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Research Article

A Modified Approach to the New Solutions of Generalized mKdV Equation Using (G'/G)-Expansion

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The modified (G'/G)-expansion method is applied for finding new solutions of the generalized mKdV equation. By taking an appropriate transformation, the generalized mKdV equation is solved in different cases and hyperbolic, trigonometric, and rational function solutions are obtained.

1. Introduction

The evolutions of the physical, engineering, and other systems always behave nonlinearly; hence many nonlinear evolution equations have been introduced to interpret the phenomena. Many kinds of mathematical methods have been established to investigate the solutions of those nonlinear evolution equations both numerically and asymptotically, while the exact solutions are of particular interests. In recent decades, with the rapid progress of computation methods, many effective calculating approaches have been developed, for example, the tanh-coth expansion [1, 2], F-expansion [3, 4], Painlevé expansion [5], Jacobi elliptic function method [6], Hirota bilinear transformation [7], Backlund/Darboux transformation [8, 9], variational method [10], the homogeneous balance method [11], exp-function expansion [12], and so on. However, a unified approach to obtain the complete solutions of the nonlinear evolution equations has not been revealed.

Within recent years, a new method called (G'/G)-expansion [13] has been proposed for finding the traveling wave solutions of the nonlinear evolution equations. Many equations have been investigated and many solutions have been found using the method, including KdV equation, Hirota-Satsuma equation [13], coupled Boussinesq equation [14], generalized Bretherton equation [15], the mKdV equation [16], the Burgers-KdV equation, the Benjamin-Bona-Mahony equation [17], the Whitham-Broer-Kaup-like equation [18], the Kolmogorov-Petrovskii-Piskunov equation [19], KdV-Burgers equation [20], and Drinfeld-Sokolov-Satsuma-Hirota equation [21].

The mKdV equation, a modified version of the Korteweg-de Vries (KdV) equation, has been investigated extensively since Zabusky showed how this equation depict the oscillations of a lattice of particles connected by nonlinear springs as the Fermi-Pasta-Ulam (FPU) model [22–25]. Afterwards, this equation has been used to describe the evolution of internal waves at the interface of two layers of equal depth [26]. Generally, the KdV theory describes the weak nonlinearity and weak dispersion while, in the study of nonlinear optics, the complex mKdV equation has even been used to describe the propagation of optical pulses in nematic optical fibers when we go beyond the usual weakly nonlinear limit of Kerr medium [27]. In some cases, the exponential order may be not a positive integer, but just a real number. After this kind of generalized mKdV equation [28] has been introduced, interests of investigating the solutions of it [29] have been inspired; then the standard expansion methods cannot be applied, and some kinds of transformation are needed.

In this paper, we modify the standard (G'/G)-expansion method and use it to solve the generalized mKdV equation. In next section, we briefly introduce the modified (G'/G)-expansion method while in Section 3, we apply it to find some types of new solutions of mKdV equation, and the last section gives the summary and conclusion.

2. An Introduction to the Modified (G'/G)-Expansion Method

Recently, a new approach called (G'/G)-expansion has been proposed dealing with the problems of finding solutions of nonlinear evolution equations [13] and some modifications to this method have been developed. Here we briefly outline the main steps of the modified (G'/G)-expansion method in the following.

Step 1. We consider a given

$$P\left(u, u_t, u_x, u_x t, u_{tt}, u_{xx}, \ldots\right) = 0, \tag{1}$$

where P is a polynomial for its arguments and $u = u(x,t)$ is the unknown function. Introducing the new variable ξ and supposing that $u(\xi) = u(x,t)$ and $\xi = x - vt$, hence, the partial differential equation (PDE) (1) is reduced to an ordinary differential equation (ODE) for $u = u(\xi)$ as

$$P\left(u, -vu', u', v^2u'', -vu'', u'', \ldots\right) = 0. \tag{2}$$

Step 2. For ODE (2) above, the solution could be expressed by a polynomial in G'/G as

$$u(\xi) = \sum_{i=-m}^{m} a_i \left(\frac{G'}{G}\right)^i, \tag{3}$$

where $G = G(\xi)$ is the solution of a second order linear ODE

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \tag{4}$$

with constants λ, μ to be determined later. Positive integer m is an index yet undetermined which should be calculated by the balance between the highest order derivatives and the nonlinear terms from ODE (2). By solving (4), it is apparent that the form of G'/G in three different cases read as follows.

