Hindawi Publishing Corporation
ISRN Astronomy and Astrophysics
Volume 2013, Article ID 293298, 5 pages
http://dx.doi.org/10.1155/2013/293298

Research Article

Dust Fluid Cosmological Model

Anita Bagora (Menaria)[1] and Rakeshwar Purohit[2]

[1] *Department of Mathematics, Jaipur National University, Jaipur 302025, India*
[2] *Department of Mathematics and Statistics, University College of Science, Mohanlal Sukhadia University, Udaipur 313001 (Raj.), India*

Correspondence should be addressed to Anita Bagora (Menaria); anita_bagora@yahoo.com

Received 30 November 2012; Accepted 10 January 2013

Academic Editors: S. Bogovalov and H. Dehnen

Bianchi type I tilted bulk viscous fluid cosmological model filled with dust fluid is investigated. We assume that $\zeta\theta = K$ (constant), where ζ is the coefficient of bulk viscosity and θ is the expansion in the model. It has been assumed that the expansion in the model is only in two directions; that is, one of the components of Hubble parameters ($H_1 = A_4/A$) is zero. The physical and geometrical aspects of the model in the presence and absence of bulk viscosity are also discussed. Also, we have discussed two special models and their physical properties. From this, we present a particular example based on dust fluid.

1. Introduction

Homogeneous and anisotropic cosmological models have been studied widely in the framework of general relativity. These models are more restricted than the inhomogeneous models. But in spite of this, they explain a number of observed phenomena quite satisfactorily. In recent years, there has been a considerable interest in investigating spatially homogeneous and anisotropic cosmological models in which matter does not move orthogonal to the hypersurface of homogeneity. Such types of models are called tilted cosmological models.

The general dynamics of these cosmological models have been studied in detail by King and Ellis [1], Ellis and King [2], and Collins and Ellis [3]. Ellis and Baldwin [4] have investigated that we are likely to be living in a tilted universe and they have indicated how we may detect it. King and Ellis [1] have found that there is no Bianchi type I tilted model if it has been obtained under the assumption that the matter takes the perfect fluid form

$$T_{ij} = (\epsilon + p)\, v_i v_j + p g_i^j,$$
$$v_i v^j = -1 \qquad \epsilon > 0,\ p > 0, \tag{1}$$

where v^i is the velocity flow vector and ϵ, p are the density and pressure of the fluid, respectively.

A realistic treatment of the problem requires the consideration of material distribution other than the perfect fluid. It is well known that in early stages of the universe when radiation in the form of photon as well as neutrino decoupled from matter, it behaved like a viscous fluid. The possibilities of bulk viscosity-driven inflationary solution of full Israel-Stewart theory in different cases are discussed by Zimdahl [5]. Murphy [6] has considered a Robertson-Walker universe with cosmological fluid possessing bulk viscosity, and the coefficient of bulk viscosity has been assumed to be proportional to the density. He has found a nonsingular solution of the Einstein's field equation claiming that the initial singularity can be avoidable by introducing the concept of bulk viscosity. Although Belinskii and Khalatnikov [7] later criticized Murphy's model for corresponding to very peculiar parameter choices, the model has nevertheless continued to attract interest during the years.

After the introduction by Guth [8] of the inflationary cosmological model, it has been pointed out that sources for viscosity are present in the cosmic continuum as the inflation inducing phase transition starts. Bulk viscosity associated with the grand unified theory (GUT) phase transition can lead to sufficient inflation (exponential or generalized), independent of the details of the phase transition, provided it is small [9]. A great amount of work has been done in order to study Bianchi type I cosmological models with both linear and nonlinear viscosity. An extensive review of the subject can be found in the paper of Gron [10], who has studied and carried out further the research on viscous cosmological models. Mainly he has studied inflationary cosmological models of Bianchi type I with shear, bulk, and nonlinear viscosity to a great extent.

Bali [11] discussed expanding and rotating magneto viscous fluid cosmological model in general relativity. He has obtained a cosmological model of Bianchi type I in which the distribution consists of an electrically neutral viscous fluid with an infinite electrical conductivity in the pressure of magnetic field. Patel and Koppar [12] derived four cosmological models having nonzero expansion and shear. One of them has nonzero constant shear viscosity coefficient.

