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Research Article

Approximate Solution for the Electrohydrodynamic Flow in a Circular Cylindrical Conduit

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This paper considers the nonlinear boundary value problem (BVP) for the electrohydrodynamic flow of a fluid in an ion drag configuration in a circular cylindrical conduit. The velocity field was solved using the new homotopy perturbation method (NHPM), considering the electrical field and strength of the nonlinearity. The approximate analytical procedure depends only on two components and polynomial initial condition. The analytical solution is obtained and the numerical results presented graphically. The effects of the Hartmann electric number $\mathcal{H}a$ and the strength of nonlinearity α are discussed and presented graphically. We also compare this method with numerical solution (N.S) and show that the present approach is less computational and is applicable for solving nonlinear boundary value problem (BVP).

1. Introduction

The electrohydrodynamic flow of a fluid in an "ion drag" configuration in a circular cylindrical conduit is governed by a nonlinear second-order ordinary differential equation [1–3]

$$\frac{d^2w(r)}{dr^2}+\frac{1}{r}\frac{dw(r)}{dr}+\mathcal{H}a^2\left(1-\frac{w(r)}{1-\alpha w(r)}\right)=0, \qquad 0<r<1 \tag{1}$$

subject to the boundary conditions

$$w'(0)=0, \qquad w(1)=0, \tag{2}$$

where $w(r)$ is the fluid velocity, r is the radial distance from the center of the cylindrical conduit, $\mathcal{H}a$ is the Hartmann electric number, and the parameter α is a measure of the strength of the nonlinearity. In [1] McKee and his colleagues developed perturbation solutions in terms of the parameter α governing a nonlinear problem. McKee and his coworkers used a Gauss-Newton finite-difference solver combined with the continuation method and Runge-Kutta shooting method to provide numerical results for the fluid velocity over a large range of values of α. This was done for both large and small values of α. Paullet [2] proved the existence and uniqueness of a solution of BVP of electrohydrodynamic flow and in addition, discovered an error in the perturbative and numerical solutions given in [1] for large values of α. Very recently Mastroberardino [3] presented the approximate solution by homotopy analysis method (HAM) for the nonlinear BVP governed by electrohydrodynamic flow of a fluid in a circular cylindrical conduit.

In the present paper, we introduce a new computational method, namely, new homotopy perturbation method [4–6] for solving electrohydrodynamic flow of a fluid in a circular cylindrical conduit. It is interesting to note that the efficiency of the approach depends only on two components of the homotopy series. The method is an improvement of classical homotopy perturbation method [7–12]. In contrast to the HAM and HPM, in this method, it is not required to solve the

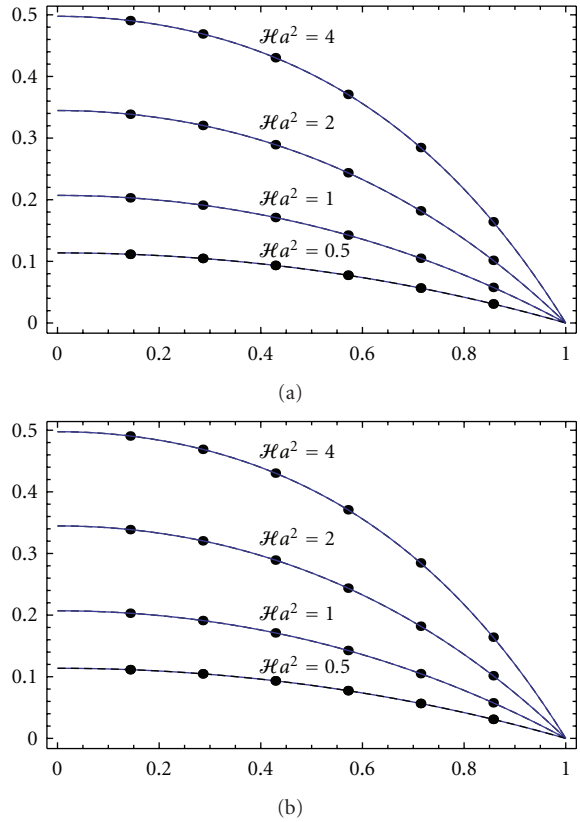

Figure 1: Comparison of NHPM solution (solid line) and numerical solution r versus w (a) $\alpha = 1$, (b) $\alpha = 0.5$.

functional equations in each iteration. Unlike the Adomian decomposition method (ADM) [13], the NHPM is free from the need to use Adomian polynomials.