(1) When $\lambda^2 - 4\mu > 0$,

$$\frac{G'}{G} = \frac{\sqrt{\lambda^2 - 4\mu}}{2} \times \left(\left(C_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi + C_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) \times \left(C_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi + C_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)^{-1}\right) - \frac{\lambda}{2}. \tag{5}$$

(2) When $\lambda^2 - 4\mu < 0$,

$$\frac{G'}{G} = \frac{\sqrt{4\mu - \lambda^2}}{2} \times \left(\left(-C_1 \sin \frac{\sqrt{4\mu - \lambda^2}}{2}\xi + C_2 \cos \frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right) \times \left(C_1 \cos \frac{\sqrt{4\mu - \lambda^2}}{2}\xi + C_2 \sin \frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right)^{-1}\right) - \frac{\lambda}{2}. \tag{6}$$

(3) When $\lambda^2 - 4\mu = 0$,

$$\frac{G'}{G} = \frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}, \tag{7}$$

where C_1 and C_2 in above three solutions (5), (6), and (7) of (4) are integrate constants.

Step 3. Substituting solution (3) into ODE (2) using (4), we have a set of differential equations. Collecting all terms together according to the same order of G'/G, the left-hand side of ODE (2) becomes a long expression in the form of polynomial of G'/G. Making all coefficients of each order of G'/G equal to zero, the solution sets of the parameters λ, μ, a_i, and v will be got after solving the algebra equations.

Step 4. Based on the last step, we now have the solutions of the coefficient algebra equations and after substituting the parameters a_i, v, and so forth into solution (3), we could reach different types of travelling wave solutions of the PDE (1).

3. Application to the Generalized mKdV Equation

Now, we consider the generalized mKdV equation with the form [28]

$$u_t + \alpha u^{\gamma} u_x + \beta u_{xxx} = 0, \tag{8}$$

while parameters α and β are real constants. Denoting the travelling wave solution as

$$u(x,t) = u(\xi), \tag{9}$$

with $\xi = x - vt$, then the PDE (8) becomes an ODE. After integrating with single variable ξ and setting the integrate constant to zero, we have

$$-vu + \frac{\alpha}{1+\gamma} u^{1+\gamma} + \beta u'' = 0. \tag{10}$$

It can be easily seen that the standard (G'/G)-expansion cannot be applied directly to this situation because of the arbitrary power index γ, which results in the noninteger power index of G'/G. So it is necessary to introduce a transformation to deal with it via assuming that

$$u(\xi) = w^{1/\gamma}(\xi); \tag{11}$$

then (10) becomes

$$\begin{aligned}&\beta\gamma(1+\gamma)ww'' + \beta\left(1-\gamma^2\right)w'^2\\&\quad - \nu\gamma^2(1+\gamma)w^2 + \alpha\gamma^2 w^3 = 0.\end{aligned} \tag{12}$$

The balancing between the highest order nonlinear term and the highest order derivative term leads to the balance parameter $m = 2$; hence the solution (3) could be expressed as

$$w(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi) + \frac{a_{-1}}{\phi(\xi)} + \frac{a_{-2}}{\phi^2(\xi)}, \tag{13}$$

with $\phi(\xi) = G'(\xi)/G(\xi)$ and a_0, a_1, a_2, a_{-1}, and a_{-2} being constants to be determined later.

Substituting (13) into (12), making use of (4), a polynomial of $\phi(\xi)$ is obtained; a set of nonlinear algebra equations about undetermined constants a_0, a_1, a_2, a_{-1}, a_{-2}, λ, μ, and ν are reached through setting the coefficients of each order of ϕ to zero. These equations are expressed as follows.

(1) ϕ^0-order:

$$\begin{aligned}
&-\nu\left(a_0^2 - 2a_1a_{-1} - 2a_2a_{-2}\right)\gamma^3\\
&\quad+\Big\{\Big[4\left(a_1a_{-1}+4a_2a_{-2}\right)\lambda^2\\
&\qquad+\left(17a_1a_{-2}+17\mu a_2a_{-1}+a_0a_{-1}+\mu a_0a_1\right)\lambda\\
&\qquad+\left(2a_0a_2-a_1^2\right)\mu^2+8\left(a_1a_{-1}+4a_2a_{-2}\right)\mu\\
&\qquad-a_{-1}^2+2a_0a_{-2}\Big]\beta\\
&\qquad+\left(\alpha a_0^3-\nu a_0^2+6\alpha\left(a_1a_{-1}+a_2a_{-2}\right)a_0\right.\\
&\qquad+\left(3\alpha a_1^2-2\nu a_2\right)a_{-2}\\
&\qquad\left.+\left(3\alpha a_2a_{-1}^2-2\nu a_1a_{-1}\right)\right\}\gamma^2\\
&\quad+\Big[2\left(a_1a_{-1}+4a_2a_{-2}\right)\lambda^2\\
&\qquad+\left(\mu a_0a_1+a_0a_{-1}+9\mu a_2a_{-1}+9a_1a_{-2}\right)\lambda\\
&\qquad+2a_0a_2\mu^2+4\left(a_1a_{-1}+4a_2a_{-2}\right)\mu+2a_0a_{-2}\Big]\beta\gamma\\
&\quad+\Big[-2\left(a_1a_{-1}+4a_2a_{-2}\right)\lambda^2\\
&\qquad-8\left(\mu a_2a_{-1}+a_1a_{-2}\right)\lambda+\mu^2a_1^2\\
&\qquad-4\left(a_1a_{-1}+4a_2a_{-2}\right)\mu+a_{-1}^2\Big]\beta=0.
\end{aligned} \tag{14}$$