Bianchi type I models with bulk and shear viscosity and an equation of state state $p = (\gamma - 1)\rho$, with $1 \leq \gamma \leq 2$ were investigated by Belinskii and Khalatnikov [13]. Models with the coefficient of shear viscosity proportional to the density have been analyzed by Banerjee and Santos [14]. Tilted Bianchi type I cosmological models in the presence a perfect and bulk viscous fluid with heat flow are derived by Pradhan and Pandey [15] and Pradhan and Rai [16], respectively.

Beside this, in general relativity, a dust solution is an exact solution of the Einstein field equation in which the gravitational field is produced entirely by the mass, momentum, and stress density of a perfect fluid which has positive mass energy density but vanishing pressure. Bali and Sharma [17] have investigated tilted Bianchi type I dust fluid and shown that model has cigar-type singularity when $T = 0$. Concerning the tilted perfect fluid models, Bradley [18] have stated that there does not exist tilted dust self-similar model. Bagora et al. [19, 20] have discussed tilted Bianchi type I cosmological models in different context with dust fluid.

Motivated by these researches, we have studied tilted homogeneous cosmological model for dust fluid in the presence and absence of bulk viscosity. It has been shown that the tilted nature of the model is preserved due to bulk viscosity. We assume that $\zeta\theta = K$ (constant), where ζ is the coefficient of bulk viscosity and θ is the expansion in the model. It has been assumed that the expansion in the model is only in two directions; that is, one of the components of Hubble parameters ($H_1 = A_4/A$) is zero. The physical and geometrical aspects of the model in the presence and absence of bulk viscosity are also discussed.

2. The Metric and Field Equations

We consider the Bianchi type I metric in the form

$$ds^2 = -dt^2 + dx^2 + B^2 dy^2 + C^2 dz^2, \quad (2)$$

where B and C are functions of "t" alone.

The energy-momentum tensor for perfect fluid distribution with heat conduction given by Ellis [21] and for bulk viscosity given by Landau and Lifshitz [22] is given by

$$T_i^j = (\rho + p) v_i v^j + p g_i^j + q_i v^j + v_i q^j - \zeta\theta \left(g_i^j + v_i v^j \right), \quad (3)$$

together with

$$g_{ij} v^i v^j = -1, \quad (4)$$

$$\begin{aligned} q_i q^j &> 0, \\ q_i v^i &= 0, \end{aligned} \quad (5)$$

In the above, p is the isotropic pressure, ρ the matter density, and q_i the heat conduction vector orthogonal to v^i. The fluid flow vector v^i has the components $(\sinh\lambda, 0, 0, \cosh\lambda)$ satisfying (4), with λ being the tilt angle.

The Einstein's field equation is

$$R_i^j - \frac{1}{2} R g_i^j = -8\pi T_i^j, \quad (c = G = 1). \quad (6)$$

The field equation for the line element (2) leads to

$$\begin{aligned} \frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} &= -8\pi \left[(\rho + p)\sinh^2\lambda \right. \\ &\quad \left. + p + 2q_1 \sinh\lambda - K\cosh^2\lambda \right], \end{aligned} \quad (7)$$

$$\frac{C_{44}}{C} = -8\pi (p - K), \quad (8)$$

$$\frac{B_{44}}{B} = -8\pi (p - K), \quad (9)$$

$$\begin{aligned} \frac{B_4 C_4}{BC} &= -8\pi \left[-(\rho + p)\cosh^2\lambda + p \right. \\ &\quad \left. - 2q_1 \sinh\lambda - K\sinh^2\lambda \right], \end{aligned} \quad (10)$$

$$\begin{aligned} &(\rho + p)\sinh\lambda\cosh\lambda + q_1\cosh\lambda \\ &\quad + q_1 \frac{\sinh^2\lambda}{\cosh\lambda} - K\sinh\lambda\cosh\lambda = 0, \end{aligned} \quad (11)$$

where the suffix "4" stands for ordinary differentiation with respect to the cosmic time "t" alone.

3. Solutions of Field Equations

Equations from (7) to (11) are five equations in six unknown B, C, ρ, p, q_1, and λ. For the complete determination of these quantities, we assume that the model is filled with dust fluid which leads to

$$p = 0. \quad (12)$$

Also, we assume that

$$\zeta\theta = K. \quad (13)$$

The condition $\zeta\theta = K$ is due to the peculiar characteristic of the bulk viscosity. It acts like a negative energy field in an expanding universe (Johri and Sudarshan [23]); that is, $\zeta\theta = K$. According to that, expansion is inversely proportional to bulk viscosity.