2. Analysis of the Method

Let us consider the nonlinear differential equation

$$\mathcal{A}(u) = f(z), \quad z \in \Omega, \tag{3}$$

where $\mathcal{A}$ is an operator, f is a known function, and u is a sought function. Assume that operator $\mathcal{A}$ can be written as

$$\mathcal{A}(u) = \mathcal{L}(u) + \mathcal{N}(u), \tag{4}$$

where $\mathcal{L}$ is the linear operator and $\mathcal{N}$ is the nonlinear operator. Hence, (3) can be rewritten as follows:

$$\mathcal{L}(u) + \mathcal{N}(u) = f(z), \quad z \in \Omega. \tag{5}$$

We define an operator H as

$$H(v;p) \equiv (1-p)(\mathcal{L}(v) - \mathcal{L}(u_0)) + p(\mathcal{A}(v) - f), \tag{6}$$

where $p \in [0,1]$ is an embedding or homotopy parameter, $v(z;p) : \Omega \times [0,1] \to \Re$, and u_0 is an initial approximation of solution of the problem in (3). Equation (6) can be written as

$$H(v;p) \equiv \mathcal{L}(v) - \mathcal{L}(u_0) + p\mathcal{L}(u_0) + p(\mathcal{N}(v) - f(z)) = 0. \tag{7}$$

We assume that the solution of equation $H(v,p)$ can be written as a power series in embedding parameter p, as follows:

$$v = v_0 + pv_1. \tag{8}$$

Now, let us write (7) in the following form:

$$\mathcal{L}(v) = u_0(z) + p(f(z) - \mathcal{N}(v) - u_0(z)). \tag{9}$$

By applying the inverse operator, $\mathcal{L}^{-1}$ to both sides of (9), we have

$$v = \mathcal{L}^{-1}u_0(z) + p(\mathcal{L}^{-1}f(z) - \mathcal{L}^{-1}\mathcal{N}(v) - \mathcal{L}^{-1}u_0(z)). \tag{10}$$

Suppose that the initial approximation of (3) has the form

$$u_0(z) = \sum_{n=0}^{\infty} a_n P_n(z), \tag{11}$$

where a_n, $n = 0,1,2,\ldots$ are unknown coefficients and $P_n(z)$, $n = 0,1,2,\ldots$ are specific functions on the problem. By substituting (8) and (11) into (10), we get

$$\begin{aligned} v_0 + v_1 p = {} & \mathcal{L}^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(z)\right) \\ & + p\left(\mathcal{L}^{-1}f(z) - \mathcal{L}^{-1}\mathcal{N}(v_0 + pv_1) \right. \\ & \left. - \mathcal{L}^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(z)\right)\right). \end{aligned} \tag{12}$$

Equating the coefficients of like powers of p, we get following set of equations:

$$\begin{aligned} & \text{Coefficient of } p^0 : v_0 = \mathcal{L}^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(z)\right), \\ & \text{Coefficient of } p^1 : v_1 = \mathcal{L}^{-1}(f(z)) - \mathcal{L}^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(z)\right) \\ & \qquad - \mathcal{L}^{-1}\mathcal{N}(v_0). \end{aligned} \tag{13}$$

Now, we solve these equations in such a way that $v_1(z) = 0$. Therefore, the approximate solution may be obtained as

$$u(z) = v_0(z) = \mathcal{L}^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(z)\right). \tag{14}$$