(2) ϕ^1-order:

$$\begin{aligned}
&-2\nu\left(a_0a_1+a_2a_{-1}\right)\gamma^3\\
&\quad+\Big\{\beta\left(a_0a_1+9a_2a_{-1}\right)\lambda^2\\
&\qquad+\Big[\beta\mu\left(6a_0a_1-a_1^2\right)+8\beta\left(a_1a_{-1}+4a_2a_{-2}\right)\Big]\lambda\\
&\qquad-2\beta\mu^2a_1a_2+2\beta\left(a_0a_1+9a_2a_{-1}\right)\mu\\
&\qquad+3\left(a_1^2a_{-1}+a_0^2a_1+2a_1a_2a_{-1}+2a_0a_2a_{-1}\right)\alpha\\
&\qquad-2\nu a_0a_1-2\nu a_2a_{-1}+8\beta a_1a_{-2}\Big\}\gamma^2\\
&\quad+\beta\left(\gamma a_0a_1+5\gamma a_2a_{-1}-4a_2a_{-1}\right)\lambda^2\\
&\quad+\beta\Big\{\Big[(2+\gamma)a_1^2+6\gamma a_0a_2\Big]\mu\\
&\qquad+4(\gamma-1)\left(a_1a_{-1}+4a_2a_{-2}\right)\Big\}\lambda\\
&\quad+\beta\Big[2(\gamma+2)a_1a_2\mu^2\\
&\qquad+2\left(\gamma a_0a_1+5\gamma a_2a_{-1}-4a_2a_{-1}\right)\mu\\
&\qquad+4(\gamma-1)a_1a_{-1}\Big]=0.
\end{aligned} \tag{15}$$

(3) ϕ^2-order:

$$\begin{aligned}
&\beta(1+\gamma)\left(a_1^2+4a_0a_2\right)\lambda^2\\
&\quad+\beta\Big[-(\gamma+1)(\gamma-8)a_1a_2\mu+3\gamma(1+\gamma)a_0a_1\\
&\qquad+(\gamma+1)(19\gamma-8)a_2a_{-1}\Big]\lambda\\
&\quad-2\beta(\gamma+1)(\gamma-2)a_2^2\mu^2\\
&\quad+2\beta(\gamma+1)\left(a_1^2+4\gamma a_0a_2\right)\mu\\
&\quad-\nu\gamma^3\left(2a_0a_2+a_1^2\right)\\
&\quad+\gamma^2\Big[4\beta\left(a_1a_{-1}+4a_2a_{-2}\right)\\
&\qquad-\nu\left(a_1^2+2a_0a_2\right)\\
&\qquad+3\alpha\Big(a_0a_1^2+a_0^2a_2\\
&\qquad+a_2^2a_{-2}+2a_1a_2a_{-1}\Big)\Big]=0.
\end{aligned} \tag{16}$$

(4) ϕ^3-order:

$$\begin{aligned}
&\beta a_1a_2(\gamma+4)(\gamma+1)\lambda^2+\beta(\gamma+1)\\
&\quad\times\Big[(\gamma+2)a_1^2+2(\gamma+4)\mu a_2^2+10\gamma a_0a_2\Big]\lambda\\
&\quad-2\nu\gamma^3a_1a_2\\
&\quad+\Big[2\beta\left(a_0a_1+5a_2a_{-1}\right)+2a_1a_2(\beta\mu-\nu)\\
&\qquad+\alpha\left(a_1^3+3a_2^2a_{-1}+6a_0a_1a_2\right)\Big]\gamma^2
\end{aligned}$$

$$+2\beta\left[(3\gamma-2)a_2a_{-1}+(5\gamma+4)a_1a_2\mu+\gamma a_0a_1\right]=0. \tag{17}$$

