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*Research Article*

# On Second-Order Differential Equations with Nonsmooth Second Member

**M. Milla Miranda,[1] A. T. Lourêdo,[1] and L. A. Medeiros[2]**

[1] *DM, UEPB, Campina Grande-PB, Brazil*
[2] *IM, UFRJ, Rio de Janeiro, RJ, Brazil*

Correspondence should be addressed to M. Milla Miranda; milla@im.ufrj.br

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In an abstract framework, we consider the following initial value problem: $u'' + \mu Au + F(u)u = f$ in $(0,T)$, $u(0) = u^0, u'(0) = u^1$, where $\mu$ is a positive function and $f$ a nonsmooth function. Given $u^0$, $u^1$, and $f$ we determine $F(u)$ in order to have a solution $u$ of the previous equation. We analyze two cases of $F(u)$. In our approach, we use the Theory of Linear Operators in Hilbert Spaces, the compactness Aubin-Lions Theorem, and an argument of Fixed Point. One of our two results provides an answer in a certain sense to an open question formulated by Lions in (1981, Page 284).

## 1. Introduction

Let $V$ and $H$ be two real separable Hilbert spaces with $V$ dense in $H$ and $V$ continuously embedding in $H$. The scalar product and norms of $V$ and $H$ are represented, respectively, by

$$(u, v), |u|, \qquad ((u, v)), \|u\|. \tag{1}$$

Let $A$ be the self-adjoint operator of $H$ defined by the triplet $\{V, H, ((u, v))\}$. Consider $\alpha \in \mathbb{R}$, $\alpha \geq 0$. We denote by $D(A^\alpha)$ the Hilbert space

$$D(A^\alpha) = \{u \in H; A^\alpha u \in H\} \tag{2}$$

equipped with the scalar product

$$(u, v)_{D(A^\alpha)} = (A^\alpha u, A^\alpha v) \tag{3}$$

(cf. Lions [1]).

Consider the following initial value problem:

$$\begin{gathered} u'' + \mu Au + F(u)u = f \quad \text{in } (0, T), \\ u(0) = u^0, \qquad u'(0) = u^1, \end{gathered} \tag{4}$$

where $\mu$ is a positive function and $f$ a nonsmooth function.

The objective of this work is to study the following inverse problem: given $u^0, u^1$, and $f \in L^2(0, T; D(A^\alpha)')$ ($D(A^\alpha)'$ dual space of $D(A^\alpha)$) to determine $F(u)$ such that Problem (4) has a solution $u$. We analyze two cases of $F(u)$, more precisely, the cases

$$F(u) = \left|A^{-\theta/2}u\right|^{2p}, \tag{5}$$

$$F(u) = \|u\|_{C^0([0,T];X)}, \tag{6}$$

where $X$ is an appropriate Hilbert space.

In Lions [2, Page 284], the following problem is formulated

$$\begin{gathered} u'' - \Delta u + \left[\int_0^t \left(\int_\Omega u^2 dx\right) ds\right] u \\ = v(t)\,\delta(x - x_0) \quad \text{in } \Omega \times (0, T), \\ u = 0 \quad \text{on } \Gamma \times (0, T), \\ u(x, 0) = 0, \quad u'(x, 0) = 0, \quad x \in \Omega, \end{gathered} \tag{7}$$

where $\Omega$ is an open bounded set of $\mathbb{R}^n$ with boundary $\Gamma$, $x_0 \in \Omega$, and $\delta(x - x_0)$ being the Dirac mass supported at $\{x_0\}$. He says not to know if this problem admits a solution.

He say also that one of the difficulties in the study of existence of solutions of the nonlinear equations lies in the difficulty in defining weak solutions, since the transposition method is essentially a linear method. This is ultimately connected to the fact that one cannot multiply distributions.

Problem (4) with $F(u)$ of the form (6) is an abstract formulation of Problem (7) with a slight modification of the nonlinear term. Theorem 3 gives the existence of solutions of this problem. In applications we give examples of Problem (7), with the modification of the nonlinear term, for $\Omega$ an open bounded set of $\mathbb{R}^n$, $n = 1, 2, 3$.

In Grotta Ragazzo [3] the following equation is studied:

$$u_{tt} - u_{xx} - au + \left(\frac{1}{\pi}\int_0^\pi u^2 dx\right)^\alpha u = 0 \quad \text{in } (0,\pi)\times\mathbb{R}. \tag{8}$$

This equation is considered as a first approximation of the Klein-Gordon equation

$$u_{tt} - u_{xx} - au + u^{1+2\alpha} = 0 \quad \text{in } (0,\pi)\times\mathbb{R}. \tag{9}$$

Observe that (8) with $a = 0$ and $\alpha = 1$ is the meson equation of Schiff [4] (cf. also Jörgens [5]).

The physical motivation of (8) with $\alpha = 1$ can be seen in Lourêdo et al. [6].

Problem (4) with $F(u)$ of the form (5) generalizes (8) when $a = 0$. The existence of solutions of this problem is studied in Theorem 1.

In Louredo et al., loc.cit., is analyzed the equation

$$\begin{aligned} &u'' - \mu(t)\Delta u + a\left(\int_\Omega u^2 dx\right)u + b\left(\int_\Omega u'^2 dx\right)u' \\ &\quad = 0 \quad \text{in } \Omega\times(0,\infty) \end{aligned} \tag{10}$$

with nonlinear boundary condition. The $F(u)$ given in (5) is different from the $a(\int_\Omega u^2 dx)$ of this equation. The term $F(u)$ is related to the nonsmoothness of $f$.

## 2. Main Results

We use the notation $D(A^\alpha)' = D(A^{-\alpha})$, $\alpha \in \mathbb{R}$, $\alpha \geq 0$. Identifying $H$ with $H'$, we have

$$D(A^\alpha) \hookrightarrow H \hookrightarrow D(A^{-\alpha}). \tag{11}$$

Here and in what follows the notation $X \hookrightarrow Y$ means that the space $X$ in dense in the space $Y$ and the embedding of $X$ in $Y$ are continuous. Note that $D(A^{-\alpha})' = D(A^\alpha)$. Also, if $\beta, \gamma \in \mathbb{R}$ with $\beta \geq \gamma$, we have

$$D(A^\beta) \hookrightarrow D(A^\gamma). \tag{12}$$

Assume that

$$\text{the embedding of } V \text{ in } H \text{ is compact.} \tag{13}$$

First we analyze Problem (2) with $F(u) = |A^{-\theta/2}u|^{2p}$, that is, the problem

$$\begin{gathered} u'' + \mu A u + \left|A^{-\theta/2}u\right|^{2p} u = f \quad \text{in } (0,T), \\ u(0) = u^0, \qquad u'(0) = u^0. \end{gathered} \tag{14}$$

**Theorem 1.** *Assume condition* (13). *Let $\theta$ and $p$ be real numbers with $p \geq 1$. Consider*

$$\begin{gathered} u^0 \in D\left(A^{(1-\theta)/2}\right), \qquad u^1 \in D\left(A^{-\theta/2}\right), \\ \mu \in W^{1,1}(0,T), \qquad \mu(t) \geq \mu_0 > 0, \\ \forall t \in [0,T]\ (\mu_0 \text{ constant}), \\ f \in W^{1,1}\left(0,T; D\left(A^{-(1+\theta)/2}\right)\right). \end{gathered} \tag{15}$$

*Then there exists a function $u$ in the class*

$$\begin{gathered} u \in L^\infty\left(0,T; D\left(A^{(1-\theta)/2}\right)\right), \\ u' \in L^\infty\left(0,T; D\left(A^{-\theta/2}\right)\right), \\ u'' \in L^\infty\left(0,T; D\left(A^{-(1+\theta)/2}\right)\right) \end{gathered} \tag{16}$$

*such that $u$ is solution of the equation*

$$\begin{aligned} &u'' + \mu A u + \left|A^{-\theta/2}u\right|^{2p} u \\ &\quad = f \quad \text{in } L^\infty\left(0,T; D\left(A^{-(1+\theta)/2}\right)\right) \end{aligned} \tag{17}$$

*and satisfies the initial conditions*

$$u(0) = u^0, \qquad u'(0) = u^1. \tag{18}$$

*Remark 2.* When $\mu = 1$ it is possible to obtain a solution $u$ of Problem (14) by using the Theory of Semigroups (cf. Pazy [7]).

