. The set of fixed points of S is denoted by $\mathrm{Fix}(S)$. It is well known that $\mathrm{Fix}(S)$ is closed and convex. There are two iterative methods for approximating fixed points of a nonexpansive mapping. One is introduced by Mann in [1] and the other by Halpern in [2]. The iteration procedure of Mann's type for approximating fixed points of a nonexpansive mapping S is the following: $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S x_n, \tag{1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. The iteration procedure of Halpern's type is the following: $u \in C$, $x_1 \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S x_n, \tag{2}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find $\overline{x} \in C$ such that $f(\overline{x}, y) \ge 0$ for all $y \in C$. The set of such solutions is denoted by $\mathrm{EP}(f)$. Numerous problems in physics, optimization, and economics reduce to finding a solution to the equilibrium problem (e.g., see [3]). For solving the equilibrium problem, we assume that the bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$,

(A2) f is monotone, that is, $f(x, y) + f(y, x) \le 0$ for all $x, y \in C$,

(A3) for every $x, y, z \in C$, $\limsup_{t\downarrow 0} f(tz + (1 - t)x, y) \le f(x, y)$,

(A4) $f(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

Equilibrium problems have been studied extensively; see [3–9].

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in H : Bx \ne \emptyset\}$. A multivalued mapping B is said to be monotone if

$$\langle x - y, u - v\rangle \ge 0 \quad \forall x, y \in \mathrm{dom}(B),\ u \in Bx,\ v \in By. \tag{3}$$

A monotone operator B is said to be maximal if its graph is not properly contained in the graph of any other monotone operator. For a maximal monotone operator B on H and $r > 0$, the operator $J_r = (I + rB)^{-1} : H \to \mathrm{dom}(B)$ is called the resolvent of B for r. It is known that J_r is firmly nonexpansive. Given a positive constant α, a mapping $A : C \to H$ is said to be α-inverse strongly monotone if

$$\langle x - y, Ax - Ay\rangle \ge \alpha \|Ax - Ay\|^2 \quad \forall x, y \in C. \tag{4}$$

Some authors have paid more attention to finding an element in the set of zeros of $A+B$. For a mapping A from C into H, we know that $(A+B)^{-1}0 = \mathrm{Fix}(J_\lambda(I-\lambda A))$; see [10]. Takahashi et al. [11] constructed the following iterative sequence. Let $u \in C$, $x_1 = x \in C$, and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J_{\lambda_n}(x_n - \lambda_n A x_n). \tag{5}$$

Under appropriate conditions they proved that the sequence $\{x_n\}$ converges strongly to a point $z_0 \in (A+B)^{-1}0$. Lin and Takahashi [12] introduced an iterative sequence that converges strongly to an element of $(A+B)^{-1}0 \cap F^{-1}0$, where F is another maximal monotone operator. Takahashi et al. [13] presented a new iterative sequence converging strongly to an element of $(A+B)^{-1}0 \cap \mathrm{Fix}(S)$.

Motivated by the above results, in this paper, we introduce a new iterative algorithm for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions to an equilibrium problem, and the set of zeros of the sum of maximal monotone operators and prove the strong convergence theorems in the Hilbert spaces. Finally, we give the applications to the variational inequality and optimization problems.

2. Preliminaries

Throughout this paper, let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$, and let C be a nonempty closed convex subset of H. We write $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Similarly, $x_n \rightharpoonup x$ will mean weak convergence. It is well known that H satisfies Opial's condition; that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \forall y \neq x. \tag{6}$$

For any $x \in H$, there exists a unique point $P_C x \in C$ such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{7}$$

P_C is called the metric projection of H onto C. Note that P_C is a nonexpansive mapping of H onto C. For $x \in H$ and $z \in C$, we have

$$z = P_C x \Longleftrightarrow \langle z - y, x - z \rangle \ge 0 \quad \text{for every } y \in C. \tag{8}$$

For $\overline{\gamma} > 0$, a mapping V on H is called $\overline{\gamma}$-strongly monotone if

$$\langle x - y, Vx - Vy \rangle \ge \overline{\gamma}\|x - y\|^2 \quad \forall x, y \in H. \tag{9}$$

Taking $L > 0$, a mapping T on H is said to be L-Lipschitzian continuous if

$$\|Tx - Ty\| \le L\|x - y\| \quad \forall x, y \in H. \tag{10}$$

It is easy to see that A is $\overline{\gamma}/L^2$-inverse strongly monotone whenever A is $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous. Now we consider inverse strongly monotone. Let $\alpha > 0$, and let $A : C \to H$ be an α-inverse strongly monotone operator. If $0 < \lambda \le 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping. Indeed, for $x, y \in C$ and $0 < \lambda \le 2\alpha$, we get

$$\begin{aligned}
&\|(I-\lambda A)x - (I-\lambda A)y\|^2 \\
&\quad = \|(x-y) - \lambda(Ax - Ay)\|^2 \\
&\quad = \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2\|Ax-Ay\|^2 \\
&\quad \le \|x-y\|^2 - 2\lambda\alpha\|Ax-Ay\|^2 + \lambda^2\|Ax-Ay\|^2 \\
&\quad \le \|x-y\|^2 + \lambda(\lambda - 2\alpha)\|Ax-Ay\|^2 \\
&\quad \le \|x-y\|^2.
\end{aligned} \tag{11}$$

Therefore, the operator $I - \lambda A$ is a nonexpansive mapping of C into H.

We need the following lemmas.

Lemma 1 (see [3]). *Let C be a nonempty closed convex subset of H, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). If $r > 0$ and $x \in H$, then there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0 \quad \forall y \in C. \tag{12}$$

Lemma 2 (see [7]). *Let C be a nonempty closed convex subset of H, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). For $r > 0$, define a mapping $T_r : H \to 2^C$ as follows:*

$$T_r(x) = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0 \ \forall y \in C \right\}. \tag{13}$$

Then the following hold:

(i) *T_r is single valued,*

(ii) *T_r is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\langle x - y, T_r x - T_r y\rangle \ge \|T_r x - T_r y\|^2, \tag{14}$$

(iii) $\mathrm{Fix}(T_r) = \mathrm{EP}(f)$,

(iv) $\mathrm{EP}(f)$ *is closed and convex.*

Lemma 3. *Let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on a real Hilbert space H with $\overline{\gamma}$, $L > 0$ and $2\overline{\gamma} - 1 < L^2 < 2\overline{\gamma}$. Suppose that $\{\beta_n\}$ is a sequence in $(0,1)$. For all $x, y \in H$, one has*

$$\|(I - \beta_n V)x - (I - \beta_n V)y\| \le (1 - \beta_n \tau)\|x - y\|, \tag{15}$$

where $\tau = \overline{\gamma} - L^2/2$.