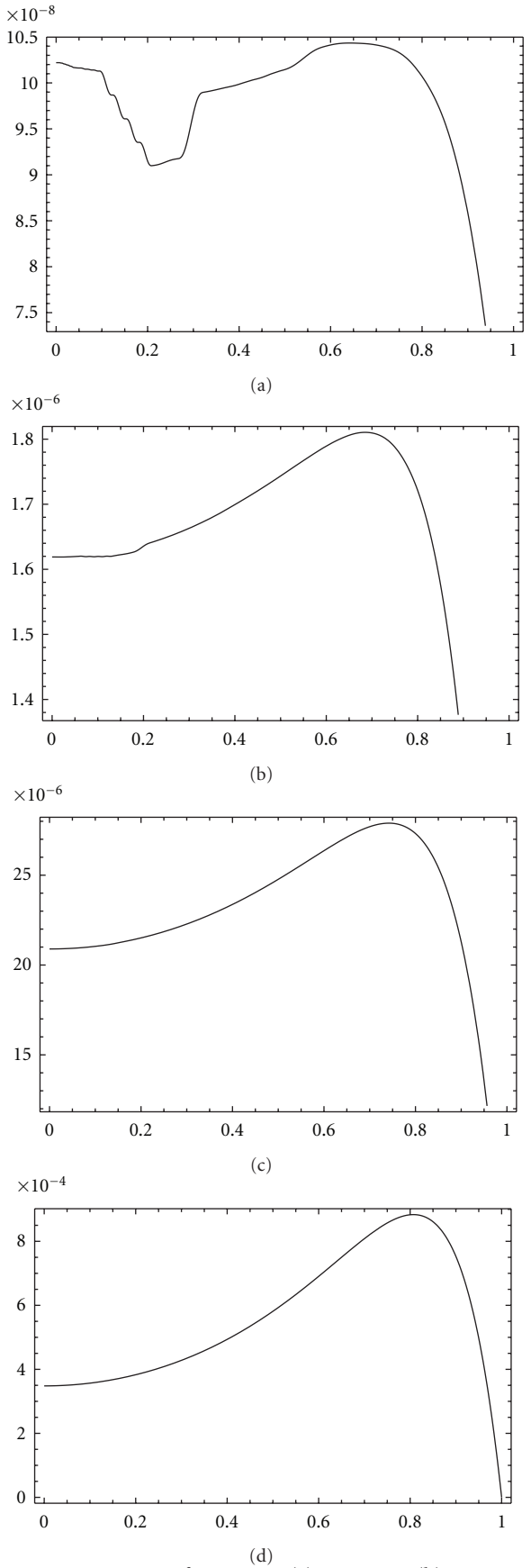

Figure 2: Error curves for $\alpha = 0.5$ (a) $\mathcal{H}a = 0.5$ (b) $\mathcal{H}a = 1$ (c) $\mathcal{H}a = 2$ (d) $\mathcal{H}a = 4$.

3. Analytical Solution

To obtain the solution of (1) by NHPM, we construct the following homotopy:

$$\begin{aligned}(1-p)&\left(\frac{d^2W(r)}{dr^2}-W_0(r)\right)\\&+p\left(\frac{d^2W(r)}{dr^2}+\frac{1}{r}\frac{dW(r)}{dr}\right.\\&\qquad-\alpha W(r)\left(\frac{d^2W(r)}{dr^2}+\frac{1}{r}\frac{dW(r)}{dr}\right)\\&\left.\qquad+\mathcal{H}a^2(1-(1+\alpha)W(r))\right)=0.\end{aligned}\tag{15}$$

Applying the inverse operator, $\mathcal{L}^{-1}(\bullet)=\int_0^r\int_0^\xi(\bullet)d\eta d\xi$ to the both sides of (15), we obtain

$$\begin{aligned}W(r)=W(0)&+rW'(0)+\int_0^r\int_0^\xi w_0(\eta)d\eta d\xi\\&-p\int_0^r\int_0^\xi\left(w_0(\eta)-\frac{1}{r}\frac{dW(r)}{dr}\right.\\&\qquad+\alpha W(r)\left(\frac{d^2W(r)}{dr^2}+\frac{1}{r}\frac{dW(r)}{dr}\right)\\&\left.\qquad-\mathcal{H}a^2\left(1-(1+\alpha)W(r)\right)\right)d\eta d\xi.\end{aligned}\tag{16}$$