To formulate the second problem, we introduce some notations. In fact, let us define

$$Y = D\left(A^{(1-\theta)/2}\right), \qquad Z = D\left(A^{-\theta/2}\right), \qquad X = D\left(A^\lambda\right), \tag{19}$$

where

$$-\frac{\theta}{2} \leq \lambda < \frac{1-\theta}{2}, \quad \lambda \in \mathbb{R}. \tag{20}$$

By (12) we have

$$Y \hookrightarrow X \hookrightarrow Z. \tag{21}$$

Consider Problem (4) with $F(u) = \|u\|_{C^0([0,T];X)}$ and $u_0 = 0$, that is, the problem

$$\begin{gathered} u'' + \mu A u + \|u\|_{C^0([0,T];X)} u = f \quad \text{in } (0,T), \\ u(0) = 0, \qquad u'(0) = u^1. \end{gathered} \tag{22}$$

**Theorem 3.** *Assume that $\theta$, $u^1$, and $f$ satisfy the hypotheses of Theorem 1 and $u^0 = 0$. Then there exists a function $u$ in the class* (16) *such that $u$ is solution of the problem*

$$\begin{aligned} &u'' + \mu A u + \|u\|_{C^0([0,T];X)} u \\ &\quad = f \quad \text{in } L^\infty\left(0,T; D\left(A^{-(1+\theta)/2}\right)\right), \\ &u(0) = 0, \qquad u'(0) = u^1. \end{aligned} \tag{23}$$

*Remark 4.* Note that if $u$ belongs to class (16) then $u \in C^0([0,T];X)$ (cf. Lions and Magenes [8]).

**Corollary 5.** *Under the same hypotheses of Theorem 1, there exists a function $u$ in the class* (16) *such that $u$ is solution of the problem*

$$u'' + \mu A u + \|u\|^p_{L^p(0,T;X)} u = f \quad \text{in } L^\infty\left(0,T;D\left(A^{-(1+\theta)/2}\right)\right), \qquad u(0) = 0, \qquad u'(0) = u^1. \tag{24}$$

We analyze the uniqueness of solutions. Consider $\theta = 0$ in Theorem 1. Then the solution $u$ gives by this theorem when $\theta = 0$ satisfies

$$u \in L^\infty\left(0,T;D\left(A^{1/2}\right)\right), \qquad u' \in L^\infty(0,T;H), \qquad u'' \in L^\infty\left(0,T;D\left(A^{-1/2}\right)\right), \tag{25}$$

$$u'' + \mu A u + |u|^{2p} u = f \quad \text{in } L^\infty\left(0,T;D\left(A^{-1/2}\right)\right), \qquad u(0) = u^0, \qquad u'(0) = u^1. \tag{26}$$

**Theorem 6.** *Let $p \geq 1$ be a real number. Consider*

$$u^0 \in D\left(A^{1/2}\right), \quad u^1 \in H, \qquad \mu \text{ satisfying } (15)_2 \text{ with } \mu' \in L^\infty(0,T), \qquad f \in W^{1,1}\left(0,T;D\left(A^{-1/2}\right)\right). \tag{27}$$

*Then there exists a unique solution $u$ of Problem* (26) *in the class* (25).

We do not know if there is uniqueness of solutions for Theorem 3, even when $\theta = 0$.

In what follows we prove the above results.

## 3. Proof of Theorem 1

Before proving the theorem, we make some considerations on the operator $A^\alpha$. Recall hypothesis (13). By solving the spectral problem $((u,v)) = \lambda(u,v)$, for all $v \in V$, we determine the eigenfunctions and eigenvalues, respectively, $(w_\nu)$ and $(\lambda_\nu)$ of the operator $A$, that is,

$$A w_\nu = \lambda_\nu w_\nu, \quad \nu = 1,2,\ldots, \qquad \lambda_\nu \longrightarrow \infty, \quad \nu \longrightarrow \infty. \tag{28}$$

Note that $(w_\nu)$ is a Hilbert basis of $H$ (cf. Brezis [9] and Komornik [10]).

Let be $\alpha \in \mathbb{R}$, $\alpha \geq 0$. Then the linear operator

$$A^\alpha : D\left(A^\alpha\right) \longrightarrow H \tag{29}$$

is continuous, bijective, and

$$A^\alpha u = \sum_{\nu=1}^{\infty} \lambda_\nu^\alpha \left(u, w_\nu\right) w_\nu, \quad \forall u \in D\left(A^\alpha\right). \tag{30}$$

Also,

$$\left(A^\alpha\right)^{-1} : H \longrightarrow D\left(A^\alpha\right) \tag{31}$$

is given by

$$\left(A^\alpha\right)^{-1} f = \sum_{\nu=1}^{\infty} \lambda_\nu^{-\alpha} \left(f, w_\nu\right) w_\nu, \quad \forall f \in H. \tag{32}$$

These results can be found in Lions [1] and Medeiros and Milla Miranda [11].

Introduce the adjoint operator $(A^\alpha)^*$ of $A^\alpha$, that is,

$$\left(A^\alpha\right)^* : H \longrightarrow D(A^\alpha)', \qquad \left\langle \left(A^\alpha\right)^* f, z\right\rangle_{\mathcal{X}'\times\mathcal{X}} = \left(f, A^\alpha z\right), \quad \forall z \in \mathcal{X}, \tag{33}$$

where $\mathcal{X} = D(A^\alpha)$. Note that $H$ is identified with $H'$. By the properties of $A^\alpha$, we have that

$$\left(A^\alpha\right)^* \text{ is linear, continuous and bijective.} \tag{34}$$

Thus, the linear operator

$$\left[\left(A^\alpha\right)^*\right]^{-1} : D(A^\alpha)' \longrightarrow H \tag{35}$$

is continuous and bijective.

**Proposition 7.** *Let $f \in H$ and $g \in D(A^\alpha)'$. Then one has the following.*

(i) $\sum_{\nu=1}^{\infty} \lambda_\nu^\alpha (f, w_\nu)_{w_\nu}$ *converges in* $D(A^\alpha)'$, *and*

$$\left(A^\alpha\right)^* f = \sum_{\nu=1}^{\infty} \lambda_\nu^\alpha \left(f, w_\nu\right) w_\nu. \tag{36}$$

(ii) $\sum_{\nu=1}^{\infty} \lambda_\nu^{-\alpha} \langle g, w_\nu\rangle_{\mathcal{X}'\times\mathcal{X}} w_\nu$ *converges in* $H$, *and*

$$\left[\left(A^\alpha\right)^*\right]^{-1} g = \sum_{\nu=1}^{\infty} \lambda_\nu^{-\alpha} \left\langle g, w_\nu\right\rangle_{\mathcal{X}'\times\mathcal{X}} w_\nu, \tag{37}$$

*where* $\mathcal{X} = D(A^\alpha)$.

*Proof.* We prove (i). As $f \in H$, we have

$$f = \sum_{\nu=1}^{\infty} \left(f, w_\nu\right) w_\nu. \tag{38}$$

Consider $z \in \mathcal{X}$. Then noting that $\langle w_\nu, w_\mu\rangle_{\mathcal{X}'\times\mathcal{X}} = (w_\nu, w_\mu)$, we obtain

$$\begin{aligned} \left\langle \sum_{\nu=m}^{n} \lambda_\nu^\alpha \left(f, w_\nu\right) w_\nu, z \right\rangle_{\mathcal{X}'\times\mathcal{X}} &= \left\langle \sum_{\nu=m}^{n} \lambda_\nu^\alpha \left(f, w_\nu\right) w_\nu, \sum_{\mu=1}^{\infty} \left(z, w_\mu\right) w_\mu \right\rangle_{\mathcal{X}'\times\mathcal{X}} \\ &= \sum_{\nu=m}^{n} \lambda_\nu^\alpha \left(f, w_\nu\right) \left(z, w_\nu\right). \end{aligned} \tag{39}$$

On the other hand, by (30) we derive

$$\left(\sum_{\nu=m}^{n}(f,w_\nu)\,w_\nu, A^\alpha z\right)=\left(\sum_{\nu=m}^{n}(f,w_\nu)\,w_\nu, \sum_{\mu=1}^{\infty}\lambda_\mu^\alpha\left(z,w_\mu\right)w_\mu\right)$$
$$=\sum_{\nu=m}^{n}\lambda_\nu^\alpha\left(f,w_\nu\right)\left(z,w_\nu\right). \tag{40}$$

The last two expressions give

$$\left\langle\sum_{\nu=m}^{n}\lambda_\nu^\alpha\left(f,w_\nu\right)w_\nu, z\right\rangle_{\mathscr{X}'\times\mathscr{X}}=\left(\sum_{\nu=m}^{n}\left(f,w_\nu\right)w_\nu, A^\alpha z\right). \tag{41}$$

This and (38) provide that

$$\sum_{\nu=m}^{n}\lambda_\nu^\alpha\left(f,w_\nu\right)w_\nu\longrightarrow 0\quad\text{in } \mathscr{X}' \text{ as } m,n\longrightarrow\infty. \tag{42}$$

So $(\mathrm{i})_1$ is proved. Taking the limit in (41) and observing $(33)_2$, we obtain (36).