From (7) and (10), we have

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{2B_4C_4}{BC} = 8\pi\left[\rho - p + K\right]. \tag{14}$$

Using the condition of dust fluid from (12) in (14), we have

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{2B_4C_4}{BC} = 8\pi\rho + 8\pi K. \tag{15}$$

Equations (8) and (9) lead to

$$\frac{B_{44}}{B} - \frac{C_{44}}{C} = 0. \tag{16}$$

Let us assume that

$$BC = \mu, \qquad \frac{B}{C} = \nu. \tag{17}$$

With the help of (17), (16) leads to

$$\frac{\nu_4}{\nu} = \frac{a}{\mu}, \tag{18}$$

where "a" is a constant of integration.

Again (8) and (9) lead to

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} = -16\pi p + 16\pi K. \tag{19}$$

Using condition (12) in (19), we have

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} = 16\pi K. \tag{20}$$

Using (17) and (18) in (20), we have

$$2\mu_{44} - \frac{\mu_4^2}{\mu} = \frac{-a^2}{\mu} + 32\pi K\mu. \tag{21}$$

This leads to

$$\mu_4^2 = a^2 + 32\pi K\mu^2 + b\mu, \tag{22}$$

where "b" is a constant of integration and $\mu_4 = f(\mu)$.

Equation (18) leads to

$$\log \nu = \int \frac{a\,d\mu}{\mu\sqrt{a^2 + 32\pi K\mu^2 + b\mu}}. \tag{23}$$

Hence, the metric (2) reduces to the form

$$ds^2 = -\frac{d\mu^2}{f^2} + dx^2 + \mu\nu dy^2 + \frac{\mu}{\nu}dz^2. \tag{24}$$

This leads to

$$ds^2 = \frac{-dT^2}{a^2 + 32\pi KT^2 + bT} + dX^2 + T\nu\, dY^2 + \frac{T}{\nu}dZ^2, \tag{25}$$

where $x = X$, $y = Y$, $z = Z$, and ν is determined by (23) with $\mu = T$.

In the absence of bulk viscosity the metric (25) reduces to

$$ds^2 = \frac{-dT^2}{a^2 + bT} + dX^2 + T\nu\, dY^2 + \frac{T}{\nu}dZ^2, \tag{26}$$

where ν is determined by (23) with $\mu = T$ and $K = 0$.

4. Some Physical and Geometrical Features

The matter density ρ for the model (25) is given by

$$8\pi\rho = \frac{1}{2T}\left(b + 48\pi KT\right). \tag{27}$$

The tilt angle λ is given by

$$\cosh\lambda = \sqrt{\frac{48\pi KT + b}{32\pi KT}}, \qquad \sinh\lambda = \sqrt{\frac{16\pi KT + b}{32\pi KT}}. \tag{28}$$

The scalar of expansion θ calculated for the flow vector v^i for the model (25) is given by

$$\theta = \frac{(96\pi KT + b)}{8T^2}\sqrt{\frac{\left(a^2 + 32\pi KT^2 + bT\right)}{2\pi K\left(48\pi KT + b\right)}}. \tag{29}$$

The components of flow vector v^i and heat conduction vector q_i for the model (25) are given by

$$\begin{gathered} v^1 = \sqrt{\frac{16\pi KT + b}{32\pi KT}}, \\ v^4 = \sqrt{\frac{48\pi KT + b}{32\pi KT}}, \\ q_1 = \frac{-\left(b + 48\pi KT\right)}{128\pi^{3/2}K^{3/2}}\sqrt{\frac{b + 16\pi KT}{2T}}, \\ q_4 = \frac{b + 16\pi KT}{128\pi^{3/2}K^{3/2}}\sqrt{\frac{b + 48\pi KT}{2T}}. \end{gathered} \tag{30}$$