(5) ϕ^4-order:

$$\begin{aligned}&4(1+\gamma)\beta a_2^2\lambda^2+(\gamma+1)(5\gamma+8)\beta a_1a_2\lambda\\&-\nu\gamma^3a_2^2+\left[3\alpha a_2\left(a_1^2+a_0a_2\right)+\beta\left(a_1^2+6a_0a_2\right)-\nu a_2^2\right]\gamma^2\\&+2\beta\gamma\left(a_1^2+3a_0a_2+4a_2^2\mu\right)\gamma\\&+8\beta\left(a_2^2\mu+a_1^2\right)=0.\end{aligned} \tag{18}$$

(6) ϕ^5-order:

$$\begin{aligned}&\gamma^2(4\beta a_1+2\beta a_2\lambda+3\alpha a_1a_2)a_2\\&+2\gamma\beta(5\lambda a_2+4a_1)+4\beta(2\lambda a_2+a_1)a_2=0.\end{aligned} \tag{19}$$

(7) ϕ^6-order:

$$\alpha\gamma^2a_2^3+2\beta\gamma(\gamma+1)(\gamma+2)a_2^2=0. \tag{20}$$

Equations for the coefficients of ϕ^{-i} ($i\in[1,6]$) are similar to the above equations and hence not shown here.

It is straight forward to give the solution sets of the algebraic equations in different cases as follows.

Case i. All the coefficients in (13) are equal to 0, and λ, μ, and ν are arbitrary.

Case ii. All the coefficients in (13) are equal to 0 except for $a_0=(1+\gamma)\nu/\alpha$, and λ, μ, ν are arbitrary.

Case iii. All the coefficients in (13) equal to 0 except for $a_2=-2\beta(\gamma^2+3\gamma+2)/\alpha\gamma^2$.

Case iv. Consider

$$\begin{aligned}a_0&=-\frac{2\beta\left(\gamma^2+3\gamma+2\right)\mu}{\alpha\gamma^2},\\a_{-2}&=-\frac{2\beta\left(\gamma^2+3\gamma+2\right)\mu^2}{\alpha\gamma^2},\\a_1&=a_2=a_{-1}=0,\\\lambda&=0,\qquad\nu=-\frac{4\beta\mu}{\gamma^2}.\end{aligned} \tag{21}$$

Case v. Consider

$$\begin{aligned}a_1&=-\frac{2\beta\left(\gamma^2+3\gamma+2\right)\lambda}{\alpha\gamma^2},\\a_2&=-\frac{2\beta\left(\gamma^2+3\gamma+2\right)}{\alpha\gamma^2},\\a_0&=a_{-1}=a_{-2}=0,\\\mu&=0,\qquad\nu=\frac{\beta\lambda^2}{\gamma^2}.\end{aligned} \tag{22}$$

Case vi. Consider

$$\begin{aligned}a_0&=-\frac{2\beta\left(\gamma^2+3\gamma+2\right)\mu}{\alpha\gamma^2},\\a_1&=-\frac{2\beta\left(\gamma^2+3\gamma+2\right)\lambda}{\alpha\gamma^2},\\a_2&=-\frac{2\beta\left(\gamma^2+3\gamma+2\right)}{\alpha\gamma^2},\\a_{-1}&=a_{-2}=0,\qquad\nu=\frac{\beta\left(\lambda^2-4\mu\right)}{\gamma^2}.\end{aligned} \tag{23}$$

Case vii. Consider

$$\begin{aligned}a_0&=-\frac{\beta\left(\gamma^2+3\gamma+2\right)\lambda^2}{2\alpha\gamma^2},\\a_1&=-\frac{2\beta\left(\gamma^2+3\gamma+2\right)\lambda}{\alpha\gamma^2},\\a_2&=-\frac{2\beta\left(\gamma^2+3\gamma+2\right)}{\alpha\gamma^2},\\a_{-1}&=a_{-2}=0,\\\mu&=\frac{\lambda^2}{4},\qquad\nu=0.\end{aligned} \tag{24}$$

Case viii. Consider

$$\begin{aligned}a_0&=-\frac{4\beta\left(\gamma^2+3\gamma+2\right)\mu}{\alpha\gamma^2},\\a_2&=-\frac{2\beta\left(\gamma^2+3\gamma+2\right)}{\alpha\gamma^2},\\a_{-2}&=-\frac{2\beta\left(\gamma^2+3\gamma+2\right)\mu^2}{\alpha\gamma^2},\\a_1&=a_{-1}=0,\\\lambda&=0,\qquad\nu=-\frac{16\beta\mu}{\gamma^2}.\end{aligned} \tag{25}$$