The solution of (16) to have the following form:

$$W(r)=W_0(r)+pW_1(r).\tag{17}$$

Substituting (17) in (16) and equating the coefficients of like powers of p, we get following set of equations:

$$W_0(r)=W(0)+rW'(0)+\int_0^r\int_0^\xi w_0(\eta)d\eta d\xi\tag{18}$$

$$\begin{aligned}W_1(r)=\int_0^r\int_0^\xi&\left(-w_0(\eta)\right.\\&+\left(\frac{1}{r}\frac{dW_0(r)}{dr}-\alpha W_0(r)\right.\\&\qquad\left.\left.\times\left(\frac{d^2W_0(r)}{dr^2}+\frac{1}{r}\frac{dW_0(r)}{dr}\right)\right)\right)d\eta d\xi\\&+\int_0^r\int_0^\xi(\mathcal{H}a^2(1-(1+\alpha)W_0(r)))d\eta d\xi.\end{aligned}\tag{19}$$

Assuming $w_0(r)=\sum_{n=0}^{7}a_nP_n$, $P_k=r^k$, $a=W(0)$, solving the above equation for $W_1(r)$ leads to the result

$$\begin{aligned}W_1(r)=&\left(-a_0-\frac{3a}{2}-a_0+\frac{2aa_1}{2}\right)r^2\\&+\left(-\frac{a_1}{4}+\frac{aa_1}{8}\right)r^4+\cdots.\end{aligned}\tag{20}$$