We prove (ii). We have that there exists a unique $f\in H$ such that

$$\left(A^\alpha\right)^* f=g. \tag{43}$$

By $(33)_2$ and $(30)_1$, we have

$$\left\langle g,w_\mu\right\rangle_{\mathscr{X}'\times\mathscr{X}}=\left(\sum_{\nu=1}^{\infty}\left(f,w_\nu\right)w_\nu, A^\alpha w_\mu\right)=\lambda_\mu^\alpha\left(f,w_\mu\right). \tag{44}$$

Then

$$\sum_{\nu=m}^{n}\lambda_\nu^{-\alpha}\left\langle g,w_\nu\right\rangle_{\mathscr{X}'\times\mathscr{X}}w_\nu=\sum_{\nu=m}^{n}\left(f,w_\nu\right)w_\nu. \tag{45}$$

This implies that

$$\sum_{\nu=m}^{n}\lambda_\nu^{-\alpha}\left\langle g,w_\nu\right\rangle_{\mathscr{X}'\times\mathscr{X}}w_\nu\longrightarrow 0\quad\text{in } H \text{ as } m,n\longrightarrow\infty. \tag{46}$$

Thus $(\mathrm{ii})_2$ is proved. By (43) and (45), we obtain

$$\sum_{\nu=1}^{\infty}\lambda_\nu^{-\alpha}\left\langle g,w_\nu\right\rangle_{\mathscr{X}'\times\mathscr{X}}w_\nu=\sum_{\nu=1}^{\infty}\left(f,w_\nu\right)w_\nu=f=\left[\left(A^\alpha\right)^*\right]^{-1}g. \tag{47}$$

This concludes the proof of the proposition. □

Motived by (37), we equip the space $D(A^\alpha)'$ with the scalar product

$$(g,h)_{D(A^\alpha)'}$$
$$=\left(\left[\left(A^\alpha\right)^*\right]^{-1}g,\left[\left(A^\alpha\right)^*\right]^{-1}h\right)$$
$$=\left(\sum_{\nu=1}^{\infty}\lambda_\nu^{-\alpha}\left\langle g,w_\nu\right\rangle_{\mathscr{X}'\times\mathscr{X}}w_\nu, \sum_{\mu=1}^{\infty}\lambda_\mu^{-\alpha}\left\langle h,w_\mu\right\rangle_{\mathscr{X}'\times\mathscr{X}}w_\mu\right), \tag{48}$$

where $\mathscr{X}=D(A^\alpha)$. This scalar product on $\mathscr{X}'$ yields a norm

$$\|g\|_{D(A^\alpha)'}=\left\|\left[\left(A^\alpha\right)^*\right]^{-1}g\right\|=\left|\sum_{\nu=1}^{\infty}\lambda_\nu^{-\alpha}\left\langle g,w_\nu\right\rangle_{\mathscr{X}'\times\mathscr{X}}w_\nu\right| \tag{49}$$

which is equivalent to the usual norm of $D(A^\alpha)'$.

By similarity between expressions (30) and (36) and between (32) and (37), respectively, we introduce the notations

$$\left(A^\alpha\right)^*=A^\alpha,\qquad\left[\left(A^\alpha\right)^*\right]^{-1}=\left(A^\alpha\right)^{-1}. \tag{50}$$

Also we use the notation

$$\left(A^\alpha\right)^{-1}=A^{-\alpha}=D\left(A^\alpha\right)'. \tag{51}$$

With these considerations and expressions (29) and (33), we obtain

$$D\left(A^\alpha\right)\xrightarrow{A^\alpha}H\xrightarrow{A^\alpha}D\left(A^{-\alpha}\right) \tag{52}$$

and by expressions (35) and (31),

$$D\left(A^{-\alpha}\right)\xrightarrow{A^{-\alpha}}H\xrightarrow{A^{-\alpha}}D\left(A^\alpha\right). \tag{53}$$

Also by (37) and (48), (49), respectively, we find

$$A^{-\alpha}g=\sum_{\nu=1}^{\infty}\lambda_\nu^{-\alpha}\left\langle g,w_\nu\right\rangle_{D(A^{-\alpha})\times D(A^\alpha)}w_\nu,\quad\forall g\in D\left(A^{-\alpha}\right), \tag{54}$$

$$(g,h)_{D(A^{-\alpha})}=\left(A^{-\alpha}g,A^{-\alpha}h\right),\qquad\|g\|_{D(A^{-\alpha})}=\left|A^{-\alpha}g\right|. \tag{55}$$

**Proposition 8.** *Consider $\beta,\gamma\in\mathbb{R}$. Then the linear operator*

$$A^\gamma: D\left(A^\beta\right)\longrightarrow D\left(A^{\gamma-\beta}\right)' \tag{56}$$

*defined by*

$$\left\langle A^\gamma y,z\right\rangle_{D(A^{\gamma-\beta})'\times D(A^{\gamma-\beta})}=\left(A^\beta y,A^{\gamma-\beta}z\right) \tag{57}$$

*is continuous.*

*Proof.* We obtain

$$\left|\left\langle A^\gamma y,z\right\rangle_{\mathscr{Y}'\times\mathscr{Y}}\right|\le\left|A^\beta y\right|\left|A^{\gamma-\beta}z\right|=\|y\|_{D(A^\beta)}\|z\|\,y, \tag{58}$$

where $\mathscr{Y}=D(A^{\gamma-\beta})$. Then

$$\left\|A^\gamma y\right\|_{\mathscr{Y}'}\le\|y\|_{D(A^\beta)},\quad\forall y\in D\left(A^\beta\right) \tag{59}$$

which proves the proposition. □

*Proof of Theorem 1.* We use the Galerkin method (cf. Lions [12] and Vicente and Frota [13]). Thus consider an approximate solution $u_m$ of Problem (14); that is,

$$u_m(t) = \sum_{j=1}^{m} g_{jm}(t)\, w_j; \tag{60}$$

$$\begin{aligned}
&\left(u_m'', w_j\right) + \mu\left(Au_m, w_j\right) + \left|A^{-\theta/2}u_m\right|^{2p}\left(u_m, w_j\right) \\
&\quad = \left\langle f, w_j\right\rangle_{D(A^{\gamma_0})'\times D(A^{\gamma_0})}, \qquad j = 1, 2, \ldots, m; \\
&u_m(0) = u_m^0 = \sum_{j=1}^{m}\left\langle u^0, w_j\right\rangle_{E'\times E} w_j; \\
&u_m'(0) = u_m^1 = \sum_{j=1}^{m}\left\langle u^1, w_j\right\rangle_{G'\times G} w_j,
\end{aligned} \tag{61}$$

where $E = D(A^{-(1-\theta)/2})$ and $G = D(A^{\theta/2})$. □

*Remark 9.* Note that

$$\begin{aligned}
&u_m^0 \longrightarrow u^0 \quad \text{in } D\left(A^{(1-\theta)/2}\right), \\
&\left|A^{(1-\theta)/2}u_m^0\right| \le \left|A^{(1-\theta)/2}u^0\right|, \\
&u_m^1 \longrightarrow u^1 \quad \text{in } D\left(A^{-\theta/2}\right), \\
&\left|A^{-\theta/2}u_m^1\right| \le \left|A^{-\theta/2}u^1\right|.
\end{aligned} \tag{62}$$

*Remark 10.* Observe that if $\gamma \le 0$ then

$$\left\langle z, w_j\right\rangle_{D(A^{-\gamma})\times D(A^{\gamma})} = \left(z, w_j\right), \qquad j = 1, 2, \ldots \tag{63}$$

Multiply both sides of $(61)_1$ by $g_{jm}'(t)\lambda_j^{-\theta}$ and add from $j = 1$ up to $j = m$. We obtain

$$\begin{aligned}
&\left(u_m'', A^{-\theta}u_m'\right) + \mu\left(Au_m, A^{-\theta}u_m'\right) \\
&\quad + \left|A^{-\theta/2}u_m\right|^{2p}\left(u_m, A^{-\theta}u_m'\right) \\
&\quad = \left\langle f, A^{-\theta}u_m'\right\rangle_{W'\times W},
\end{aligned} \tag{64}$$

where

$$W = D\left(A^{\gamma_0}\right), \qquad \gamma_0 = \frac{1+\theta}{2}. \tag{65}$$

Then,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|A^{-\theta/2}u_m'\right|^2 + \frac{1}{2}\frac{d}{dt}\left(\mu\left|A^{(1-\theta)/2}u_m\right|^2\right) \\
&\quad + \frac{1}{2(p+1)}\frac{d}{dt}\left|A^{-\theta/2}u_m\right|^{2(p+1)} \\
&\quad = \frac{1}{2}\mu'\left|A^{(1-\theta)/2}u_m\right|^2 + \left\langle f, A^{-\theta}u_m'\right\rangle_{W'\times W}.
\end{aligned} \tag{66}$$

By Proposition 8 with $\gamma = 0$ and $\beta = -\gamma_0$ and noting that $\gamma_0 - \theta = (1-\theta)/2$, we obtain

$$\left\langle f, A^{-\theta}w_j\right\rangle_{W'\times W} = \left(A^{-\gamma_0}f, A^{\gamma_0-\theta}w_j\right) = \left(A^{-\gamma_0}f, A^{(1-\theta)/2}w_j\right). \tag{67}$$