The nonvanishing components of shear tensor (σ_{ij}) and rotation tensor (ω_{ij}) are given by

$$\begin{gathered} \sigma_{11} = \frac{-\left(b + 24\pi KT\right)}{192}\sqrt{\frac{\left(b + 48\pi KT\right)\left(a^2 + 32\pi KT^2 + bT\right)}{\pi^3K^3T^5}}, \\ \sigma_{14} = \frac{-\left(b + 24\pi KT\right)}{192}\sqrt{\frac{\left(b + 16\pi KT\right)\left(a^2 + 32\pi KT^2 + bT\right)}{\pi^3K^3T^5}}, \\ \omega_{14} = \frac{1}{8T^{3/2}}\sqrt{\frac{a^2 + 32\pi KT^2 + bT}{2\pi K\left(b + 16\pi KT\right)}}. \end{gathered} \tag{31}$$

Thus,

$$\sigma_{11}v^1 + \sigma_{14}v^4 = 0. \tag{32}$$

Similarly,

$$\omega_{11}v^1 + \omega_{14}v^4 = 0. \tag{33}$$

The physical significance of conditions (32) and (33) is explained by Ellis [24]. The shear tensor (σ_{ij}) determines the distortion arising in the fluid flow, leaving the volume invariant. The direction of principal axis is unchanged by the distortion, but all other directions are changed. Thus, we have $\sigma_{ij}v^j = 0$, which leads to

$$\sigma_{11}v^1 + \sigma_{14}v^4 = 0 \quad (\because v_1 \neq 0, v_4 \neq 0). \tag{34}$$

Shear (σ) is given by

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij}. \tag{35}$$

Thus, $\sigma^2 \geq 0$ and $\sigma = 0 \Leftrightarrow \sigma_{ij} = 0$.

The vorticity tensor (ω_{ij}) determines a rigid rotation of cluster of galaxies with respect to a local inertial rest frame. Thus, we have

$$\omega_{ij} = \eta_{ijk\ell}\omega^k\omega^\ell, \tag{36}$$

where $\eta_{ijk\ell}$ is pseudotensor and $\omega^i = (1/2)\eta_{ijk\ell}v_j\omega_{k\ell}$.

Thus, $\omega_{ij}v^j = 0$.

This leads to

$$\omega_{11}v^1 + \omega_{14}v^4 = 0 \quad (\because v_1 \neq 0, v_4 \neq 0). \tag{37}$$

The magnitude of ω_{ij} is ω and is defined as

$$\omega^2 = \frac{1}{2}\omega_{ij}\omega^{ij}. \tag{38}$$

Also $\omega = 0 \Leftrightarrow \omega_{ij} = 0$.

4.1. Special Model I. When the bulk viscosity is absent, that is, $K = 0$, (22) reduces to

$$\mu_4^2 = a^2 + b\mu. \tag{39}$$

Putting $a^2 + b\mu = \xi$ in (39), we have

$$\mu = \frac{b\left(t^2 - M^2\right)}{4}, \tag{40}$$

where $M = 2a/b$.

Again by (23), we have

$$v = N\left(\frac{tb + 2a}{tb - 2a}\right), \tag{41}$$

where "N" is a constant of integration.

The metric (2) reduces to

$$ds^2 = \frac{-dT^2}{a^2 + bT} + dX^2 + \frac{Nb\left(t^2 - M^2\right)(tb + 2a)}{4(tb - 2a)}dY^2 + \frac{b\left(t^2 - M^2\right)(tb - 2a)}{4N(tb + 2a)}dZ^2. \tag{42}$$

For the model (42), density ρ and tilt angle λ are given by

$$\begin{aligned}\rho &= \frac{2}{8\pi\left(t^2 - M^2\right)},\\ \cosh\lambda &= \frac{1}{b}\sqrt{\frac{b^2 - 4a^2}{2\left(1 - t^2\right)}},\\ \sinh\lambda &= \frac{1}{b}\sqrt{\frac{2b^2t^2 - b^2 - 4a^2}{2\left(1 - t^2\right)}}.\end{aligned} \tag{43}$$

The components of flow vector v^i and heat conduction vector q_i for the model (42) are given by

$$\begin{aligned}v^1 &= \sqrt{\frac{16\pi KT + b}{32\pi KT}},\\ v^4 &= \sqrt{\frac{48\pi KT + b}{32\pi KT}},\\ q_1 &= \frac{-(b + 48\pi KT)}{128\pi^{3/2}K^{3/2}}\sqrt{\frac{b + 16\pi KT}{2T}},\\ q_4 &= \frac{b + 16\pi KT}{128\pi^{3/2}K^{3/2}}\sqrt{\frac{b + 48\pi KT}{2T}}.\end{aligned} \tag{44}$$