Proof. Observe that

$$\begin{aligned}
&\|(I-\beta_n V)x-(I-\beta_n V)y\|^2\\
&\quad=\|x-y\|^2+\beta_n^2\|Vx-Vy\|^2-2\beta_n\langle x-y,Vx-Vy\rangle\\
&\quad\le\|x-y\|^2+\beta_n^2L^2\|x-y\|^2-2\beta_n\overline{\gamma}\|x-y\|^2\\
&\quad=\left(1+\beta_n^2L^2-2\beta_n\overline{\gamma}\right)\|x-y\|^2\\
&\quad=\left[1+\beta_n^2L^2-2\beta_n\left(\tau+\frac{L^2}{2}\right)\right]\|x-y\|^2\\
&\quad=\left(1-2\beta_n\tau-\beta_nL^2+\beta_n^2L^2\right)\|x-y\|^2\\
&\quad\le\left(1-2\beta_n\tau+\beta_n^2\tau^2-\beta_nL^2+\beta_n^2L^2\right)\|x-y\|^2\\
&\quad=\left[(1-\beta_n\tau)^2-\beta_n\left(L^2-\beta_nL^2\right)\right]\|x-y\|^2.
\end{aligned}\tag{16}$$

Since the sequence $\{\beta_n\}\subset(0,1)$ and $2\overline{\gamma}-1<L^2<2\overline{\gamma}$, we obtain

$$\|(I-\beta_n V)x-(I-\beta_n V)y\|\le(1-\beta_n\tau)\|x-y\|.\tag{17}$$

□

Lemma 4 (see [8]). *Suppose that (A1)–(A4) hold. If $x,y\in H$ and $r_1,r_2>0$, then*

$$\left\|T_{r_2}y-T_{r_1}x\right\|\le\|y-x\|+\frac{|r_2-r_1|}{r_2}\left\|T_{r_2}y-y\right\|.\tag{18}$$

Lemma 5 (see [13]). *Let H be a real Hilbert space, and let B be a maximal monotone operator on H. Then the following holds:*

$$\frac{s-t}{s}\langle J_sx-J_tx,J_sx-x\rangle\ge\|J_sx-J_tx\|^2\tag{19}$$

for all $s,t>0$ and $x\in H$.

Lemma 6 (see [14, 15]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1}\le(1-\alpha_n)s_n+\alpha_n\beta_n+\gamma_n,\quad n\ge0,\tag{20}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\}\subset[0,1]$, $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(ii) $\limsup_{n\to\infty}\beta_n\le0$,

(iii) $\gamma_n\ge0$, $\sum_{n=1}^{\infty}\gamma_n<\infty$.

Then $\lim_{n\to\infty}s_n=0$.

The following lemma is an immediate consequence of the inner product on H.

Lemma 7. *For all $x,y\in H$, the inequality $\|x+y\|^2\le\|x\|^2+2\langle y,x+y\rangle$ holds.*

Lemma 8 (see [16] (demiclosedness principle)). *Let C be a nonempty closed convex subset of H, $S:C\to H$ a nonexpansive mapping, and x a point in H, the sequence $\{x_n\}$ in C. Suppose that $x_n\rightharpoonup x$ and that $x_n-Sx_n\to0$. Then $x\in\mathrm{Fix}(S)$.*

3. Strong Convergence Theorems

In this section, we present a new iterative method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions to an equilibrium problem, and the set of zeros of the sum of maximal monotone operators.

Theorem 9. *Let C be a nonempty closed convex subset of a real Hilbert space H and A an α-inverse strongly monotone operator of C into H. Let B be a maximal monotone operator on H such that the domain of B is included in C. Let $J_\lambda=(I+\lambda B)^{-1}$ be the resolvent of B for $\lambda>0$, and let S be a nonexpansive mapping of C into itself. Suppose that V is a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H with $\overline{\gamma}$, $L>0$ and $2\overline{\gamma}-1<L^2<2\overline{\gamma}$. Assume that $f:C\times C\to\mathbb{R}$ satisfies (A1)–(A4). Suppose that $(A+B)^{-1}0\cap\mathrm{Fix}(S)\cap\mathrm{EP}(f)\neq\emptyset$. Let $\omega\in C$ and $x_1\in C$, and let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
&u_n\in C,\ \text{such that } f(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge0\quad\forall y\in C,\\
&x_{n+1}=\beta_n\omega+(I-\beta_nV)\times\left[\alpha_nx_n+(1-\alpha_n)J_{\lambda_n}(I-\lambda_nA)Su_n\right],
\end{aligned}\tag{21}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$, and $\{r_n\}$ satisfy the following conditions:

(1) $\{\alpha_n\}\subset(0,1)$ *and* $\sum_{n=1}^{\infty}\alpha_n<\infty$,

(2) $\{\beta_n\}\subset(0,1)$, $\beta_n\to0$, $\sum_{n=1}^{\infty}\beta_n=\infty$, *and* $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$,

(3) $0<a\le\lambda_n\le b<2\alpha$ *and* $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$,

(4) $0<c\le r_n$ *and* $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$.

Then the sequence $\{x_n\}$ converges strongly to an element of $(A+B)^{-1}0\cap\mathrm{Fix}(S)\cap\mathrm{EP}(f)$.

Proof. The proof will be completed by eight steps.

Step 1. Show that the sequences $\{x_n\}$ and $\{u_n\}$ are bounded.

Note that $(A+B)^{-1}0\cap\mathrm{Fix}(S)\cap\mathrm{EP}(f)$ is a closed convex subset of H since $(A+B)^{-1}0$, $\mathrm{Fix}(S)$, and $\mathrm{EP}(f)$ are closed and convex. For simplicity, we write

$$\Omega:=(A+B)^{-1}0\cap\mathrm{Fix}(S)\cap\mathrm{EP}(f).\tag{22}$$

From Lemmas 1 and 2, we have $u_n=T_{r_n}x_n$, and for any $z\in\Omega$,

$$\|u_n-z\|=\left\|T_{r_n}x_n-T_{r_n}z\right\|\le\|x_n-z\|.\tag{23}$$

Set $y_n = \alpha_n x_n + (1-\alpha_n) J_{\lambda_n}(I-\lambda_n A)Su_n$. It follows that

$$\begin{aligned}\|y_n - z\| &= \left\|\alpha_n x_n + (1-\alpha_n) J_{\lambda_n}(I-\lambda_n A)Su_n - z\right\| \\ &= \left\|\alpha_n (x_n - z) + (1-\alpha_n)\left[J_{\lambda_n}(I-\lambda_n A)Su_n - z\right]\right\| \\ &\le \alpha_n \|x_n - z\| + (1-\alpha_n)\|x_n - z\| \\ &= \|x_n - z\|.\end{aligned} \tag{24}$$

Lemma 3 implies that

$$\begin{aligned}\|x_{n+1} - z\| &= \big\|\beta_n \omega + (I-\beta_n V) \\ &\quad \times \left[\alpha_n x_n + (1-\alpha_n) J_{\lambda_n}(I-\lambda_n A)Su_n\right] - z\big\| \\ &= \|\beta_n(\omega - Vz) + (I-\beta_n V)y_n - (I-\beta_n V)z\| \\ &\le \beta_n\|\omega - Vz\| + \|(I-\beta_n V)y_n - (I-\beta_n V)z\| \\ &\le \beta_n\|\omega - Vz\| + (1-\beta_n\tau)\|y_n - z\| \\ &\le \beta_n\|\omega - Vz\| + (1-\beta_n\tau)\|x_n - z\| \\ &\le \beta_n\tau\frac{\|\omega - Vz\|}{\tau} + (1-\beta_n\tau)\|x_n - z\| \\ &\le \max\left\{\|x_n - z\|, \frac{\|\omega - Vz\|}{\tau}\right\}, \quad \text{where } \tau = \overline{\gamma} - \frac{L^2}{2}.\end{aligned} \tag{25}$$

From a simple inductive process, it follows that

$$\|x_{n+1} - z\| \le \max\left\{\|x_1 - z\|, \frac{\|\omega - Vz\|}{\tau}\right\}, \tag{26}$$

which yields that $\{x_n\}$ is bounded, so is the sequence $\{u_n\}$.