Substituting this equality into (66) and integrating on $[0, t]$, we obtain

$$\begin{aligned}
&\frac{1}{2}\left|A^{-\theta/2}u_m'(t)\right|^2 + \frac{1}{2}\mu(t)\left|A^{(1-\theta)/2}u_m(t)\right|^2 \\
&\quad + \frac{1}{2(p+1)}\left|A^{-\theta/2}u_m(t)\right|^{2(p+1)} \\
&\quad = \frac{1}{2}\int_0^t \mu'(s)\left|A^{(1-\theta)/2}u_m(s)\right|^2 ds \\
&\quad + \int_0^t \left(A^{-\gamma_0}f(s), A^{(1-\theta)/2}u_m'(s)\right) ds \\
&\quad + \frac{1}{2}\left|A^{-\theta/2}u_m^1\right|^2 + \frac{1}{2}\mu(0)\left|A^{(1-\theta)/2}u_m^0\right|^2 \\
&\quad + \frac{1}{2(p+1)}\left|A^{-\theta/2}u_m^0\right|^{2(p+1)}.
\end{aligned} \tag{68}$$

Note that

$$\begin{aligned}
&\frac{1}{2}\int_0^t \mu'(s)\left|A^{(1-\theta)/2}u_m(s)\right|^2 ds \\
&\quad \le 2\int_0^t \frac{\left|\mu'(s)\right|}{\mu_0}\left[\frac{1}{4}\mu(s)\right]\left|A^{(1-\theta)/2}u_m(s)\right|^2 ds.
\end{aligned} \tag{69}$$

Also

$$\begin{aligned}
&\int_0^t \left(A^{-\gamma_0}f(s), A^{(1-\theta)/2}u_m'(s)\right) ds \\
&\quad = \left(A^{-\gamma_0}f(t), A^{(1-\theta)/2}u_m(t)\right) \\
&\quad - \left(A^{-\gamma_0}f(0), A^{(1-\theta)/2}u_m^0\right) \\
&\quad - \int_0^t \left(A^{-\gamma_0}f'(s), A^{(1-\theta)/2}u_m(s)\right) ds,
\end{aligned} \tag{70}$$

$$\begin{aligned}
&\left|\left(A^{-\gamma_0}f(t), \mathrm{A}^{(1-\theta)/2}u_m(t)\right)\right| \\
&\quad \le \frac{1}{\mu_0}\|f\|^2_{L^\infty(0,T;D(A^{-\gamma_0}))} + \frac{1}{4}\mu(t)\left|A^{(1-\theta)/2}u_m(t)\right|^2, \\
&\left|\int_0^t \left(A^{-\gamma_0}f'(s), A^{(1-\theta)/2}u_m(s)\right) ds\right| \\
&\quad \le \frac{1}{\mu_0}\int_0^T \left|A^{-\gamma_0}f'(s)\right|^2 ds \\
&\quad + \frac{1}{4}\int_0^t \left|A^{-\gamma_0}f'(s)\right|\left[\mu(s)\left|A^{(1-\theta)/2}u_m(s)\right|^2 ds\right].
\end{aligned} \tag{71}$$

Substituting (69)–(71) into (68), we find

$$\begin{aligned}
&\frac{1}{2}\left|A^{-\theta/2}u_m'(t)\right|^2+\frac{1}{4}\mu(t)\left|A^{(1-\theta)/2}u_m(t)\right|^2\\
&\quad+\frac{1}{2(p+1)}\left|A^{-\theta/2}u_m(t)\right|^{2(p+1)}\\
&\le 2\int_0^t\frac{\left|\mu'(s)\right|}{\mu_0}\left[\frac{1}{4}\mu(s)\left|A^{(1-\theta)/2}u_m(s)\right|^2ds\right]\\
&\quad+M+\left|A^{-\gamma_0}f(0)\right|\left|A^{(1-\theta)/2}u^0\right|+N\\
&\quad+\frac{1}{4}\int_0^t\left|A^{-\gamma_0}f'(s)\right|\left[\mu(s)\left|A^{(1-\theta)/2}u_m(s)\right|^2ds\right]\\
&\quad+E(0),
\end{aligned}\tag{72}$$

where

$$\begin{aligned}
M&=\frac{1}{\mu_0}\|f\|^2_{L^\infty(0,T;D(A^{-\gamma_0}))},\\
N&=\frac{1}{\mu_0}\int_0^T\left|A^{-\theta/2}f'(s)\right|ds,\\
E(0)&=\frac{1}{2}\left|A^{-\theta/2}u^1\right|^2+\frac{1}{2}\mu(0)\left|A^{(1-\theta)/2}u^0\right|^2\\
&\quad+\frac{1}{2(p+1)}\left|A^{-\theta/2}u^0\right|^{2(p+1)}.
\end{aligned}\tag{73}$$

Applying Gronwall inequality in (72), we deduce

$$\begin{aligned}
&\frac{1}{2}\left|A^{-\theta/2}u_m'(t)\right|^2+\frac{1}{4}\mu(t)\left|A^{(1-\theta)/2}u_m(t)\right|^2\\
&\quad+\frac{1}{2(p+1)}\left|A^{-\theta/2}u_m(t)\right|^{2(p+1)}\\
&\le\left(E(0)+M+\left|A^{-\gamma_0}f(0)\right|\left|A^{(1-\theta)/2}u^0\right|+N\right)\\
&\quad\times\exp^{\int_0^T[(2/\mu_0)|\mu'(t)|+|A^{-\gamma_0}f'(t)|]dt},\\
&\qquad\forall t\in[0,T].
\end{aligned}\tag{74}$$

With this inequality, we determine a subsequence of $(u_m)$, still denoted by $(u_m)$, and a function $u$ such that

$$\begin{aligned}
&u_m\longrightarrow u\ \text{weak star in}\ L^\infty\left(0,T;D\left(A^{(1-\theta)/2}\right)\right),\\
&u_m'\longrightarrow u'\ \text{weak star in}\ L^\infty\left(0,T;D\left(A^{-\theta/2}\right)\right).
\end{aligned}\tag{75}$$

By (13) we have that

$$D\left(A^{(1-\theta)/2}\right)\ \text{is compactly embedding in}\ D\left(A^{-\theta/2}\right)\tag{76}$$

(cf. [1] and [11]). Then convergences (75) and Aubin-Lions Theorem (cf. [14]) imply

$$u_m\longrightarrow u\quad\text{in}\ L^\infty\left(0,T;D\left(A^{-\theta/2}\right)\right).\tag{77}$$

Therefore,

$$\left|A^{-\theta/2}u_m\right|^{2p}\longrightarrow\left|A^{-\theta/2}u\right|^{2p}\quad\text{in}\ L^\infty(0,T).\tag{78}$$

This convergence and convergence $(75)_1$ provide

$$\begin{aligned}
&\left|A^{-\theta/2}u_m\right|^{2p}u_m\longrightarrow\left|A^{-\theta/2}u\right|^{2p}u\\
&\text{weak star in}\ L^\infty\left(0,T;D\left(A^{(1-\theta)/2}\right)\right)
\end{aligned}\tag{79}$$

which implies

$$\begin{aligned}
&\left|A^{-\theta/2}u_m\right|^{2p}u_m\longrightarrow\left|A^{-\theta/2}u\right|^{2p}u\\
&\text{weak star in}\ L^\infty\left(0,T;D\left(A^{-\gamma_0}\right)\right)
\end{aligned}\tag{80}$$

since $(1-\theta)/2>-\gamma_0$, $\gamma_0$ defined in (65).

In order to obtain an estimate for $(u_m'')$, we apply the method of projections to the approximate equation (61) (cf. Lions [12]). Thus, we consider the orthogonal projection

$$P_m:H\longrightarrow V_m\subset H,\qquad P_mz=\sum_{j=1}^m\left(z,w_j\right)w_j,\tag{81}$$

where $V_m$ is the subspace generated by $w_1,w_2,\dots,w_m$.

By similar arguments employed to obtain (67) and by (54) or (30), we find

$$\left\langle f,w_j\right\rangle_{D(A^{-\gamma_0})\times D(A^{\gamma_0})}=\left(A^{-\gamma_0}f,A^{\gamma_0}w_j\right)=\lambda_j^{\gamma_0}\left(A^{-\gamma_0}f,w_j\right).\tag{82}$$

Multiply both members of $(61)_1$ by $\lambda_j^{-\gamma_0}w_j$ and add from $j=1$ up to $j=m$. Then, applying to this result, expression (82), affirmation (54), or (30) and noting that $A^{-\gamma_0}u_m''$, $A^{-\gamma_0}(Au_m)$, $|A^{-\theta/2}u_m|^{2p}A^{-\gamma_0}u_m$ belong to $V_m$, we obtain

$$\begin{aligned}
&A^{-\gamma_0}u_m''+\mu A^{-\gamma_0}\left(Au_m\right)+\left|A^{-\theta/2}u_m\right|^{2p}A^{-\gamma_0}u_m\\
&\quad=P_m\left(A^{-\gamma_0}f\right)
\end{aligned}\tag{83}$$

which gives

$$\begin{aligned}
\left|A^{-\gamma_0}u_m''\right|&\le\mu\left|A^{(1-\theta)/2}u_m\right|\\
&\quad+C\left|A^{-\theta/2}u_m\right|^{2p}\left|A^{(1-\theta)/2}u_m\right|+\left|A^{-\gamma_0}f\right|.
\end{aligned}\tag{84}$$

Then estimates (74) and (80) provide

$$\left(u_m''\right)\ \text{is bounded in}\ L^\infty\left(0,T;D\left(A^{-\gamma_0}\right)\right).\tag{85}$$

Thus, there exists a subsequence of $(u_m'')$, still denoted by $(u_m'')$, such that

$$u_m''\longrightarrow u''\ \text{weak star in}\ L^\infty\left(0,T;D\left(A^{-\gamma_0}\right)\right).\tag{86}$$

Expressions (75) and (86) tell us that $u$ belongs to class (16). Convergences $(75)_1$, (80), and (86) allow us to pass to limit in (83) and to obtain

$$A^{-\gamma_0}u''+A^{-\gamma_0}\left(\mu Au\right)+\left|A^{-\theta/2}u\right|^{2p}A^{-\gamma_0}u=A^{-\gamma_0}f\tag{87}$$

which provides (17). Initial conditions (18) follow from convergences (75) and (86).