Case ix. Consider

$$\begin{aligned}a_0&=-\frac{\beta\left(\gamma^2+3\gamma+2\right)\lambda^2}{2\alpha\gamma^2},\\a_{-1}&=-\frac{\beta\left(\gamma^2+3\gamma+2\right)\lambda^3}{2\alpha\gamma^2},\end{aligned}$$

$$a_{-2} = -\frac{\beta(\gamma^2+3\gamma+2)\lambda^4}{8\alpha\gamma^2},$$
$$a_1 = a_2 = 0,$$
$$\mu = \frac{\lambda^2}{4}, \qquad \nu = 0. \tag{26}$$

Case x. Consider

$$a_0 = -\frac{2\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2},$$
$$a_{-1} = -\frac{2\beta(\gamma^2+3\gamma+2)\lambda\mu}{\alpha\gamma^2},$$
$$a_{-2} = -\frac{2\beta(\gamma^2+3\gamma+2)\mu^2}{\alpha\gamma^2}, \tag{27}$$
$$a_1 = a_2 = 0, \qquad \nu = \frac{\beta(\lambda^2-4\mu)}{\gamma^2}.$$

Cases i to iii are trivial and of no interest, hence not discussed here. We focus our attention to cases from iv to x. Using solutions (21) to (27), solution (13) can be expressed in different forms corresponding to different cases listed above.

For Case iv, there are two solution types with $\xi = x + 4\beta\mu t/\gamma^2$.

(iv-1) When $\mu < 0$, we obtain the hyperbolic function solution:

$$w(\xi) = -\frac{2\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2} + \frac{2\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2} \cdot \Big(\big((C_1^2+C_2^2)\cosh^2\sqrt{-\mu}\xi + 2C_1C_2\sinh\sqrt{-\mu}\xi\cosh\sqrt{-\mu}\xi - C_2^2 \big) \times \big((C_1^2+C_2^2)\cosh^2\sqrt{-\mu}\xi + 2C_1C_2\sinh\sqrt{-\mu}\xi\cosh\sqrt{-\mu}\xi - C_1^2 \big)^{-1} \Big), \tag{28}$$

where C_1 and C_2 are integration constants. Recalling (11) we get

$$u(\xi) = \Big[-\frac{2\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2} + \frac{2\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2} \cdot \Big(\big((C_1^2+C_2^2)\cosh^2\sqrt{-\mu}\xi + 2C_1C_2\sinh\sqrt{-\mu}\xi\cosh\sqrt{-\mu}\xi - C_2^2 \big) \times \big((C_1^2+C_2^2)\cosh^2\sqrt{-\mu}\xi + 2C_1C_2\sinh\sqrt{-\mu}\xi\cosh\sqrt{-\mu}\xi - C_1^2 \big)^{-1} \Big) \Big]^{1/\gamma}. \tag{29}$$

For simplicity, we only show expression for w rather than u in the following cases.

(iv-2) When $\mu > 0$, we have the trigonometric function solution:

$$w(\xi) = -\frac{2\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2} + \frac{2\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2} \cdot \Big(\big((C_1^2-C_2^2)\cos^2\sqrt{\mu}\xi + 2C_1C_2\sin\sqrt{\mu}\xi\cos\sqrt{\mu}\xi + C_2^2 \big) \times \big((C_1^2-C_2^2)\cos^2\sqrt{\mu}\xi + 2C_1C_2\sin\sqrt{\mu}\xi\cos\sqrt{\mu}\xi - C_1^2 \big)^{-1} \Big). \tag{30}$$

For Case v, there are also two types of solution with $\xi = x - \beta\lambda^2 t/\gamma^2$.

(v-1) When $\lambda \neq 0$:

$$w(\xi) = -\frac{2\beta(\gamma^2+3\gamma+2)\lambda}{\alpha\gamma^2} \times \left(\frac{\lambda}{2} \frac{C_1\sinh(\lambda/2)\xi + C_2\cosh(\lambda/2)\xi}{C_1\cosh(\lambda/2)\xi + C_2\sinh(\lambda/2)\xi} - \frac{\lambda}{2} \right) - \frac{2\beta(\gamma^2+3\gamma+2)}{\alpha\gamma^2} \times \left(\frac{\lambda}{2} \frac{C_1\sinh(\lambda/2)\xi + C_2\cosh(\lambda/2)\xi}{C_1\cosh(\lambda/2)\xi + C_2\sinh(\lambda/2)\xi} - \frac{\lambda}{2} \right)^2. \tag{31}$$

(v-2) When $\lambda = 0$:

$$w(\xi) = -\frac{2\beta(\gamma^2+3\gamma+2)}{\alpha\gamma^2} \frac{C_2^2}{(C_1+C_2\xi)^2}. \tag{32}$$

For Case vi, there are three types of solution with $\xi = x - \beta(\lambda^2-4\mu)t/\gamma^2$.