Step 2. Show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Since

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| &= \big\|\beta_{n+1}\omega + (I-\beta_{n+1}V) \\ &\quad \times \left[\alpha_{n+1}x_{n+1} + (1-\alpha_{n+1}) J_{\lambda_{n+1}}(I-\lambda_{n+1}A)Su_{n+1}\right] \\ &\quad - \beta_n\omega - (I-\beta_n V) \\ &\quad \times \left[\alpha_n x_n + (1-\alpha_n) J_{\lambda_n}(I-\lambda_n A)Su_n\right]\big\| \\ &= \|(\beta_{n+1}-\beta_n)\omega + (I-\beta_{n+1}V)y_{n+1} \\ &\quad - (I-\beta_{n+1}V)y_n + (I-\beta_{n+1}V)y_n - (I-\beta_n V)y_n\| \\ &\le |\beta_{n+1}-\beta_n|\,\|\omega\| + (1-\beta_{n+1}\tau)\|y_{n+1} - y_n\| \\ &\quad + |\beta_{n+1}-\beta_n|\,\|Vy_n\| \le |\beta_{n+1}-\beta_n|(\|\omega\| + \|Vy_n\|) \\ &\quad + (1-\beta_{n+1}\tau) \\ &\quad \times \big\|\alpha_{n+1}x_{n+1} + (1-\alpha_{n+1}) J_{\lambda_{n+1}}(I-\lambda_{n+1}A)Su_{n+1} \\ &\qquad - \alpha_n x_n - (1-\alpha_n) J_{\lambda_n}(I-\lambda_n A)Su_n\big\| \\ &\le |\beta_{n+1}-\beta_n|(\|\omega\| + \|Vy_n\|) + (1-\beta_{n+1}\tau) \\ &\quad \times \big[\alpha_{n+1}\|x_{n+1} - J_{\lambda_{n+1}}(I-\lambda_{n+1}A)Su_{n+1}\| \\ &\qquad + \alpha_n\|x_n - J_{\lambda_n}(I-\lambda_n A)Su_n\| \\ &\qquad + \|J_{\lambda_{n+1}}(I-\lambda_{n+1}A)Su_{n+1} - J_{\lambda_n}(I-\lambda_n A)Su_n\|\big] \\ &\le |\beta_{n+1}-\beta_n|(\|\omega\| + \|Vy_n\|) + (1-\beta_{n+1}\tau) \\ &\quad \times \big[\alpha_{n+1}\|x_{n+1} - J_{\lambda_{n+1}}(I-\lambda_{n+1}A)Su_{n+1}\| \\ &\qquad + \alpha_n\|x_n - J_{\lambda_n}(I-\lambda_n A)Su_n\| \\ &\qquad + \big\|J_{\lambda_{n+1}}(I-\lambda_{n+1}A)Su_{n+1} \\ &\qquad\quad - J_{\lambda_{n+1}}(I-\lambda_{n+1}A)Su_n + J_{\lambda_{n+1}}(I-\lambda_{n+1}A)Su_n \\ &\qquad\quad - J_{\lambda_{n+1}}(I-\lambda_n A)Su_n + J_{\lambda_{n+1}}(I-\lambda_n A)Su_n \\ &\qquad\quad - J_{\lambda_n}(I-\lambda_n A)Su_n\big\|\big] \\ &\le |\beta_{n+1}-\beta_n|(\|\omega\| + \|Vy_n\|) + (1-\beta_{n+1}\tau) \\ &\quad \times \big[\alpha_{n+1}\|x_{n+1} - J_{\lambda_{n+1}}(I-\lambda_{n+1}A)Su_{n+1}\| \\ &\qquad + \alpha_n\|x_n - J_{\lambda_n}(I-\lambda_n A)Su_n\| \\ &\qquad + \|u_{n+1} - u_n\| + |\lambda_{n+1} - \lambda_n|\,\|ASu_n\| \\ &\qquad + \|J_{\lambda_{n+1}}(I-\lambda_n A)Su_n - J_{\lambda_n}(I-\lambda_n A)Su_n\|\big],\end{aligned} \tag{27}$$

it follows from Lemmas 4 and 5 that

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| &\le |\beta_{n+1}-\beta_n|(\|\omega\| + \|Vy_n\|) + (1-\beta_{n+1}\tau) \\ &\quad \times \Bigg[\alpha_{n+1}\|x_{n+1} - J_{\lambda_{n+1}}(I-\lambda_{n+1}A)Su_{n+1}\| \\ &\qquad + \alpha_n\|x_n - J_{\lambda_n}(I-\lambda_n A)Su_n\| \\ &\qquad + |\lambda_{n+1} - \lambda_n|\,\|ASu_n\| + \|x_{n+1} - x_n\| \\ &\qquad + \frac{|r_{n+1} - r_n|}{r_{n+1}}\|u_{n+1} - x_{n+1}\| + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \\ &\qquad \times \|J_{\lambda_{n+1}}(I-\lambda_n A)Su_n - (I-\lambda_n A)Su_n\|\Bigg]\end{aligned}$$

$$\begin{aligned}
&\le |\beta_{n+1}-\beta_n|\left(\|\omega\|+\|Vy_n\|\right)+\left(1-\beta_{n+1}\tau\right)\\
&\quad\times\Bigg[\alpha_{n+1}\left\|x_{n+1}-J_{\lambda_{n+1}}\left(I-\lambda_{n+1}A\right)Su_{n+1}\right\|\\
&\qquad+\alpha_n\left\|x_n-J_{\lambda_n}\left(I-\lambda_nA\right)Su_n\right\|\\
&\qquad+|\lambda_{n+1}-\lambda_n|\,\|ASu_n\|+\|x_{n+1}-x_n\|\\
&\qquad+\frac{|r_{n+1}-r_n|}{c}\|u_{n+1}-x_{n+1}\|+\frac{|\lambda_{n+1}-\lambda_n|}{a}\\
&\qquad\times\left\|J_{\lambda_{n+1}}\left(I-\lambda_nA\right)Su_n-\left(I-\lambda_nA\right)Su_n\right\|\Bigg]\\
&\le\left(1-\beta_{n+1}\tau\right)\|x_{n+1}-x_n\|\\
&\quad+\alpha_{n+1}\left\|x_{n+1}-J_{\lambda_{n+1}}\left(I-\lambda_{n+1}A\right)Su_{n+1}\right\|\\
&\quad+\alpha_n\left\|x_n-J_{\lambda_n}\left(I-\lambda_nA\right)Su_n\right\|+|\lambda_{n+1}-\lambda_n|\,\|ASu_n\|\\
&\quad+\frac{|r_{n+1}-r_n|}{c}\|u_{n+1}-x_{n+1}\|\\
&\quad+\frac{|\lambda_{n+1}-\lambda_n|}{a}\left\|J_{\lambda_{n+1}}\left(I-\lambda_nA\right)Su_n-\left(I-\lambda_nA\right)Su_n\right\|\\
&\quad+|\beta_{n+1}-\beta_n|\left(\|\omega\|+\|Vy_n\|\right).
\end{aligned}\tag{28}$$