## 4. Proof of Theorem 3

The idea is to apply a fixed point argument to the problem

$$\begin{aligned} u_k'' + \mu A u_k + k u_k &= f \quad \text{in } (0,T), \\ u_k(0) = 0, \qquad u_k'(0) &= u^1, \end{aligned} \tag{88}$$

where $k \in \mathbb{R}, k \geq 0$.

We solve (88). Consider an approximate solution $u_{km}$ of (88) given by

$$u_{km}(t) = \sum_{j=1}^{m} g_{jkm}(t)\, w_j, \tag{89}$$

$$\begin{gathered} \left(u_{km}'', w_j\right) + \mu\left(A u_{km}, w_j\right) + k\left(u_{km}, w_j\right) \\ = \left\langle f, w_j \right\rangle_{D(A^{\gamma_0})' \times D(A^{\gamma_0})}, \quad j = 1, 2, \ldots, m; \\ u_{km}(0) = 0, \qquad u_{km}'(0) = u_m^1. \end{gathered} \tag{90}$$

By similar arguments used to obtain (74), we derive

$$\begin{aligned} &\frac{1}{2}\left|A^{-\theta/2} u_{km}'(t)\right|^2 + \frac{1}{4}\mu(t)\left|A^{(1-\theta)/2} u_{km}(t)\right|^2 \\ &\quad + \frac{k}{2}\left|A^{-\theta/2} u_{km}(t)\right|^2 \\ &\quad \leq L \exp^{\int_0^T a(t)dt} = R, \\ &\qquad \forall t \in [0, \mathrm{T}], \forall k \geq 0, \forall m, \end{aligned} \tag{91}$$

where

$$\begin{gathered} L = \frac{1}{2}\left|A^{-\theta/2} u^1\right|^2 + M + N \quad (M \text{ and } N \text{ defined in } (73)), \\ a(t) = \frac{2}{\mu_0}\left|\mu'(t)\right| + \left|A^{-\gamma_0} f'(t)\right|. \end{gathered} \tag{92}$$

The preceding inequality gives

$$\begin{aligned} &\left(u_{km}\right) \text{ is bounded in } L^\infty\left(0, T; D\left(A^{(1-\theta)/2}\right)\right), \\ &\qquad \forall m, \forall k \geq 0; \\ &\left(u_{km}'\right) \text{ is bounded in } L^\infty\left(0, T; D\left(A^{-\theta/2}\right)\right), \\ &\qquad \forall m, \forall k \geq 0. \end{aligned} \tag{93}$$

By the projection method, we obtain, as in (83),

$$A^{-\gamma_0} u_{km}'' + \mu A^{-\gamma_0}\left(A u_{km}\right) + k A^{-\gamma_0} u_{km} = P_m\left(A^{-\gamma_0} f\right). \tag{94}$$

This and estimate (91) provide

$$\left(u_{km}''\right) \text{ is bounded in } L^\infty\left(0, T; D\left(A^{-\gamma_0}\right)\right), \quad \forall m. \tag{95}$$

Estimates (93) and (95) allow us to find a subsequence of $(u_{km})$, still denoted by $(u_{km})$, and a function $u_k$ such that, by passing to limit in (94), we obtain

$$\begin{aligned} &A^{-\gamma_0} u_k'' + A^{-\gamma_0}\mu\left(A u_k\right) + k A^{-\gamma_0} u_k \\ &\quad = A^{-\gamma_0} f \quad \text{in } L^\infty(0, T; H). \end{aligned} \tag{96}$$

This, initial conditions $(90)_2$, and estimates (95) imply

$$\begin{gathered} u_k'' + \mu A u_k + k u_k = f \quad \text{in } L^\infty\left(0, T; D\left(A^{-\gamma_0}\right)\right); \\ u_k(0) = 0, \qquad u_k'(0) = u^1. \end{gathered} \tag{97}$$

By taking the lim inf in both side of (91), we obtain

$$\left\|u_k'\right\|_{L^2(0,T;Z)}^2 + \left\|u_k\right\|_{L^2(0,T;Y)}^2 \leq P, \quad \forall k \geq 0. \tag{98}$$

As $Y \hookrightarrow X \hookrightarrow Z$ and the embedding $Y$ in $X$ are compact ($\lambda < (1-\theta)/2$), it follows from of Aubin-Lions Theorem (see Simon [14]) that

$$\left\|u_k\right\|_{C^0([0,T];X)} \leq P_1, \quad \forall k \geq 0. \tag{99}$$

Define the map

$$\psi : [0, \infty) \longrightarrow \mathbb{R}, \qquad \psi(k) = \left\|u_k\right\|_{C^0([0,T];X)}, \tag{100}$$

where $u_k$ is the solution of Problem (97). We will prove that $\psi$ has a fixed point. Consider only the case $f \neq 0$. The case $f = 0$ is outside of our attention. We will prove the following results.

(I) One has $\psi(0) > 0$.

In fact if $\psi(0) = 0$, we have that $u_0 = 0$ is a solution of (97) with $k = 0$, but this a contradiction since $f \neq 0$.

(II) One has $\psi$ is continuous on $[0, \infty)$.

Let $k_0 > 0$. Consider $k > 0$. By (94) and $(90)_2$ we obtain

$$\begin{gathered} u_{km}'' + \mu A u_{km} + k u_{km} = A^{\gamma_0} P_m\left(A^{-\gamma_0} f\right); \\ u_{km}(0) = 0, \qquad u_{km}'(0) = u_m^1, \\ u_{k_0 m}'' + \mu A u_{k_0 m} + k_0 u_{k_0 m} = A^{\gamma_0} P_m\left(A^{-\gamma_0} f\right), \\ u_{k_0 m}(0) = 0, \qquad u_{k_0 m}'(0) = u_m^1. \end{gathered} \tag{101}$$

Use the notation $z_m = u_{km} - u_{k_0 m}$. Then the preceding problems give

$$\begin{gathered} z_m'' + \mu A z_m + k_0 z_m = -\left(k - k_0\right) u_{km}; \\ z_m(0) = 0, \qquad z_m'(0) = 0. \end{gathered} \tag{102}$$

Taking the scalar product of $H$ of both sides of this equation with $A^{-\theta} z_m'$, we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left|A^{-\theta/2} z_m'\right|^2 + \frac{1}{2}\frac{d}{dt}\left[\mu\left|A^{(1-\theta)/2} z_m\right|^2\right] \\ &\quad + \frac{k_0}{2}\frac{d}{dt}\left|A^{-\theta/2} z_m\right|^2 \\ &\quad = \frac{1}{2}\mu'\left|A^{(1-\theta)/2} z_m\right|^2 - \left(k - k_0\right)\left(u_{km}, A^{-\theta} z_m'\right). \end{aligned} \tag{103}$$

We have

$$\left(u_{km}, A^{-\theta} z_m'\right) = \left(A^{-\gamma_0} u_{km}, A^{(1-\theta)/2} z_m'\right). \tag{104}$$

Integrating on $[0,t]$ both sides of the last two expressions, we derive

$$\begin{aligned}
&\frac{1}{2}\left|A^{-\theta/2}z_m'(t)\right|^2+\frac{1}{2}\mu(t)\left|A^{(1-\theta)/2}z_m(t)\right|^2\\
&\qquad+\frac{k_0}{2}\left|A^{-\theta/2}z_m(t)\right|^2\\
&\qquad=\frac{1}{2}\int_0^t\mu'(s)\left|A^{(1-\theta)/2}z_m(s)\right|^2ds\\
&\qquad-(k-k_0)\int_0^t\left(A^{-\gamma_0}u_{km}(s),A^{(1-\theta)/2}z_m'(s)\right)ds.
\end{aligned}\tag{105}$$

We obtain

$$\begin{aligned}
&\int_0^t\left(A^{-\gamma_0}u_{km}(s),A^{(1-\theta)/2}z_m'(s)\right)ds\\
&\qquad=\left(A^{-\gamma_0}u_{km}(t),A^{(1-\theta)/2}z_m(t)\right)\\
&\qquad-\int_0^t\left(A^{-\gamma_0}u_{km}'(s),A^{(1-\theta)/2}z_m(s)\right)ds.
\end{aligned}\tag{106}$$