(vi-1) When $\lambda^2 - 4\mu > 0$:

$$w(\xi) = -\frac{2\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2} - \frac{2\beta(\gamma^2+3\gamma+2)\lambda}{\alpha\gamma^2} \times \left(\frac{\sqrt{\lambda^2-4\mu}}{2} \cdot \left(\left(C_1 \sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2 \cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) \times \left(C_1 \cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2 \sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)^{-1}\right) - \frac{\lambda}{2}\right) - \frac{2\beta(\gamma^2+3\gamma+2)}{\alpha\gamma^2} \cdot \left(\frac{\sqrt{\lambda^2-4\mu}}{2} \times \left(\left(C_1 \sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2 \cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) \times \left(C_1 \cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + C_2 \sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)^{-1}\right) - \frac{\lambda}{2}\right)^2. \tag{33}$$

(vi-2) When $\lambda^2 - 4\mu < 0$:

$$w(\xi) = -\frac{2\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2} - \frac{2\beta(\gamma^2+3\gamma+2)\lambda}{\alpha\gamma^2} \times \left(\frac{\sqrt{4\mu-\lambda^2}}{2} \cdot \left(\left(-C_1 \sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2 \cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) \times \left(C_1 \cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2 \sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)^{-1}\right) - \frac{\lambda}{2}\right) - \frac{2\beta(\gamma^2+3\gamma+2)}{\alpha\gamma^2} \cdot \left(\frac{\sqrt{4\mu-\lambda^2}}{2} \times \left(\left(-C_1 \sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2 \cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) \times \left(C_1 \cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + C_2 \sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)^{-1}\right) - \frac{\lambda}{2}\right)^2. \tag{34}$$

(vi-3) When $\lambda^2 - 4\mu = 0$:

$$w(\xi) = -\frac{2\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2} - \frac{2\beta(\gamma^2+3\gamma+2)\lambda}{\alpha\gamma^2}\left(\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}\right) - \frac{2\beta(\gamma^2+3\gamma+2)}{\alpha\gamma^2}\left(\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}\right)^2. \quad (35)$$

For Case vii, we only have one type of solution:

$$w(\xi) = -\frac{\beta(\gamma^2+3\gamma+2)\lambda^2}{2\alpha\gamma^2} - \frac{2\beta(\gamma^2+3\gamma+2)\lambda}{\alpha\gamma^2}\left(\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}\right) - \frac{2\beta(\gamma^2+3\gamma+2)}{\alpha\gamma^2}\left(\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}\right)^2, \quad (36)$$

where $\xi = x$.

For Case viii, there are three types of solution with $\xi = x + 16\beta\mu t/\gamma^2$.

(viii-1) When $\mu < 0$:

$$\begin{aligned} w(\xi) &= -\frac{4\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2} + \frac{2\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2} \\ &\quad\cdot\Big(\Big(\big(C_1^2+C_2^2\big)\cosh^2\sqrt{-\mu}\xi + 2C_1C_2\sinh\sqrt{-\mu}\xi\cosh\sqrt{-\mu}\xi - C_1^2\Big) \\ &\quad\times\Big(\big(C_1^2+C_2^2\big)\cosh^2\sqrt{-\mu}\xi + 2C_1C_2\sinh\sqrt{-\mu}\xi\cosh\sqrt{-\mu}\xi - C_2^2\Big)^{-1}\Big) \\ &\quad+\frac{2\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2} \\ &\quad\cdot\Big(\Big(\big(C_1^2+C_2^2\big)\cosh^2\sqrt{-\mu}\xi + 2C_1C_2\sinh\sqrt{-\mu}\xi\cosh\sqrt{-\mu}\xi - C_2^2\Big) \\ &\quad\times\Big(\big(C_1^2+C_2^2\big)\cosh^2\sqrt{-\mu}\xi + 2C_1C_2\sinh\sqrt{-\mu}\xi\cosh\sqrt{-\mu}\xi - C_1^2\Big)^{-1}\Big). \end{aligned} \quad (37)$$

(viii-2) When $\mu > 0$:

$$\begin{aligned} w(\xi) &= -\frac{4\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2} + \frac{2\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2} \\ &\quad\cdot\Big(\Big(\big(C_1^2-C_2^2\big)\cos^2\sqrt{\mu}\xi + 2C_1C_2\sin\sqrt{\mu}\xi\cos\sqrt{\mu}\xi - C_1^2\Big) \\ &\quad\times\Big(\big(C_1^2-C_2^2\big)\cos^2\sqrt{\mu}\xi + 2C_1C_2\sin\sqrt{\mu}\xi\cos\sqrt{\mu}\xi + C_2^2\Big)^{-1}\Big) \\ &\quad+\frac{2\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2} \\ &\quad\cdot\Big(\Big(\big(C_1^2-C_2^2\big)\cos^2\sqrt{\mu}\xi + 2C_1C_2\sin\sqrt{\mu}\xi\cos\sqrt{\mu}\xi + C_2^2\Big) \\ &\quad\times\Big(\big(C_1^2-C_2^2\big)\cos^2\sqrt{\mu}\xi + 2C_1C_2\sin\sqrt{\mu}\xi\cos\sqrt{\mu}\xi - C_1^2\Big)^{-1}\Big). \end{aligned} \quad (38)$$