Set $M=\sup_{n\in\mathbb{N}}\{\|x_n-J_{\lambda_n}(I-\lambda_nA)Su_n\|, \|ASu_n\|, \|u_{n+1}-x_{n+1}\|, \|J_{\lambda_{n+1}}(I-\lambda_nA)Su_n-(I-\lambda_nA)Su_n\|, (\|\omega\|+\|Vy_n\|)\}$. We have

$$\begin{aligned}
&\|x_{n+2}-x_{n+1}\|\\
&\quad\le\left(1-\beta_{n+1}\tau\right)\|x_{n+1}-x_n\|\\
&\qquad+M\Bigg(\alpha_{n+1}+\alpha_n+|\lambda_{n+1}-\lambda_n|+\frac{|r_{n+1}-r_n|}{c}\\
&\qquad\qquad+\frac{|\lambda_{n+1}-\lambda_n|}{a}+|\beta_{n+1}-\beta_n|\Bigg).
\end{aligned}\tag{29}$$

By the assumptions $\sum_{n=1}^{\infty}\beta_n=\infty$, $\sum_{n=1}^{\infty}\alpha_n<\infty$, $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$, and $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$, it follows from Lemma 6 that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{30}$$

Step 3. Show that $\lim_{n\to\infty}\|x_n-u_n\|=0$.

For any $z\in\Omega$, we have

$$\begin{aligned}
&\|u_n-z\|^2\\
&\quad=\left\|T_{r_n}x_n-T_{r_n}z\right\|^2\le\langle x_n-z,u_n-z\rangle\\
&\quad=\frac{1}{2}\left[\|x_n-z\|^2+\|u_n-z\|^2-\|x_n-u_n\|^2\right],
\end{aligned}\tag{31}$$

which implies that

$$\|u_n-z\|^2\le\|x_n-z\|^2-\|x_n-u_n\|^2.\tag{32}$$

With the help of Lemma 7, we get

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
&\quad=\Big\|\beta_n(\omega-Vz)+(I-\beta_nV)\\
&\qquad\times\left[\alpha_nx_n+(1-\alpha_n)J_{\lambda_n}(I-\lambda_nA)Su_n\right]\\
&\qquad-(I-\beta_nV)z\Big\|^2\\
&\quad\le\Big\|(I-\beta_nV)\left[\alpha_nx_n+(1-\alpha_n)J_{\lambda_n}(I-\lambda_nA)Su_n\right]\\
&\qquad-(I-\beta_nV)z\Big\|^2+2\langle\beta_n(\omega-Vz),x_{n+1}-z\rangle\\
&\quad\le(1-\beta_n\tau)\Big\|\alpha_n(x_n-z)+(1-\alpha_n)\\
&\qquad\qquad\times(J_{\lambda_n}(I-\lambda_nA)Su_n-z)\Big\|^2\\
&\qquad+2\beta_n\langle\omega-Vz,x_{n+1}-z\rangle.
\end{aligned}\tag{33}$$

Consequently,

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
&\quad\le(1-\beta_n\tau)\left[\alpha_n\|x_n-z\|^2+(1-\alpha_n)\|u_n-z\|^2\right]\\
&\qquad+2\beta_n\langle\omega-Vz,x_{n+1}-z\rangle\\
&\quad\le(1-\beta_n\tau)\Big[\alpha_n\|x_n-z\|^2\\
&\qquad\qquad+(1-\alpha_n)\left(\|x_n-z\|^2-\|x_n-u_n\|^2\right)\Big]\\
&\qquad+2\beta_n\langle\omega-Vz,x_{n+1}-z\rangle\\
&\quad\le(1-\beta_n\tau)\left[\|x_n-z\|^2-(1-\alpha_n)\|x_n-u_n\|^2\right]\\
&\qquad+2\beta_n\langle\omega-Vz,x_{n+1}-z\rangle\\
&\quad\le\|x_n-z\|^2-(1-\beta_n\tau)(1-\alpha_n)\|x_n-u_n\|^2\\
&\qquad+2\beta_n\langle\omega-Vz,x_{n+1}-z\rangle.
\end{aligned}\tag{34}$$

Hence,

$$\begin{aligned}
&(1-\beta_n\tau)(1-\alpha_n)\|x_n-u_n\|^2\\
&\quad\le\|x_n-z\|^2-\|x_{n+1}-z\|^2\\
&\qquad+2\beta_n\langle\omega-Vz,x_{n+1}-z\rangle\\
&\quad\le\|x_n-x_{n+1}\|\left(\|x_n-z\|+\|x_{n+1}-z\|\right)\\
&\qquad+2\beta_n\|\omega-Vz\|\,\|x_{n+1}-z\|.
\end{aligned}\tag{35}$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\beta_n=0$, we get

$$\lim_{n\to\infty}\|x_n-u_n\|=0.\tag{36}$$

Step 4. Show that $\lim_{n\to\infty}\|ASu_n - Az\| = 0$, for all $z \in \Omega$.