As $(1-\theta)/2>-\gamma_0$, we have

$$\begin{aligned}
&\left|\left(A^{-\gamma_0}u_{km}(t),A^{(1-\theta)/2}z_m(t)\right)\right|\\
&\qquad\le C\left|A^{(1-\theta)/2}u_{km}(t)\right|\left|A^{(1-\theta)/2}z_m(t)\right|\\
&\qquad\le\frac{C^2}{2\mu_0}\left|A^{(1-\theta)/2}u_{km}(t)\right|+\frac{1}{2}\mu(t)\left|A^{(1-\theta)/2}z_m(t)\right|^2.
\end{aligned}\tag{107}$$

Also,

$$\left(A^{-\gamma_0}u_{km}',A^{(1-\theta)/2}z_m\right)=\left(A^{-\theta/2}u_{km}',A^{-\theta/2}z_m\right),\tag{108}$$

$$\begin{aligned}
&\left|\int_0^t\left(A^{-\gamma_0}u_{km}'(s),A^{(1-\theta)/2}z_m(s)\right)ds\right|\\
&\qquad\le\int_0^t\left|A^{-\theta/2}u_{km}'(s)\right|\left|A^{-\theta/2}z_m(s)\right|ds\\
&\qquad\le\int_0^t C\left|A^{-\theta/2}u_{km}'(s)\right|\left|A^{(1-\theta)/2}z_m(t)\right|\ ds\\
&\qquad\le\frac{C^2}{2\mu_0}\int_0^T\left|A^{-\theta/2}u_{km}'(t)\right|^2dt\\
&\qquad+\frac{1}{2}\int_0^t\mu(s)\left|A^{(1-\theta)/2}z_m(s)\right|^2ds.
\end{aligned}\tag{109}$$

Taking into account estimate (91) in (107) and (109), we find

$$\begin{aligned}
&\left|\left(A^{-\gamma_0}u_{km}(t),A^{(1-\theta)/2}z_m(t)\right)\right|\\
&\qquad\le\frac{2C^2R}{\mu_0^2}+\frac{1}{2}\mu(t)\left|A^{(1-\theta)/2}z_m(t)\right|^2,\\
&\left|\int_0^t\left(A^{-\gamma_0}u_{km}'(s),A^{(1-\theta)/2}z_m(s)\right)ds\right|\\
&\qquad\le\frac{C^2RT}{\mu_0}+\frac{1}{2}\int_0^t\mu(s)\left|A^{(1-\theta)/2}z_m(s)\right|^2ds.
\end{aligned}\tag{110}$$

Substituting the last two inequalities into (106), we obtain

$$\begin{aligned}
&|k-k_0|\int_0^t\left|\left(A^{-\gamma_0}u_{km}(s),A^{(1-\theta)/2}z_m'(s)\right)\right|ds\\
&\quad\le\frac{2C^2R}{\mu_0^2}|k-k_0|\\
&\qquad+\frac{1}{2}|k-k_0|\,\mu(t)\left|A^{(1-\theta)/2}z_m(t)\right|^2+\frac{C^2RT}{\mu_0}|k-k_0|\\
&\qquad+\frac{1}{2}|k-k_0|\int_0^t\mu(s)\left|A^{(1-\theta)/2}z_m(s)\right|^2ds.
\end{aligned}\tag{111}$$

Combining this inequality with (105), we derive

$$\begin{aligned}
&\frac{1}{2}\left|A^{-\theta/2}z_m'(t)\right|^2+\frac{(1-|k-k_0|)}{2}\mu(t)\left|A^{(1-\theta)/2}z_m(t)\right|^2\\
&\qquad+\frac{k_0}{2}\left|A^{-\theta/2}z_m(t)\right|^2\\
&\quad\le\frac{1}{2\mu_0}\int_0^t\left|\mu'(s)\right|\left[\mu(s)\left|A^{(1-\theta)/2}z_m(s)\right|^2\right]ds\\
&\qquad+\left[\frac{2C^2R}{\mu_0^2}+\frac{C^2}{\mu_0}RT\right]|k-k_0|\\
&\qquad+\frac{1}{2}|k-k_0|\int_0^t\mu(s)\left|A^{(1-\theta)/2}z_m(s)\right|^2ds.
\end{aligned}\tag{112}$$

Considering $|k-k_0|<1/2$ and using the Gronwall inequality, this expression gives

$$\begin{aligned}
&\frac{1}{2}\left|A^{-\theta/2}z_m'(t)\right|^2+\frac{1}{4}\mu_0\left|A^{(1-\theta)/2}z_m(t)\right|^2\\
&\qquad+\frac{k_0}{2}\left|A^{-\theta/2}z_m(t)\right|^2\le R_1|k-k_0|,\\
&\qquad\forall t\in[0,T],
\end{aligned}\tag{113}$$

where the constant $R_1 > 0$ is independent of $m$ and $k$. Taking the lim inf in both sides of this inequality, we find

$$\begin{aligned} &\frac{1}{2}\left|A^{-\theta/2}\left(u_k'(t)-u_{k_0}'(t)\right)\right|^2 \\ &\qquad+\frac{\mu_0}{4}\left|A^{(1-\theta)/2}\left(u_k(t)-u_{k_0}(t)\right)\right|^2 \\ &\quad\le R_1\left|k-k_0\right|,\quad \forall t\in[0,T]. \end{aligned} \tag{114}$$

By Simon [14] and noting that the embedding of $Y$ in $X$ is compact, we derive

$$u_k \longrightarrow u_{k_0} \quad \text{in } C^0([0,T];X), \quad k\longrightarrow k_0. \tag{115}$$

Thus.

$$\|u_k\|_{C^0([0,T];X)} \longrightarrow \|u_{k_0}\|_{C^0([0,T];X)}, \quad k\longrightarrow k_0 \tag{116}$$

which proves the continuity of $\psi$ at $k_0 > 0$. In similar way we prove the continuity of $\psi$ at $k_0 = 0$.

(III) One has $\psi(k) \to 0$ as $k \to \infty$.

Let $(k_l)$ be a sequence of positive numbers with $k_l \to \infty$. It follows from (98) and the compactness of the embedding of $Y$ in $X$ that there exists a subsequence of $(k_l)$, still denoted by $(k_l)$, and a function $\chi$ such that

$$u_{k_l} \longrightarrow \chi \quad \text{in } C^0([0,T];X). \tag{117}$$

This implies

$$u_{k_l} \longrightarrow \chi \quad \text{in } L^2(0,T;Z). \tag{118}$$

By estimate (91), we obtain

$$\frac{k}{2}\|u_k\|^2_{L^2(0,T;Z)} \le R, \quad \forall k\ge 0. \tag{119}$$

Then

$$u_{k_l} \longrightarrow 0 \quad \text{in } L^2(0,T;Z). \tag{120}$$

Convergences (118) and (120) provide

$$\chi = 0. \tag{121}$$

Thus by (117) we find

$$u_{k_l} \longrightarrow 0 \quad \text{in } C^0([0,T];X). \tag{122}$$

As the sequence $(k_l)$ was arbitrary and the limit of $(u_{k_l})$ is always the same, we conclude that

$$u_k \longrightarrow 0 \quad \text{in } C^0([0,T];X), \quad k\longrightarrow\infty. \tag{123}$$

Thus

$$\|u_k\|_{C^0([0,T];X)} \longrightarrow 0, \quad k\longrightarrow\infty \tag{124}$$

which proves part (III).

By (I)–(III), we deduce that there exists $k > 0, k \in \mathbb{R}$ such that

$$\psi(k) = k. \tag{125}$$

Considering this $k$ in (97), we obtain a solution $u$ of (22) that satisfies all conditions of the theorem.

The proof of Corollary 5 follows by defining the map

$$\psi : [0,\infty) \longrightarrow \mathbb{R}, \quad \psi(k) = \|u_k\|^p_{L^p(0,T;X)}, \tag{126}$$

where $u_k$ is the solution of the problem

$$\begin{aligned} &u_k'' + \mu A u_k + k u_k = f \quad \text{in } L^\infty\left(0,T;D\left(A^{-(1+\theta)/2}\right)\right); \\ &\qquad u(0) = 0, \qquad u'(0) = u^1, \end{aligned} \tag{127}$$

and applying similar arguments to those used in the proof of Theorem 3.