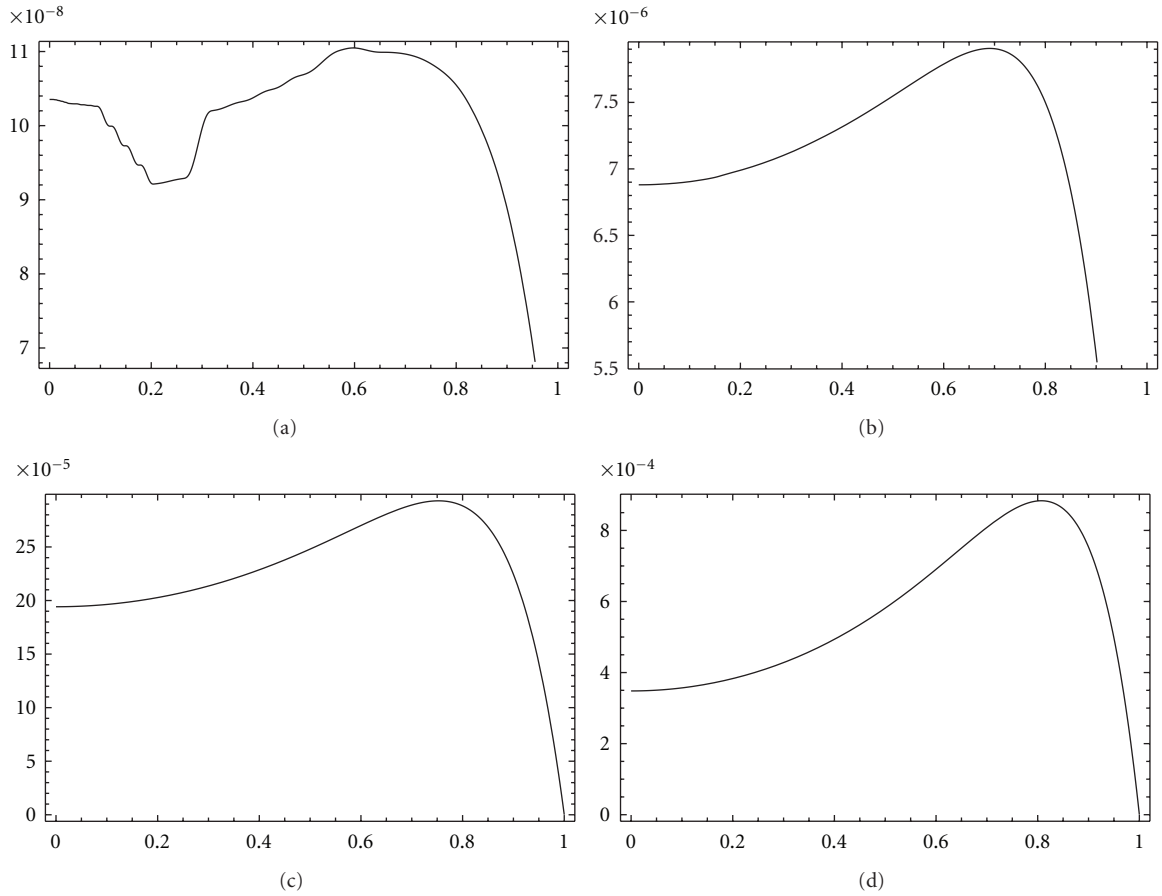

Figure 3: Error curves for $\alpha = 1$ (a) $\mathcal{H}a = 0.5$ (b) $\mathcal{H}a = 1$ (c) $\mathcal{H}a = 2$ (d) $\mathcal{H}a = 4$.

With vanishing $W_1(r)$, we have the following values for coefficients $a_i,\ i = 0, 1, \ldots, 7$

$$a_0 = \frac{-\mathcal{H}a^2(-1 + a + \alpha a)}{2(-1 + \alpha a)},$$
$$a_1 = \frac{-2\mathcal{H}a^4(-1 + a + \alpha a)}{16(-1 + \alpha a)^4}, \ldots. \tag{21}$$

Therefore, we obtain the solutions of (1) as

$$w(r) = a + \frac{\mathcal{H}a^2(-1 + a + \alpha a)}{4(1 - \alpha a)} r^2 - \frac{\mathcal{H}a^4(-1 + a + \alpha a)}{64(-1 + \alpha a)^3} r^4 + \cdots. \tag{22}$$

4. Numerical Results and Concluding Remarks

In this paper we have studied electrohydrodynamic flow of a fluid in an ion drag configuration in a circular cylindrical conduit by using two-component homotopy perturbation method. Figures 1(a) and 1(b) and Table 1 clearly show that the results by NHPM are in good agreement with the results of numerical solution (N.S). The main interest in this section is to investigate the effects of Hartmann electric number $\mathcal{H}a$ and the strength of nonlinearity α on the velocity emerging in the electrohydrodynamics flows. For all of the cases considered, the maximum difference between the analytical solution and the numerical solution was determined to be less than 10^{-3} as shown in Figures 2 and 3. Unlike the Adomian decomposition method (ADM), the NHPM is free from the need to use Adomian polynomials. In this method we do not need the Lagrange multiplier, correction functional, stationary conditions, and calculating integrals, which eliminate the complications that exist in the variational iteration method VIM. In contrast to the HPM and HAM, in this method, it is not required to solve the functional equations in each iteration. The efficiency of HAM is very much depending on choosing auxiliary parameter $\hbar$.

It has been noted that the nonlinearity confronted in this problem is in the form of a rational function and, thus, poses a significant challenge in regard to obtaining analytical solutions. Despite this fact, we have shown that the solutions obtained are convergent and that they compare extremely well with numerical solutions (N.S). It is interesting to note that NHPM yields convergent solutions for all of the cases considered. However, HPM yields divergent solutions for

Table 1

α	0.5	0.5	0.5	0.5	1	1	1	1
$\mathcal{H}a^2$	0.5	1	2	4	0.5	1	2	4
a N.S	0.1137	0.2070	0.3447	0.4975	0.1132	0.2034	0.3254	0.4123
a NHPM	0.1132	0.2034	0.3447	0.4975	0.1132	0.2034	0.3252	0.4297

all of the cases considered. The NHPM improves the performance of standard HPM. It was shown that NHPM requires less computational work and less consuming time when compared with the standard HPM.