For $z \in \Omega$, we get

$$\begin{aligned}
&\left\|J_{\lambda_n}(I-\lambda_n A)Su_n - z\right\|^2 \\
&\quad = \left\|J_{\lambda_n}(I-\lambda_n A)Su_n - J_{\lambda_n}(I-\lambda_n A)Sz\right\|^2 \\
&\quad \le \left\|(Su_n - Sz) - \lambda_n\left(ASu_n - ASz\right)\right\|^2 \\
&\quad \le \left\|u_n - z\right\|^2 - 2\lambda_n\left\langle Su_n - Sz, ASu_n - ASz\right\rangle \\
&\qquad + \lambda_n^2\left\|ASu_n - ASz\right\|^2 \\
&\quad \le \left\|x_n - z\right\|^2 - 2\lambda_n\alpha\left\|ASu_n - ASz\right\|^2 \\
&\qquad + \lambda_n^2\left\|ASu_n - ASz\right\|^2 \\
&\quad \le \left\|x_n - z\right\|^2 + \lambda_n\left(\lambda_n - 2\alpha\right)\left\|ASu_n - ASz\right\|^2.
\end{aligned} \tag{37}$$

This together with (33) deduces that

$$\begin{aligned}
&\left\|x_{n+1} - z\right\|^2 \\
&\quad \le \left(1-\beta_n\tau\right) \\
&\qquad \times \left[\alpha_n\left\|x_n - z\right\|^2 + \left(1-\alpha_n\right)\right. \\
&\qquad\qquad \left.\times\left(\left\|x_n - z\right\|^2 + \lambda_n\left(\lambda_n - 2\alpha\right)\left\|ASu_n - ASz\right\|^2\right)\right] \\
&\qquad + 2\beta_n\langle \omega - Vz, x_{n+1} - z\rangle \\
&\quad \le \left(1-\beta_n\tau\right) \\
&\qquad \times \left[\left\|x_n - z\right\|^2 + \left(1-\alpha_n\right)\lambda_n\left(\lambda_n - 2\alpha\right)\left\|ASu_n - ASz\right\|^2\right] \\
&\qquad + 2\beta_n\langle \omega - Vz, x_{n+1} - z\rangle \\
&\quad \le \| x_n - z\|^2 + \left(1-\beta_n\tau\right)\left(1-\alpha_n\right)\lambda_n\left(\lambda_n - 2\alpha\right) \\
&\qquad \times \left\|ASu_n - Az\right\|^2 + 2\beta_n\langle \omega - Vz, x_{n+1} - z\rangle.
\end{aligned} \tag{38}$$

Thus,

$$\begin{aligned}
&\left(1-\beta_n\tau\right)\left(1-\alpha_n\right)\lambda_n\left(2\alpha - \lambda_n\right)\left\|ASu_n - ASz\right\|^2 \\
&\quad \le \left\|x_n - z\right\|^2 - \left\|x_{n+1} - z\right\|^2 \\
&\qquad + 2\beta_n\left\|\omega - Vz\right\|\left\|x_{n+1} - z\right\| \\
&\quad \le \left\|x_n - x_{n+1}\right\|\left(\left\|x_n - z\right\| + \left\|x_{n+1} - z\right\|\right) \\
&\qquad + 2\beta_n\left\|\omega - Vz\right\|\left\|x_{n+1} - z\right\|.
\end{aligned} \tag{39}$$

Since $0 < a \le \lambda_n \le b < 2\alpha$ and $\sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$, the sequence $\{\lambda_n\}$ is a Cauchy sequence. Assume that $\lambda_n \to \lambda_0 \in [a,b]$. It follows that

$$\lim_{n\to\infty}\left\|ASu_n - Az\right\| = 0. \tag{40}$$

Step 5. Show that $\lim_{n\to\infty}\|J_{\lambda_n}(I-\lambda_n A)Su_n - Su_n\| = 0$.

Set $h_n = J_{\lambda_n}(I-\lambda_n A)Su_n$. For $z \in \Omega$, we have

$$\begin{aligned}
&\left\|h_n - z\right\|^2 \\
&\quad = \left\|J_{\lambda_n}\left(I-\lambda_n A\right)Su_n - J_{\lambda_n}\left(I-\lambda_n A\right)Sz\right\|^2 \\
&\quad \le \left\langle\left(I-\lambda_n A\right)Su_n - \left(I-\lambda_n A\right)Sz, h_n - z\right\rangle \\
&\quad = \frac{1}{2}\left[\left\|\left(I-\lambda_n A\right)Su_n - \left(I-\lambda_n A\right)Sz\right\|^2 + \left\|h_n - z\right\|^2\right. \\
&\qquad \left. - \left\|\left(I-\lambda_n A\right)Su_n - \left(I-\lambda_n A\right)Sz - \left(h_n - z\right)\right\|^2\right] \\
&\quad \le \frac{1}{2}\left[\left\|u_n - z\right\|^2 + \left\|h_n - z\right\|^2\right. \\
&\qquad \left. - \left\|\left(Su_n - h_n\right) - \lambda_n\left(ASu_n - ASz\right)\right\|^2\right].
\end{aligned} \tag{41}$$

Therefore,

$$\begin{aligned}
&\left\|h_n - z\right\|^2 \\
&\quad \le \left\|u_n - z\right\|^2 - \left\|\left(Su_n - h_n\right) - \lambda_n\left(ASu_n - ASz\right)\right\|^2 \\
&\quad \le \left\|x_n - z\right\|^2 - \left\|Su_n - h_n\right\|^2 - \lambda_n^2\left\|ASu_n - ASz\right\|^2 \\
&\qquad + 2\lambda_n\left\langle Su_n - h_n, ASu_n - ASz\right\rangle.
\end{aligned} \tag{42}$$

Using (33) again, we obtain that

$$\begin{aligned}
&\left\|x_{n+1} - z\right\|^2 \\
&\quad \le \left(1-\beta_n\tau\right) \\
&\qquad \times \left[\alpha_n\left\|x_n - z\right\|^2 + \left(1-\alpha_n\right)\left\|J_{\lambda_n}\left(I-\lambda_n A\right)Su_n - z\right\|^2\right] \\
&\qquad + 2\beta_n\left\langle \omega - Vz, x_{n+1} - z\right\rangle \\
&\quad \le \left(1-\beta_n\tau\right)\left[\alpha_n\left\|x_n - z\right\|^2 + \left(1-\alpha_n\right)\right. \\
&\qquad\qquad \times\left(\left\|x_n - z\right\|^2 - \left\|Su_n - h_n\right\|^2\right. \\
&\qquad\qquad\quad - \lambda_n^2\left\|ASu_n - ASz\right\|^2 \\
&\qquad\qquad\quad \left.\left. + 2\lambda_n\left\langle Su_n - h_n, ASu_n - ASz\right\rangle\right)\right] \\
&\qquad + 2\beta_n\left\langle \omega - Vz, x_{n+1} - z\right\rangle \\
&\quad \le \left(1-\beta_n\tau\right)\left[\left\|x_n - z\right\|^2 - \left(1-\alpha_n\right)\left\|Su_n - h_n\right\|^2\right. \\
&\qquad\qquad - \left(1-\alpha_n\right)\lambda_n^2\left\|ASu_n - ASz\right\|^2 \\
&\qquad\qquad \left. + 2\left(1-\alpha_n\right)\lambda_n\left\langle Su_n - h_n, ASu_n - ASz\right\rangle\right] \\
&\qquad + 2\beta_n\left\langle \omega - Vz, x_{n+1} - z\right\rangle
\end{aligned}$$

$$\begin{aligned}&\leq \|x_n - z\|^2 - (1-\beta_n\tau)(1-\alpha_n)\|Su_n - h_n\|^2\\&\quad + 2\lambda_n(1-\alpha_n)\|Su_n - h_n\|\|ASu_n - ASz\|\\&\quad + 2\beta_n\|\omega - Vz\|\|x_{n+1} - z\|.\end{aligned} \tag{43}$$

Thus,

$$\begin{aligned}&(1-\beta_n\tau)(1-\alpha_n)\|Su_n - h_n\|^2\\&\quad\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2\\&\qquad + 2\lambda_n\|Su_n - h_n\|\|ASu_n - ASz\|\\&\qquad + 2\beta_n\|\omega - Vz\|\|x_{n+1} - z\|\\&\quad\leq \|x_n - x_{n+1}\|(\|x_n - z\| + \|x_{n+1} - z\|)\\&\qquad + 2\lambda_n\|Su_n - h_n\|\|ASu_n - Az\|\\&\qquad + 2\beta_n\|\omega - Vz\|\|x_{n+1} - z\|.\end{aligned} \tag{44}$$

It follows from (30), (40), and $\lim_{n\to\infty}\beta_n = 0$ that

$$\lim_{n\to\infty}\|J_{\lambda_n}(I-\lambda_n A)Su_n - Su_n\| = 0. \tag{45}$$

Step 6. Show that $\lim_{n\to\infty}\|u_n - Su_n\| = 0$.