## 5. Proof of Theorem 6

Let $u$ and $z$ be solutions of Problem (26) with $u$ and $z$ in class (25). Consider $w = u - z$. Then by (26) we have

$$\begin{aligned} &w'' + \mu A w + |u|^{2p} w + \left(|u|^{2p} - |z|^{2p}\right) z \\ &\quad = 0 \quad \text{in } L^\infty\left(0,T;D\left(A^{-1/2}\right)\right), \\ &\quad w(0) = 0, \qquad w'(0) = 0. \end{aligned} \tag{128}$$

Fix $0 < s \le T$. Consider $0 \le t \le s$. Introduce the function

$$y(t) = \int_s^t w(\tau)\,d\tau. \tag{129}$$

We have

$$\begin{aligned} &y \in C^0\left([0,s];D\left(A^{1/2}\right)\right); \\ &y(s) = 0, \quad y'(t) = w(t) \quad \text{a.e. } t\in(0,s); \\ &A^{1/2}y(t) = \int_s^t A^{1/2}w(\tau)\,d\tau, \qquad A^{1/2}y'(t) = A^{1/2}w(t); \\ &y(t) = -\int_0^s w(\tau)\,d\tau + \int_0^t w(\tau)\,d\tau. \end{aligned} \tag{130}$$

Apply the operator given by the first member of $(128)_1$ to $y(t)$. We obtain

$$\begin{aligned} &\left\langle w''(t), y(t)\right\rangle_{E'\times E} + \mu(t)\left\langle A w(t), y(t)\right\rangle_{E'\times E} \\ &\quad + |u(t)|^{2p}\left(w(t), y(t)\right) \\ &\quad + \left(|u(t)|^{2p} - |z(t)|^{2p}\right)\left(z(t), y(t)\right) \\ &\quad = 0, \quad \text{a.e. } t\in(0,s), \end{aligned} \tag{131}$$

where $E = D(A^{1/2})$. By $(130)_2$, we find

$$\left\langle w''(t), y(t)\right\rangle_{E'\times E} = \frac{d}{dt}\left\langle w'(t), w(t)\right\rangle - \frac{1}{2}\frac{d}{dt}|w(t)|^2. \tag{132}$$

Also by $(130)_3$,

$$\mu(t)\langle Aw(t), y(t)\rangle_{E'\times E} = \frac{1}{2}\frac{d}{dt}\left[\mu(t)\left|A^{1/2}y(t)\right|^2\right] - \frac{1}{2}\mu'(t)\left|A^{1/2}y(t)\right|^2. \tag{133}$$

Integrate (131) on $[0,s]$ and use $(128)_1$, (130), (132), and (133). We deduce

$$\begin{aligned}\frac{1}{2}|w(s)|^2 &+ \frac{1}{2}\mu(0)\left|\int_0^s A^{1/2}w(t)\,dt\right|^2 \\ &= -\frac{1}{2}\int_0^s \mu'(t)\left|A^{1/2}y(t)\right|^2 dt \\ &\quad + \int_0^s |u(t)|^{2p}(w(t), y(t))\,dt \\ &\quad + \int_0^s \left[|u(t)|^{2p} - |z(t)|^{2p}\right](z(t), y(t))\,dt.\end{aligned} \tag{134}$$

We modify each term of the second member of (134). We have

$$A^{1/2}y(t) = \int_0^t A^{1/2}w(\tau)\,d\tau - \int_0^s A^{1/2}w(\tau)\,d\tau. \tag{135}$$

Then

$$\begin{aligned}\int_0^s &\left|\mu'(t)\right|\left|A^{1/2}y(t)\right|^2 dt \\ &\le 2\int_0^s \left|\mu'(t)\right|\left|\int_0^t A^{1/2}w(\tau)\,d\tau\right|^2 dt \\ &\quad + 2\int_0^s \left|\mu'(t)\right|\left|\int_0^s A^{1/2}w(\tau)\,d\tau\right|^2 dt \\ &\le 2\int_0^s \frac{\left|\mu'(t)\right|}{\mu_0}\mu(0)\left|\int_0^t A^{1/2}w(\tau)\,d\tau\right|^2 dt \\ &\quad + 2\left|\int_0^s A^{1/2}w(\tau)\,d\tau\right|^2 \int_0^s \left|\mu'(t)\right| dt.\end{aligned} \tag{136}$$

First, we assume that $\mu' \neq 0$. In this case, we choose $s_0 = \min\{\mu(0)/8\|\mu'\|_{L^\infty(0,T)}, 1, T\} > 0$. We have

$$\begin{aligned}2&\left|\int_0^s A^{1/2}w(\tau)\,d\tau\right|^2 \int_0^s \left|\mu'(t)\right| dt \\ &\le \frac{1}{4}\mu(0)\left|\int_0^s A^{1/2}w(\tau)\,d\tau\right|^2, \\ &\qquad 0 \le s \le s_0.\end{aligned} \tag{137}$$

Combining this last inequality with (136), we find

$$\begin{aligned}\int_0^s &\left|\mu'(t)\right|\left|A^{1/2}y(t)\right|^2 dt \\ &\le 2\int_0^s \frac{\left|\mu'(t)\right|}{\mu_0}\left[\mu(0)\left|\int_0^t A^{1/2}w(\tau)\,d\tau\right|^2\right] dt \\ &\quad + \frac{1}{4}\mu(0)\left|\int_0^s A^{1/2}w(\tau)\,d\tau\right|^2, \quad 0 \le s \le s_0.\end{aligned} \tag{138}$$

We introduce the notations

$$\|u\|_{L^\infty(0,T;H)} \le S_1, \qquad \|z\|_{L^\infty(0,T;H)} \le S_2. \tag{139}$$

(Note that $w \in C^0([0,T]; H)$ since $w$ belongs to class (25)). We have

$$\begin{aligned}&\left|\int_0^s |u(t)|^{2p}(w(t), y(t))\,dt\right| \\ &\quad \le \frac{1}{2}S_1^{2p}\int_0^s |w(t)|^2 dt + \frac{1}{2}S_1^{2p}\int_0^s \left|y(t)\right|^2 dt.\end{aligned} \tag{140}$$

By $(130)_4$, we obtain

$$\begin{aligned}\left|y(t)\right|^2 &\le 2\left[\int_0^s |w(\tau)|\,d\tau\right]^2 + 2\left[\int_0^t |w(\tau)|\,d\tau\right]^2 \\ &\le 4s\int_0^s |w(\tau)|^2 d\tau.\end{aligned} \tag{141}$$

Therefore

$$\int_0^s \left|y(t)\right|^2 dt \le 4s^2\int_0^s |w(t)|^2 dt. \tag{142}$$

Combining (140) and (142), we deduce

$$\begin{aligned}&\left|\int_0^s |u(t)|^{2p}(w(t), y(t))\,dt\right| \\ &\quad \le S_1^{2p}\int_0^s \frac{1}{2}|w(t)|^2 dt + 4S_1^{2p}s^2\int_0^s \frac{1}{2}|w(t)|^2 dt.\end{aligned} \tag{143}$$

The preceding inequality with $0 \le s \le s_0$ gives

$$\left|\int_0^s |u(t)|^{2p}(w(t), y(t))\,dt\right| \le 5S_1^{2p}\int_0^s \frac{1}{2}|w(t)|^2 dt. \tag{144}$$

On the other hand,

$$\begin{aligned}|u(t)|^{2p} - |z(t)|^{2p} &= \left(|u(t)|^p + |z(t)|^p\right)\left(|u(t)|^p - |z(t)|^p\right), \\ \left||u(t)|^p - |z(t)|^p\right| &\le p\left(S_1 + S_2\right)^{p-1}|w(t)|.\end{aligned} \tag{145}$$

Hence,

$$\left||u(t)|^{2p} - |z(t)|^{2p}\right| \le N|w(t)|, \tag{146}$$

where

$$N = p\left(S_1^p + S_2^p\right)\left(S_1 + S_2\right)^{p-1}. \tag{147}$$

Thus estimate (146) gives

$$\left|\int_0^s \left(|u(t)|^{2p} - |z(t)|^{2p}\right)(z(t), y(t))\,dt\right| \leq NS_2 \int_0^s |w(t)|\,|y(t)|\,dt \leq NS_2 \int_0^s \frac{1}{2}|w(t)|^2 dt + \frac{1}{2} NS_2 \int_0^s |y(t)|^2 dt. \tag{148}$$

This expressions, (142), and $0 \leq s \leq s_0$ provide

$$\left|\int_0^s \left(|u(t)|^{2p} - |z(t)|^{2p}\right)(z(t), y(t))\,dt\right| \leq 5NS_2 \int_0^s \frac{1}{2}|w(t)|^2 dt. \tag{149}$$

Combining inequalities (138), (144), and (149) with inequality (134), we obtain

$$\begin{aligned}\frac{1}{2}|w(s)|^2 + \frac{1}{4}\mu(0)\left|\int_0^s A^{1/2} w(\tau)\,d\tau\right|^2 &\leq 2\int_0^s \frac{|\mu'(t)|}{\mu_0}\left[\mu(0)\left|\int_0^t A^{1/2} w(\tau)\,d\tau\right|^2\right] dt \\ &\quad + \left(5S_1^{2p} + 5NS_2\right)\int_0^s \frac{1}{2}|w(t)|^2 dt, \quad 0 \leq s \leq s_0.\end{aligned} \tag{150}$$

This implies

$$w(s) = 0, \quad 0 \leq s \leq s_0. \tag{151}$$

We will prove that $w'(s_0) = 0$. In fact as $w$ belongs to class (25) we have that $w' \in C_s^0([0, s_0]; H)$; that is, $(w'(t), \xi)$ is continuous on $[0, s_0]$ for all $\xi \in H$. Consider $\xi \in D(A)$. Then by $(128)_1$ we obtain

$$\begin{aligned}&\left\langle w''(t), \xi\right\rangle_{E'\times E} + \left\langle Aw(t), \xi\right\rangle_{E'\times E} + |u(t)|^{2p}(w(t), \xi) \\ &\quad + \left(|u(t)|^{2p} - |z(t)|^{2p}\right)(z(t), \xi) \\ &= 0, \quad \text{a.e. } t \in (0, s_0).\end{aligned} \tag{152}$$