(viii-3) When $\mu = 0$:

$$w(\xi) = -\frac{2\beta(\gamma^2+3\gamma+2)}{\alpha\gamma^2}\left(\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}\right)^2. \quad (39)$$

For Case ix, only one type of solution exits; it is

$$w(\xi) = -\frac{\beta(\gamma^2+3\gamma+2)\lambda^2}{2\alpha\gamma^2} + \frac{\beta(\gamma^2+3\gamma+2)\lambda^3}{\alpha\gamma^2}\frac{C_1+C_2\xi}{\lambda(C_1+C_2\xi)-2C_2} - \frac{\beta(\gamma^2+3\gamma+2)\lambda^4}{2\alpha\gamma^2}\frac{(C_1+C_2\xi)^2}{[\lambda(C_1+C_2\xi)-2C_2]^2}, \quad (40)$$

where $\xi = x$.

For Case x, there are three types of solution with $\xi = x - \beta(\lambda^2-4\mu)t/\gamma^2$.

(x-1) When $\lambda^2 - 4\mu > 0$:

$$w(\xi) = -\frac{2\beta(\gamma^2+3\gamma+2)\mu}{\alpha\gamma^2} - \frac{4\beta(\gamma^2+3\gamma+2)\lambda\mu}{\alpha\gamma^2}$$

$$
\begin{aligned}
&\times\left(\left(C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)\right.\\
&\quad\times\left(\left(\sqrt{\lambda^2-4\mu}C_1-\lambda C_2\right)\right.\\
&\qquad\times\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\\
&\qquad+\left(\sqrt{\lambda^2-4\mu}C_2-\lambda C_1\right)\\
&\qquad\left.\left.\times\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)^{-1}\right)\\
&-\frac{8\beta\left(\gamma^2+3\gamma+2\right)\mu^2}{\alpha\gamma^2}\\
&\cdot\left(\left(C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)^2\right.\\
&\quad\times\left(\left[\left(\sqrt{\lambda^2-4\mu}C_1-\lambda C_2\right)\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right.\right.\\
&\qquad+\left(\sqrt{\lambda^2-4\mu}C_2-\lambda C_1\right)\\
&\qquad\left.\left.\left.\times\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right]^2\right)^{-1}\right).
\end{aligned}
\tag{41}
$$

(x-2) When $\lambda^2-4\mu<0$:

$$
\begin{aligned}
w(\xi)=&-\frac{2\beta\left(\gamma^2+3\gamma+2\right)\mu}{\alpha\gamma^2}-\frac{4\beta\left(\gamma^2+3\gamma+2\right)\lambda\mu}{\alpha\gamma^2}\\
&\times\left(\left(C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)\right.\\
&\quad\times\left(-\left(\sqrt{4\mu-\lambda^2}C_1+\lambda C_2\right)\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right.\\
&\qquad+\left(\sqrt{4\mu-\lambda^2}C_2-\lambda C_1\right)\\
&\qquad\left.\left.\times\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)^{-1}\right)\\
&-\frac{8\beta\left(\gamma^2+3\gamma+2\right)\mu^2}{\alpha\gamma^2}\\
&\cdot\left(\left(C_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+C_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)^2\right.\\
&\quad\times\left(\left[-\left(\sqrt{4\mu-\lambda^2}C_1-\lambda C_2\right)\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right.\right.\\
&\qquad+\left(\sqrt{4\mu-\lambda^2}C_2-\lambda C_1\right)\\
&\qquad\left.\left.\left.\times\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right]^2\right)^{-1}\right).
\end{aligned}
\tag{42}
$$

(x-3) When $\lambda^2-4\mu=0$:

$$
\begin{aligned}
w(\xi)=&-\frac{2\beta\left(\gamma^2+3\gamma+2\right)\mu}{\alpha\gamma^2}\\
&+\frac{4\beta\left(\gamma^2+3\gamma+2\right)\lambda\mu}{\alpha\gamma^2}\frac{C_1+C_2\xi}{\lambda\left(C_1+C_2\xi\right)-2C_2}\\
&-\frac{8\beta\left(\gamma^2+3\gamma+2\right)\mu^2}{\alpha\gamma^2}\frac{\left(C_1+C_2\xi\right)^2}{\left[\lambda\left(C_1+C_2\xi\right)-2C_2\right]^2}.
\end{aligned}
\tag{43}
$$