Since

$$\begin{aligned}&\|x_{n+1} - Su_n\|\\&\quad= \|\beta_n(\omega - VSu_n) + (I-\beta_n V)\\&\qquad\times[\alpha_n x_n + (1-\alpha_n)J_{\lambda_n}(I-\lambda_n A)Su_n]\\&\qquad - (I-\beta_n V)Su_n\|\\&\quad\leq \beta_n\|\omega - VSu_n\| + (1-\beta_n\tau)\\&\qquad\times\|\alpha_n x_n + (1-\alpha_n)J_{\lambda_n}(I-\lambda_n A)Su_n - Su_n\|\\&\quad\leq \beta_n\|\omega - VSu_n\| + (1-\beta_n\tau)\\&\qquad\times[\alpha_n\|x_n - Su_n\| + (1-\alpha_n)\\&\qquad\times\|J_{\lambda_n}(I-\lambda_n A)Su_n - Su_n\|],\end{aligned} \tag{46}$$

equality (45) implies that

$$\lim_{n\to\infty}\|x_{n+1} - Su_n\| = 0. \tag{47}$$

As

$$\|u_n - Su_n\| \leq \|u_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - Su_n\|, \tag{48}$$

it follows from (30), (36), and (47) that

$$\lim_{n\to\infty}\|u_n - Su_n\| = 0. \tag{49}$$

Step 7. Show that $\limsup_{n\to\infty}\langle\omega - Vz_0, x_{n+1} - z_0\rangle \leq 0$, where $z_0 = P_\Omega[\omega + (I-V)z_0]$.

Observe that the mapping $x \mapsto P_\Omega[\omega + (I-V)x]$ is a contraction. Indeed, for any $x, y \in H$,

$$\begin{aligned}&\|P_\Omega[\omega + (I-V)x] - P_\Omega[\omega + (I-V)y]\|^2\\&\quad\leq \|(I-V)x - (I-V)y\|^2\\&\quad= \|(x-y) - (Vx - Vy)\|^2\\&\quad= \|x-y\|^2 - 2\langle x-y, Vx - Vy\rangle + \|Vx - Vy\|^2\\&\quad\leq \|x-y\|^2 - 2\overline{\gamma}\|x-y\|^2 + L^2\|x-y\|^2\\&\quad= (1 - 2\overline{\gamma} + L^2)\|x-y\|^2.\end{aligned} \tag{50}$$

As $2\overline{\gamma} - 1 < L^2 < 2\overline{\gamma}$, we have $0 < 1 - 2\overline{\gamma} + L^2 < 1$. The Banach contraction mapping principle guarantees that the mapping $x \to P_\Omega[\omega + (I-V)x]$ has a unique fixed point z_0; that is, $z_0 = P_\Omega[\omega + (I-V)z_0]$.

In order to show this inequality, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle\omega - Vz_0, x_{n+1} - z_0\rangle = \lim_{i\to\infty}\langle\omega - Vz_0, x_{n_i} - z_0\rangle. \tag{51}$$

In view of the boundedness of $\{x_{n_i}\}$, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \rightharpoonup p$. Without loss of generality, we assume that $x_{n_i} \rightharpoonup p$. It follows from (36) that $u_{n_i} \rightharpoonup p$. Since $\{u_{n_i}\} \subset C$ and C is closed and convex, we get $p \in C$. Now we show that $p \in \Omega$.

First we prove that $p \in \mathrm{EP}(f)$. By (21),

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \forall y \in C. \tag{52}$$

The monotonicity of f implies that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq f(y, u_n) \quad \forall y \in C. \tag{53}$$

Replacing n by n_i, we obtain

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq f(y, u_{n_i}) \quad \forall y \in C. \tag{54}$$

Applying (36) and (A4), we have

$$f(y, p) \leq 0 \quad \forall y \in C. \tag{55}$$

For $0 < t \leq 1$, $y \in C$, set $y_t = ty + (1-t)p$. Then $y_t \in C$ and $f(y_t, p) \leq 0$. Thus,

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y). \tag{56}$$

Dividing by t, we see that

$$f(y_t, y) \geq 0. \tag{57}$$

Letting $t \downarrow 0$, we get

$$f(p, y) \geq 0 \quad \forall y \in C. \tag{58}$$

That is, $p \in \mathrm{EP}(f)$.

Now we prove that $p \in \mathrm{Fix}(S)$. Otherwise, assume that $p \notin \mathrm{Fix}(S)$, that is, $p \neq Sp$. Opial's condition and (49) imply that

$$\begin{aligned}\liminf_{i\to\infty}\left\|u_{n_i}-p\right\| &< \liminf_{i\to\infty}\left\|u_{n_i}-Sp\right\| \\ &= \liminf_{i\to\infty}\left\|u_{n_i}-Su_{n_i}+Su_{n_i}-Sp\right\| \\ &= \liminf_{i\to\infty}\left\|Su_{n_i}-Sp\right\| \\ &\leq \liminf_{i\to\infty}\left\|u_{n_i}-p\right\|.\end{aligned} \tag{59}$$

This is a contradiction. Thus, $p \in \mathrm{Fix}(S)$.

Next we will show that $p \in (A+B)^{-1}0$.