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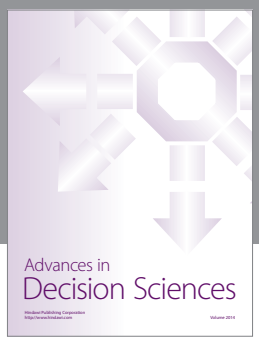

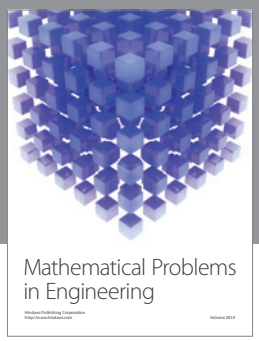

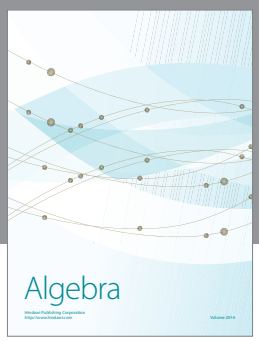

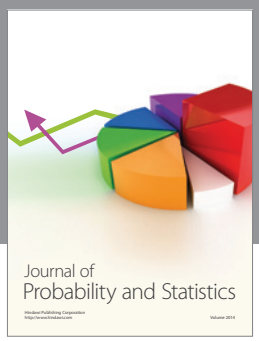

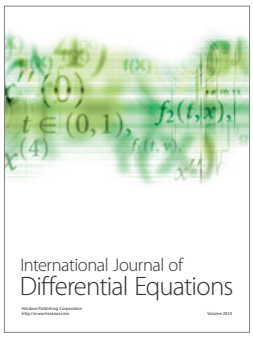

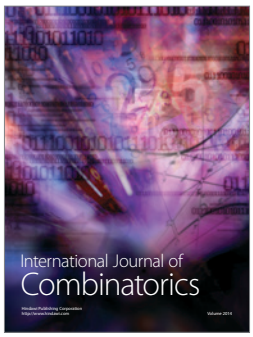

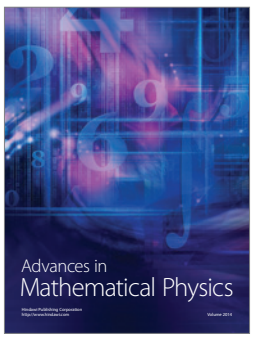

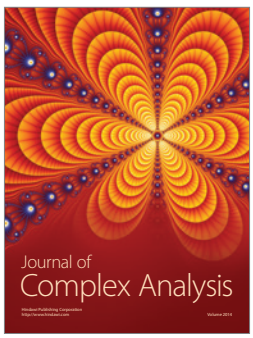

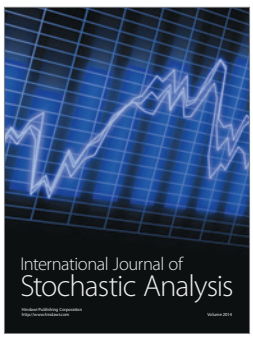

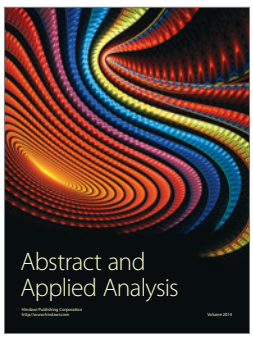

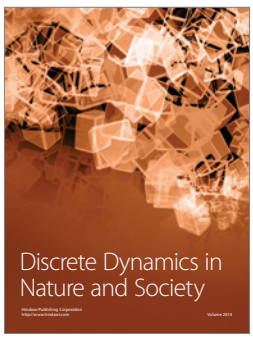

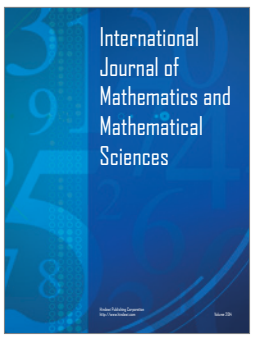

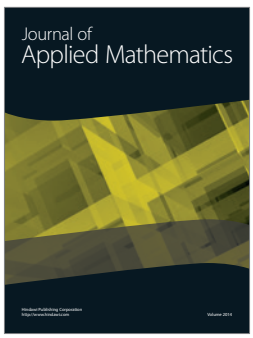

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