As an example, we consider solutions of Case (iv-1) when $C_1=0$ and $\mu=-1$; then the solution reduces to the soliton form as

$$
u(\xi)=\frac{12\beta}{\alpha}\operatorname{sech}^2\left(x-4\beta t\right)
\tag{44}
$$

for $\gamma=1$ as the standard KdV case and

$$
u(\xi)=\sqrt{\frac{6\beta}{\alpha}}\operatorname{sech}\left(x-\beta t\right)
\tag{45}
$$

for the standard mKdV case when $\gamma=2$. We show the diagrams of Case (iv-1) in Figure 1 to illustrate the behaviors of the solutions for different power index γ. We choose coefficients of the generalized mKdV equation as $\alpha=6$ and $\beta=1$; the constants of the solutions C_1 and C_2 equal 0.5 and 2, respectively, while the coefficient μ in the equation equals -0.2. It is found that (i) this kind of solution is a type of soliton solution; the amplitude decreases with the increase of power index γ; (ii) it gives that the width of the wave packet broadened when increasing γ; (iii) in addition, we find that the velocity of the wave slowed down when γ is bigger.

4. Summary and Conclusion

In this paper, we use the modified (G'/G)-expansion method to construct some types of solutions of the mKdV equation

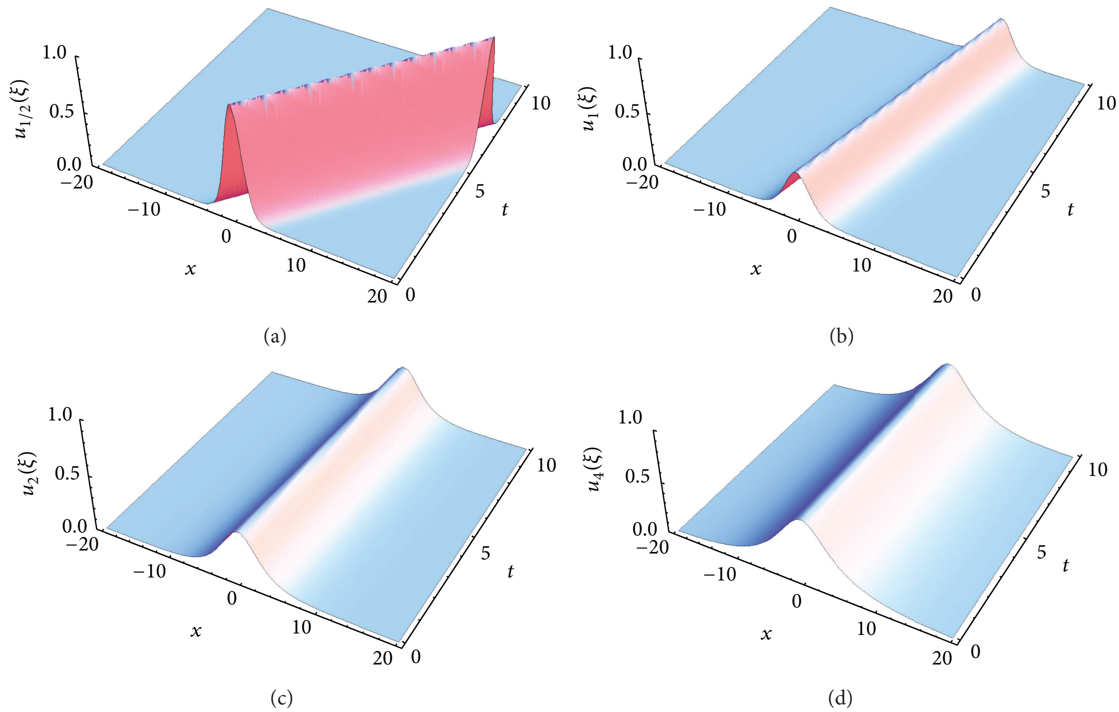

Figure 1: Figures of solutions of Case (iv-1). The solution $u(\xi)$ evolutes with spatial coordinate x and time t. The subscript of u indicates the four kinds of different power index γ of the generalized mKdV equation.

through the introduction of a proper transformation. Some new solutions are given, including the hyperbolic, trigonometric, and rational function solutions. It is shown that using the modified (G'/G)-expansion method we can deal with the nonlinear evolution equations effectively and directly and abundant solutions could be obtained.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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