In fact, let $\lambda_n \to \lambda_0 \in [a, b]$, and let $v_n = Su_n$. It follows from Lemma 5 that

$$\begin{aligned}&\left\|J_{\lambda_0}(I-\lambda_0 A)v_n - J_{\lambda_n}(I-\lambda_n A)v_n\right\| \\ &\quad = \big\|J_{\lambda_0}(I-\lambda_0 A)v_n - J_{\lambda_0}(I-\lambda_n A)v_n \\ &\qquad + J_{\lambda_0}(I-\lambda_n A)v_n - J_{\lambda_n}(I-\lambda_n A)v_n\big\| \\ &\quad \leq \left|\lambda_n-\lambda_0\right|\left\|Av_n\right\| + \frac{\left|\lambda_0-\lambda_n\right|}{\lambda_0} \\ &\qquad \times \left\|J_{\lambda_0}(I-\lambda_n A)v_n - (I-\lambda_n A)v_n\right\|.\end{aligned} \tag{60}$$

Thus,

$$\lim_{n\to\infty}\left\|J_{\lambda_0}(I-\lambda_0 A)v_n - J_{\lambda_n}(I-\lambda_n A)v_n\right\| = 0. \tag{61}$$

Since

$$\begin{aligned}&\left\|v_n - J_{\lambda_0}(I-\lambda_0 A)v_n\right\| \\ &\quad \leq \left\|v_n - J_{\lambda_n}(I-\lambda_n A)v_n\right\| \\ &\qquad + \left\|J_{\lambda_n}(I-\lambda_n A)v_n - J_{\lambda_0}(I-\lambda_0 A)v_n\right\|,\end{aligned} \tag{62}$$

equalities (45) and (61) imply that

$$\lim_{n\to\infty}\left\|v_n - J_{\lambda_0}(I-\lambda_0 A)v_n\right\| = 0. \tag{63}$$

Therefore,

$$\lim_{i\to\infty}\left\|v_{n_i} - J_{\lambda_0}(I-\lambda_0 A)v_{n_i}\right\| = 0. \tag{64}$$

It follows from $u_{n_i} \rightharpoonup p$ and (49) that $v_{n_i} \rightharpoonup p$. As $J_{\lambda_0}(I-\lambda_0 A)$ is nonexpansive, Lemma 8 implies that $p = J_{\lambda_0}(I-\lambda_0 A)p$. That is, $p \in (A+B)^{-1}0$. Hence, $p \in \Omega$. By (51) and the property of metric projection, we have

$$\begin{aligned}&\limsup_{n\to\infty}\left\langle \omega - Vz_0, x_{n+1}-z_0\right\rangle \\ &\quad = \lim_{i\to\infty}\left\langle \omega - Vz_0, x_{n_i}-z_0\right\rangle \\ &\quad = \left\langle \omega - Vz_0, p-z_0\right\rangle \\ &\quad = \left\langle \left[\omega + (I-V)z_0\right] - z_0, p-z_0\right\rangle \leq 0.\end{aligned} \tag{65}$$

Step 8. Show that $x_n \to z_0$, where $z_0 = P_\Omega[\omega + (I-V)z_0]$.

According to (21), we get

$$\begin{aligned}&\left\|x_{n+1}-z_0\right\|^2 \\ &\quad = \big\|\beta_n(\omega - Vz_0) + (I-\beta_n V) \\ &\qquad \times \left[\alpha_n x_n + (1-\alpha_n)J_{\lambda_n}(I-\lambda_n A)Su_n\right] \\ &\qquad - (I-\beta_n V)z_0\big\|^2 \\ &\quad \leq (1-\beta_n\tau)\big\|\alpha_n(x_n-z_0) \\ &\qquad + (1-\alpha_n)\left(J_{\lambda_n}(I-\lambda_n A)Su_n - z_0\right)\big\|^2 \\ &\qquad + 2\beta_n\left\langle \omega - Vz_0, x_{n+1}-z_0\right\rangle \\ &\quad \leq (1-\beta_n\tau)\left[\alpha_n\left\|x_n-z_0\right\|^2 \right. \\ &\qquad \left. + (1-\alpha_n)\left\|x_n-z_0\right\|^2\right] \\ &\qquad + 2\beta_n\langle \omega - Vz_0, x_{n+1}-z_0\rangle \\ &\quad \leq (1-\beta_n\tau)\left\|x_n-z_0\right\|^2 + 2\beta_n\langle \omega - Vz_0, x_{n+1}-z_0\rangle.\end{aligned} \tag{66}$$

It follows from (65) and Lemma 6 that $\{x_n\}$ converges strongly to $z_0 \in \Omega$. □

Remark 10. By an examination of the proof of Theorem 9, the conclusion still holds in the case that $\alpha_n \equiv 0$.

Remark 11. Consider the following quadratic optimization problem:

$$\min_{x\in H}\frac{1}{2}\langle Vx, x\rangle - \langle x, \omega\rangle, \tag{67}$$

where H is a real Hilbert space, V is a self-adjoint bounded linear operator on H such that

$$\langle Vx, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H \text{ and some } \overline{\gamma} > 0. \tag{68}$$

Letting $A = 0$, $B = \partial_{\delta_C}$ (i.e., the subdifferential of the indicator function of C), $C = H$, $f(x, y) \equiv 0$, and $\alpha_n \equiv 0$, algorithm (21) reduces to

$$x_{n+1} = \beta_n\omega + (I-\beta_n V)x_n. \tag{69}$$

Xu [17] showed that the sequence in algorithm (69) converges strongly to the solution of problem (67).

Remark 12. Consider the setting of Theorem 9 with $f(x, y) \equiv 0$, $\alpha_n \equiv 0$, and $V = S = I$. Then algorithm (21) corresponds to the algorithm in [11, Theorem 9].

The corollaries below are the direct consequences of Theorem 9.

Corollary 13. *Let C be a nonempty closed convex subset of a real Hilbert space H and A an α-inverse strongly monotone operator of C into H. Let B be a maximal monotone operator on H such that the domain of B is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$, and let S be a nonexpansive mapping of C into itself. Suppose that V is a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H with $\overline{\gamma}$, $L > 0$ and $2\overline{\gamma} - 1 < L^2 < 2\overline{\gamma}$. Suppose that $(A + B)^{-1}0 \cap \mathrm{Fix}(S) \neq \emptyset$. Let $\omega \in C$ and $x_1 \in C$, and let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} u_n &= P_C x_n, \\ x_{n+1} &= \beta_n \omega + (I - \beta_n V)\left[\alpha_n x_n + (1 - \alpha_n) J_{\lambda_n}(I - \lambda_n A) S u_n\right], \end{aligned} \tag{70}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ satisfy the following conditions:

(1) $\{\alpha_n\} \subset (0, 1)$ *and* $\sum_{n=1}^{\infty} \alpha_n < \infty$,
(2) $\{\beta_n\} \subset (0, 1)$, $\beta_n \to 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
(3) $0 < a \leq \lambda_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to an element of $(A + B)^{-1}0 \cap \mathrm{Fix}(S)$.

Proof. Letting $f(x, y) \equiv 0$ for all $x, y \in C$ and $r_n = 1$ in Theorem 9, we get the result. □

Corollary 14. *Let C be a nonempty closed convex subset of a real Hilbert space H and A an α-inverse strongly monotone operator of C into H. Let B be a maximal monotone operator on H such that the domain of B is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Assume that $f : C \times C \to \mathbb{R}$ satisfies (A1)–(A4). Suppose that $(A + B)^{-1}0 \cap \mathrm{EP}(f) \neq \emptyset$. Let $\omega \in C$ and $x_1 \in C$, and let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} &u_n \in C, \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \forall y \in C, \\ &x_{n+1} = \beta_n \omega + (1 - \beta_n)\left[\alpha_n x_n + (1 - \alpha_n) J_{\lambda_n}(I - \lambda_n A) u_n\right], \end{aligned} \tag{71}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$, and $\{r_n\}$ satisfy the following conditions:

(1) $\{\alpha_n\} \subset (0, 1)$ *and* $\sum_{n=1}^{\infty} \alpha_n < \infty$,
(2) $\{\beta_n\} \subset (0, 1)$, $\beta_n \to 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
(3) $0 < a \leq \lambda_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,
(4) $0 < c \leq r_n$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to an element of $(A + B)^{-1}0 \cap \mathrm{EP}(f)$.