Integrating this equality on $[0, s_0]$ and using (151), we derive

$$\left(w'(s_0), \xi\right) + \int_0^{s_0} \left(|u(t)|^{2p} - |z(t)|^{2p}\right)(z(t), \xi)\,dt = 0. \tag{153}$$

This, (146), and (151) give

$$\left|\left(w'(s_0), \xi\right)\right| \leq \int_0^{s_0} \left|\left(|u(t)|^{2p} - |z(t)|^{2p}\right)\|z(t)\|\,\xi\right| dt \leq \int_0^1 N\left|w(t)\,\|z(t)\|\,\xi\right| dt = 0. \tag{154}$$

Therefore, $(w'(s_0), \xi) = 0$ for all $\xi \in D(A)$. By density we obtain $(w'(s_0), \xi) = 0$ for all $\xi \in H$. Thus $w'(s_0) = 0$. Note that the constants $S_1$, $S_2$, and $N$ given, respectively, by (139) and (147) are independent of $0 \leq s \leq s_0$.

We apply similar arguments to the problem

$$\begin{gathered} w'' + Aw + |u|^{2p} w + \left(|u|^{2p} - |z|^{2p}\right) z = 0 \quad \text{in } L^\infty\left(s_0, 2s_0; D\left(A^{-1/2}\right)\right), \\ w(s_0) = 0, \qquad w'(s_0) = 0 \end{gathered} \tag{155}$$

and we obtain $w(s) = 0$ for $s_0 \leq s \leq 2s_0$. After a finite number of steps we proof that $w(s) = 0$ for $0 \leq s \leq T$.

When $\mu' = 0$, that is, $\mu(t) = \mu_0$, for all $t \in [0, T]$, expressions (144), (149), and similar arguments used to obtain the preceding result, allow us to deduce the uniqueness of solutions in this case.

## 6. Applications

Let $\Omega$ be an open bounded set of $\mathbb{R}^n$ with boundary $\Gamma$ of class $C^\infty$. Let $A = -\Delta$ be the self-adjoint operator defined by the triplet $\{H_0^1(\Omega), L^2(\Omega), ((u, v))\}$, where $((u, v))$ denotes the usual scalar product of $H_0^1(\Omega)$. The norm of $L^2(\Omega)$ is denoted by $|u|$.

**Lemma 11.** *Let $\theta \geq -1$ be a real number. Then, $D(A^{(1+\theta)/2})$ is contained in $H^{1+\theta}(\Omega)$ and*

$$\text{the embedding of } D\left(A^{(1+\theta)/2}\right) \text{ in } H^{1+\theta}(\Omega) \text{ is continuous.} \tag{156}$$

*Proof.* First we prove (156) when $(1 + \theta)/2 = m$, $m$ a natural number. More precisely, we prove that if $u \in D(A^m)$, then $u \in H^{2m}(\Omega)$ and

$$\|u\|_{H^{2m}(\Omega)} \leq C_m \left|A^m u\right|. \tag{157}$$

To prove (157), we use the method of mathematical induction. Consider $m = 1$ and $u \in D(A)$. Then, by the regularity of solutions of elliptic problems, we obtain $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \leq C_1 |Au|. \tag{158}$$

Assume that (157) holds for $m = h$. Consider $u \in D(A^{h+1})$. Then, $Au \in D(A^h)$. By induction hypothesis it follows that $Au \in H^{2h}(\Omega)$ and

$$\|Au\|_{H^{2h}(\Omega)} \leq C_h \left|A^h (Au)\right| = \left|A^{h+1} u\right|. \tag{159}$$

Consider the problem

$$\begin{aligned} Au &= g \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{160}$$

where $g = Au$. As $g \in H^{2h}(\Omega)$, by the regularity of solutions of elliptic problems, we have $u \in H^{2h+2}(\Omega)$ and

$$\|u\|_{H^{2h+2}(\Omega)} \leq C^* \|Au\|_{H^{2h}(\Omega)}. \tag{161}$$

Inequalities (159) and (161) provide

$$\|u\|_{H^{2h+2}(\Omega)} \le C^* C_h \left|A^{h+1}u\right| = C_{h+1}\left|A^{h+1}u\right|. \tag{162}$$

This and (158) prove (157).

Next we use the interpolation of Hilbert spaces. Consider a natural number $m$ such that $2m > 1+\theta$. By results of intermediate spaces, we have

$$\left[H^{2m}(\Omega), L^2(\Omega)\right]_\gamma = H^{(1-\gamma)2m}(\Omega), \quad 0 \le \gamma \le 1. \tag{163}$$

(cf. Lions and Magenes [8]). We have by (157) that the injections

$$\begin{gathered} D\left(A^m\right) \longrightarrow H^{2m}(\Omega), \\ D\left(A^0\right) = L^2(\Omega) \longrightarrow L^2(\Omega) \end{gathered} \tag{164}$$

are continuous. Then, by interpolation of Hilbert spaces, we have that the injection

$$\left[D\left(A^m\right), L^2(\Omega)\right]_\gamma \longrightarrow \left[H^{2m}(\Omega), L^2(\Omega)\right]_\gamma, \quad 0 \le \gamma \le 1 \tag{165}$$

is continuous. We choose $\gamma_0 = 1-(1+\theta)/2m$. Then, by (163), we obtain

$$\left[H^{2m}(\Omega), L^2(\Omega)\right]_{\gamma_0} = H^{1+\theta}(\Omega). \tag{166}$$

Also

$$\left[D\left(A^m\right), L^2(\Omega)\right]_{\gamma_0} = D\left(A^{(1+\theta)/2}\right). \tag{167}$$

These last two equalities and (165) give the lemma. □

Consider the operator $\delta(x-x_0)$, $x_0 \in \Omega$. For $\theta > (n/2)-1$; we have that

$$H^{1+\theta}(\Omega) \hookrightarrow C^0\left(\overline{\Omega}\right). \tag{168}$$

This embedding and (156) imply that

$$\begin{gathered} \text{the embedding of } D\left(\mathrm{A}^{(1+\theta)/2}\right) \text{ in } C^0\left(\overline{\Omega}\right) \\ \text{is continuous } \left(\theta > \frac{n}{2}-1\right). \end{gathered} \tag{169}$$

Define

$$\left\langle \delta\left(x-x_0\right), \varphi\right\rangle = \varphi\left(x_0\right), \quad \varphi \in D\left(A^{(1+\theta)/2}\right). \tag{170}$$

By (169) we have that $\delta(x-x_0) \in D(A^{-(1+\theta)/2})$. Thus, for

$$\begin{gathered} f = \nu\delta\left(x-x_0\right) \quad \text{with } \nu \in W^{1,1}(0,T), \\ \delta\left(x-x_0\right) \in D\left(A^{-(1+\theta)/2}\right) \quad \left(\theta > \frac{n}{2}-1\right), \end{gathered} \tag{171}$$

Theorems 1 and 3 give, respectively, solutions $u$ of problems (14) and (22).

Consider (171) for the particular case $\theta = 0$. Then Theorems 1 and 6 provide a unique solution $u$ of the problem

$$\begin{gathered} u_{tt} - \mu u_{xx} + \left(\int_a^b |u|^2 dx\right)^p u \\ = \nu\delta\left(x-x_0\right) \quad \text{in } (a,b)\times(0,T) \quad (p \ge 1); \\ u(a,t) = u(b,t) = 0, \quad t \in (0,T); \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \quad x \in (a,b). \end{gathered} \tag{172}$$

On the other hand, we have $X = L^2(\Omega)$ if $\lambda = 0$ ($X$ defined in (21)). In this case $-\theta/2 \le 0 < (1-\theta)/2$, that is, $0 \le \theta < 1$. The two restrictions $\theta > n/2 - 1$ and $0 \le \theta < 1$ give, respectively, for $n = 1, 2$, and 3 the variations $0 \le \theta < 1$, $0 < \theta < 1$, and $1/2 < \theta < 1$. In all three cases, Corollary 5 gives a solution $u$ of the problem

$$\begin{gathered} u'' - \mu\Delta u + \left[\int_0^T \left(\int_\Omega |u|^2 dx\right)^{p/2} dt\right] u \\ = \nu\delta\left(x-x_0\right) \quad \text{in } \Omega\times(0,T) \quad (p \ge 1); \\ u = 0 \quad \text{in } \Gamma\times(0,T); \\ u(x,0) = u^0(x), \quad u'(x,0) = u^1(x), \quad x \in \Omega, \end{gathered} \tag{173}$$

where $\Omega$ is an open bounded set of $\mathbb{R}^n$, $n = 1, 2, 3$.

## Conflict of Interests

The authors report that there is no conflict of interests in the publication of this paper.

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