The scalar of expansion θ and the nonvanishing components of (σ_{ij}) and (ω_{ij}) are given by

$$\begin{aligned}\theta &= \frac{-t\left(4a^2 - 2b^2 + b^2t^2\right)}{b\left(1 - t^2\right)\left(b^2t^2 - 4a^2\right)}\sqrt{\frac{b^2 - 4a^2}{2\left(1 - t^2\right)}},\\ \sigma_{11} &= \frac{t\left(b^2 - 4a^2\right)\left(2b^2t^2 - b^2 - 4a^2\right)}{3b^3\left(1 - t^2\right)^2\left(b^2t^2 - 4a^2\right)}\sqrt{\frac{b^2 - 4a^2}{2\left(1 - t^2\right)}},\\ \sigma_{14} &= \frac{-t\left(b^2 - 4a^2\right)\left(2b^2t^2 - b^2 - 4a^2\right)}{3b^3\left(1 - t^2\right)^2\left(b^2t^2 - 4a^2\right)}\sqrt{\frac{2b^2t^2 - b^2 - 4a^2}{2\left(1 - t^2\right)}},\\ \omega_{14} &= \frac{\left(b^2 - 4a^2\right)\left(3b^2 - 2b^2t^2 - 4a^2\right)}{4b^3\left(1 - t^2\right)^2\sqrt{2\left(2b^2t^2 - b^2 - 4a^2\right)\left(1 - t^2\right)}}.\end{aligned} \tag{45}$$

4.2. Special Model II. When bulk viscosity is present then we find the model in terms of "t"; for this we assume that $b = 0$, $a = 0$.

Then we have

$$\begin{aligned}\mu &= k_1 e^{(\sqrt{32\pi K})t},\\ v &= \text{constant}.\end{aligned} \tag{46}$$

The metric (2) becomes

$$ds^2 = -dt^2 + dx^2 + k_1k_2e^{(\sqrt{32\pi K})t}dy^2 + \frac{k_1e^{(\sqrt{32\pi K})t}}{k_2}dz^2. \tag{47}$$

The expansion θ is given by

$$\theta = \frac{\sqrt{32\pi K}}{2\left(32\pi K e^{(\sqrt{32\pi K})t} + 1\right)^{3/2}} \sqrt{\frac{1}{2}}. \tag{48}$$

The tilt angle λ is given by

$$\cosh\lambda = \sqrt{\frac{1}{2\left(32\pi K e^{(\sqrt{32\pi K})t} + 1\right)}}. \tag{49}$$

Here, at $t = 0$, the expansion is constant and at $t = \infty$ it vanishes, that is, zero.

5. Discussion

The model starts with a big bang from its initial stage at $T = 0$ and continues to expand till $T = \infty$. The model has point-type singularity at $T = 0$ (MacCallum [25]). The model represents shearing and rotating universe in general and rotation goes on decreasing as time increases. Since $\mathrm{Lim}_{T\to\infty}\sigma/\theta \neq 0$, then the model does not approach isotropy for large values of T. Density $\rho \to 0$ as $T \to \infty$ and $\rho \to \infty$ as $T \to 0$. When $T \to 0$, $q^1 \to \infty$ and $q^4 \to \infty$. Also q^1 and q^4 tend to zero as $T \to 0$. At $T = 0$, the Hubble parameters tend to infinite at the time of initial singularity of vanishing as $T \to \infty$.

The effect of bulk viscosity is to produce change in perfect fluid and exhibit essential influence on the character of the solution. All the physical characters of the model (42) remain finite and regular for the entire range of variable $-\infty < t < \infty$. Therefore, this model is free of singularity and in this case the model has singularity-free solution. Also this model is throughout a rotating model.

References

[1] A. R. King and G. F. R. Ellis, "Tilted homogeneous cosmological models," *Communications in Mathematical Physics*, vol. 31, pp. 209–242, 1973.

[2] G. F. R. Ellis and A. R. King, "Was the big bang a whimper?" *Communications in Mathematical Physics*, vol. 38, no. 2, pp. 119–156, 1974.