Proof. Putting $V = S = I$ in Theorem 9, we can obtain the desired result. □

4. Applications

In this section, we apply the results in the preceding section to variational inequality and optimization problems. Now we consider the variational inequality problem. Let H be a real Hilbert space, and let f be a proper lower semicontinuous convex function of H into $(-\infty, +\infty]$. Then the subdifferential ∂f of f is defined as

$$\partial f(x) = \{z \in H : f(y) - f(x) \geq \langle z, y - x\rangle, \forall y \in H\} \tag{72}$$

for all $x \in H$. Rockafellar [18] claimed that ∂f is a maximal monotone operator. Let C be a nonempty closed convex subset of H, and let δ_C be the indicator function of C. That is,

$$\delta_C(x) = \begin{cases} 0 & x \in C, \\ +\infty & x \notin C. \end{cases} \tag{73}$$

Since δ_C is a proper lower semicontinuous convex function on H, the subdifferential ∂_{δ_C} of δ_C is a maximal monotone operator. The resolvent J_λ of ∂_{δ_C} for $\lambda > 0$ is defined by

$$J_\lambda x = \left(I + \lambda \partial_{\delta_C}\right)^{-1} x \quad \forall x \in H. \tag{74}$$

We have

$$\begin{aligned} u = J_\lambda x = \left(I + \lambda \partial_{\delta_C}\right)^{-1} x &\Longleftrightarrow x \in u + \lambda \partial_{\delta_C} u \\ &\Longleftrightarrow x \in u + \lambda N_C u \Longleftrightarrow x - u \in \lambda N_C u \\ &\Longleftrightarrow \frac{1}{\lambda}\langle x - u, y - u\rangle \leq 0 \quad \forall y \in C \\ &\Longleftrightarrow \langle x - u, y - u\rangle \leq 0 \quad \forall y \in C \\ &\Longleftrightarrow u = P_C x, \end{aligned} \tag{75}$$

where $N_C u = \{z \in H : \langle z, y - u\rangle \leq 0, \forall y \in C\}$. The variational inequality problem for nonlinear operator A is to find $z \in C$ such that

$$\langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{76}$$

The set of its solutions is denoted by $VI(C, A)$. Then we have

$$\begin{aligned} & z \in VI(C, A) \\ & \Longleftrightarrow \langle Az, y - z\rangle \geq 0, \quad \forall y \in C \\ & \Longleftrightarrow \langle -Az, y - z\rangle \leq 0 \quad \forall y \in C \\ & \Longleftrightarrow -Az \in N_C z \\ & \Longleftrightarrow 0 \in Az + N_C z \Longleftrightarrow 0 \in Az + \partial_{\delta_C} z \\ & \Longleftrightarrow z \in \left(A + \partial_{\delta_C}\right)^{-1} 0. \end{aligned} \tag{77}$$

Using Theorem 9, we obtain the strong convergence theorem for the variational inequality problem.

Theorem 15. *Let C be a nonempty closed convex subset of a real Hilbert space H and A an α-inverse strongly monotone operator of C into H, and let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H with $\overline{\gamma}$, $L > 0$ and $2\overline{\gamma} - 1 < L^2 < 2\overline{\gamma}$. Suppose that $VI(C, A) \neq \emptyset$. Let $\omega \in C$ and $x_1 \in C$, and let $\{x_n\}$ be a sequence generated by*

$$\begin{gathered} u_n = P_C x_n, \\ x_{n+1} = \beta_n \omega + (I - \beta_n V)\left[\alpha_n x_n + (1 - \alpha_n) P_C (I - \lambda_n A) u_n\right], \end{gathered} \tag{78}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ satisfy the following conditions:

(1) $\{\alpha_n\} \subset (0, 1)$ *and* $\sum_{n=1}^{\infty} \alpha_n < \infty$,

(2) $\{\beta_n\} \subset (0, 1)$, $\beta_n \to 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(3) $0 < a \leq \lambda_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to an element of $VI(C, A)$.

Proof. Notice that $VI(C, A) = (A + \partial_{\delta_C})^{-1} 0$. Letting $f(x, y) \equiv 0$ for all x, $y \in C$, $r_n = 1$, and $S = I$, Theorem 9 yields that the sequence $\{x_n\}$ converges strongly to an element of $VI(C, A)$. □

Next we study the optimization problem

$$\begin{gathered} \min g(x) \\ x \in C, \end{gathered} \tag{79}$$

where $g(x)$ is a proper lower semicontinuous convex function of H into $(-\infty, +\infty]$ such that C is included in $\mathrm{dom} g = \{x \in H : g(x) < +\infty\}$. We denote by $\mathrm{Sol}(g, C)$ the set of solutions to problem (79). Let $f : C \times C \to \mathbb{R}$ be a bifunction defined by

$$f(x, y) = g(y) - g(x). \tag{80}$$

It is clear that $f(x, y)$ satisfies (A1)–(A4) and $\mathrm{EP}(f) = \mathrm{Sol}(g, C)$. Therefore, by Theorem 9, the following result is obtained.

Theorem 16. *Let $g(x)$ be a proper lower semicontinuous convex function of H into $(-\infty, +\infty]$ and C a nonempty closed convex subset of H such that C is included in $\mathrm{dom} g$, and let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H with $\overline{\gamma}$, $L > 0$ and $2\overline{\gamma} - 1 < L^2 < 2\overline{\gamma}$. Suppose that $\mathrm{Sol}(g, C) \neq \emptyset$. Let $\omega \in C$ and $x_1 \in C$, and let $\{x_n\}$ be a sequence generated by*

$$\begin{gathered} u_n = \arg\min_{y \in C} \left\{ g(y) + \frac{1}{2r_n} \|y - x_n\|^2 \right\}, \\ x_{n+1} = \beta_n \omega + (I - \beta_n V)\left[\alpha_n x_n + (1 - \alpha_n) u_n\right], \end{gathered} \tag{81}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ satisfy the following conditions:

(1) $\{\alpha_n\} \subset (0, 1)$ *and* $\sum_{n=1}^{\infty} \alpha_n < \infty$,

(2) $\{\beta_n\} \subset (0, 1)$, $\beta_n \to 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(3) $0 < c \leq r_n$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to an element of $\mathrm{Sol}(g, C)$.

Proof. Letting $S = I$, $A = 0$, $B = \partial_{\delta_C}$, and $f(x, y) = g(y) - g(x)$ in Theorem 9, we get the conclusion. □

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