[3] C. B. Collins and G. F. R. Ellis, "Singularities in Bianchi cosmologies," *Physics Reports*, vol. 56, no. 2, pp. 63–105, 1979.

[4] G. F. R. Ellis and J. E. Baldwin, "On the expected anisotropy of radio source content," *Monthly Notices of the Royal Astronomical Society*, vol. 206, pp. 377–381, 1984.

[5] W. Zimdahl, "Bulk viscous cosmology," *Physical Review D*, vol. 53, no. 10, pp. 5483–5493, 1996.

[6] G. L. Murphy, "Big-bang model without singularities," *Physical Review D*, vol. 8, no. 12, pp. 4231–4233, 1973.

[7] V. A. Belinskii and I. M. Khalatnikov, "Viscosity effects in isotropic cosmologies," *Soviet Physics*, vol. 45, no. 1, pp. 1–9, 1977.

[8] A. H. Guth, "Inflationary universe: a possible solution to the horizon and flatness problems," *Physical Review D*, vol. 23, no. 2, pp. 347–356, 1981.

[9] I. I. Waga, R. C. Falcão, and R. Chanda, "Bulk-viscosity-driven inflationary model," *Physical Review D*, vol. 33, no. 6, pp. 1839–1841, 1986.

[10] O. Gron, "Viscous inflationary universe models," *Astrophysics and Space Science*, vol. 173, no. 2, pp. 191–225, 1990.

[11] R. Bali, "Expanding and shearing magneto-viscous fluid cosmological model in general relativity," *Astrophysics and Space Science*, vol. 116, no. 1, pp. 81–88, 1985.

[12] L. K. Patel and S. S. Koppar, "Some Bianchi type VI_0 viscous fluid cosmological models," *Journal of the Australian Mathematical Society B*, vol. 33, no. 1, pp. 77–84, 1991.

[13] V. A. Belinskii and I. M. Khalatnikov, "Effect of viscosity on nature of cosmological evolution," *Soviet Physics*, vol. 42, no. 2, pp. 205–210, 1976.

[14] A. K. Banerjee and N. O. Santos, "Spatially homogeneous cosmological models," *General Relativity and Gravitation*, vol. 16, no. 3, pp. 217–224, 1984.

[15] A. Pradhan and O. P. Pandey, "Tilted Bianchi type I universe for barotropic perfect fluid in general relativity," *Spacetime and Substance*, vol. 5, no. 4, pp. 149–153, 2004.

[16] A. Pradhan and A. Rai, "Tilted Bianchi type I cosmological models filled with disordered radiation in general relativity revisited," *Astrophysics and Space Science*, vol. 286, no. 3-4, pp. 347–358, 2003.

[17] R. Bali and K. Sharma, "Tilted Bianchi type I dust fluid cosmological model in general relativity," *Pramana*, vol. 58, no. 3, pp. 457–463, 2002.

[18] M. Bradley, "Dust EPL cosmologies," *Classical and Quantum Gravity*, vol. 5, pp. L15–L19, 1988.

[19] G. S. Rathore and A. Bagora, "Magnetized dust fluid tilted universe for perfect fluid distribution in general relativity," *Advanced Studies in Theoretical Physics*, vol. 2, no. 17, pp. 817–824, 2008.

[20] A. Bagora, G. S. Rathore, and P. Bagora, "Tilted Bianchi type I dust fluid magnetized cosmological model in general relativity," *Turkish Journal of Physics*, vol. 33, no. 3, pp. 167–177, 2009.

[21] G. F. R. Ellis, in *General Relativity and Cosmology*, R. K. Sachs, Ed., p. 116, Academy Press, New York, NY, USA, 1971.

[22] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, vol. 6, Pergamon Press, Oxford, UK, 1963.

[23] V. B. Johri and R. Sudarshan, "FRW cosmological models with bulk viscosity and entropy production," in *Proceedings of the International Conference on Mathematical Modelling in Science and Technology*, vol. 2, Tata McGraw-Hill, 1988.

[24] G. F. R. Ellis, in *General Relativity and Cosmology*, R. K. Sachs, Ed., p. 113, Academy Press, New York, NY, USA, 1971.

[25] M. A. H. MacCallum, "A class of homogeneous cosmological models III: asymptotic behaviour," *Communications in Mathematical Physics*, vol. 20, no. 1, pp. 57–84, 1971.