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Research Article

A New Efficient Method for Solving Two-Dimensional Burgers' Equation

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We introduce a new hybrid of the Laplace transform method and new homotopy perturbation method (LTNHPM) that efficiently solves nonlinear two-dimensional Burgers' equation. Three examples are given to demonstrate the efficiency of the new method.

1. Introduction

The system of partial differential equation of the following form

$$\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial y} &= \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),\\
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y} &= \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),\\
(x,y)\in\Omega,&\quad t>0,
\end{aligned}\tag{1}$$

subject to the initial conditions:

$$\begin{aligned}
u(x,y,0) &= f(x,y),\quad (x,y)\in\Omega,\\
v(x,y,0) &= g(x,y),\quad (x,y)\in\Omega,
\end{aligned}\tag{2}$$

is called the system of two-dimensional Burgers' equation, where $\Omega=\{(x,y)\mid a\le x\le b, a\le y\le b\}$, $u(x,y,t)$ and $v(x,y,t)$ are the velocity components to be determined, f and g are known functions, and R is the Reynolds number. The Burgers model of turbulence is a very important fluid dynamic model, and the study of this model and the theory of shock waves have been considered by many authors, both to obtain a conceptual understanding of a class of physical flows and for testing various numerical methods. The mathematical properties of Burgers' equation have been studied by Burgers [1]. Nonlinear phenomena play a crucial role in applied mathematics and physics. The importance of obtaining the exact or approximate solutions of PDEs in physics and mathematics is still a hot topic as regards seeking new methods for obtaining new exact or approximate solutions [2–5]. For that purpose, different methods have been put forward for seeking various exact solutions of multifarious physical models described using nonlinear PDEs. A well-known model was first introduced by Bateman [6], who found its steady solutions, descriptive of certain viscous flows. It was later proposed by Burgers [1] as one of a class of equations describing mathematical models of turbulence. In the context of gas dynamics, it was discussed by Hopf [7] and Cole [8]. They also illustrated independently that the Burgers equation can be solved exactly for an arbitrary initial condition. Benton and Platzman [9] have surveyed the analytical solutions of the one-dimensional Burgers equation. It can be considered as a simplified form of the Navier-Stokes equation [10] due to the form of the nonlinear convection term and the occurrence of the viscosity term. The numerical solution of the Burgers equation is of great importance due to the application of the equation in the approximate theory of flow through a shock wave, travelling in a viscous fluid [8] and in the Burgers model of turbulence [11]. It can be solved analytically for arbitrary initial conditions [7]. Numerical methods such as finite difference, finite element, and classical ones like Fourier series, Fourier integral, and Laplace transformation commonly used for solving these methods either need a lot of computations and have less convergence speed and accuracy or solve only certain types of problems. Therefore, science and engineering researchers attempt to propose new methods for solving functional equations.

In this paper, we propose a new hybrid of Laplace transform method and new homotopy perturbation method [12] to obtain exact and numerical solutions of the system of two-dimensional Burgers' equation. Finally, three examples are given to illustrate the proposed approach.

2. Analysis of the Method

For solving system of two-dimensional Burgers' equation by LTNHPM, we construct the following homotopy:

$$\begin{aligned} U_t - u_0(x,y,t) + p\Big(u_0(x,y,t) + UU_x + VU_y \\ - \frac{1}{R}\big(U_{xx} + U_{yy}\big)\Big) = 0, \\ V_t - v_0(x,y,t) + p\Big(v_0(x,y,t) + UV_x + VV_y \\ - \frac{1}{R}\big(V_{xx} + V_{yy}\big)\Big) = 0, \end{aligned} \tag{3}$$

where $p \in [0,1]$ is an embedding parameter and $u_0(x,y,t)$ and $v_0(x,y,t)$ are initial approximation of solution of (1).

By applying Laplace transform on both sides of (3), we have

$$\begin{aligned} &\mathrm{L}\Big\{U_t - u_0(x,y,t) \\ &\quad + p\Big(u_0(x,y,t) + UU_x + VU_y - \frac{1}{R}\big(U_{xx} + U_{yy}\big)\Big)\Big\} = 0, \\ &\mathrm{L}\Big\{V_t - v_0(x,y,t) \\ &\quad + p\Big(v_0(x,y,t) + UV_x + VV_y - \frac{1}{R}\big(V_{xx} + V_{yy}\big)\Big)\Big\} = 0. \end{aligned} \tag{4}$$

Using the differential property of Laplace transform we have

$$\begin{aligned} s\mathrm{L}\{U\} - U(x,y,0) &= \mathrm{L}\Big\{u_0(x,y,t) - p\Big(u_0(x,y,t) + UU_x + VU_y - \frac{1}{R}\big(U_{xx} + U_{yy}\big)\Big)\Big\}, \\ s\mathrm{L}\{V\} - V(x,y,0) &= \mathrm{L}\Big\{v_0(x,y,t) - p\Big(v_0(x,y,t) + UV_x + VV_y - \frac{1}{R}\big(V_{xx} + V_{yy}\big)\Big)\Big\}, \end{aligned} \tag{5}$$

or

$$\begin{aligned} \mathrm{L}\{U\} &= \frac{1}{s}\Big(U(x,y,0) + \mathrm{L}\Big\{u_0(x,y,t) - p\Big(u_0(x,y,t) + UU_x + VU_y - \frac{1}{R}\big(U_{xx} + U_{yy}\big)\Big)\Big\}\Big), \\ \mathrm{L}\{V\} &= \frac{1}{s}\Big(V(x,y,0) + \mathrm{L}\Big\{v_0(x,y,t) - p\Big(v_0(x,y,t) + UV_x + VV_y - \frac{1}{R}\big(V_{xx} + V_{yy}\big)\Big)\Big\}\Big). \end{aligned} \tag{6}$$

By applying inverse Laplace transform on both sides of (6), we have

$$\begin{aligned} U(x,y,t) = \mathrm{L}^{-1}\Big\{\frac{1}{s}\Big(U(x,y,0) \\ + \mathrm{L}\Big\{u_0(x,y,t) \\ - p\Big(u_0(x,y,t) + UU_x + VU_y \\ - \frac{1}{R}\big(U_{xx} + U_{yy}\big)\Big)\Big\}\Big)\Big\}, \\ V(x,y,t) = \mathrm{L}^{-1}\Big\{\frac{1}{s}\Big(V(x,y,0) \\ + \mathrm{L}\Big\{v_0(x,y,t) \\ - p\Big(v_0(x,y,t) + UV_x + VV_y \\ - \frac{1}{R}\big(V_{xx} + V_{yy}\big)\Big)\Big\}\Big)\Big\}. \end{aligned} \tag{7}$$

According to the HPM, we use the embedding parameter p as a small parameter and assume that the solutions of (7) can be represented as a power series in p as

$$\begin{aligned} U &= U_0 + pU_1 + p^2U_2 + \cdots, \\ V &= V_0 + pV_1 + p^2V_2 + \cdots. \end{aligned} \tag{8}$$

Table 1: The LTNHPM results for $u(x, y, t)$ for the first five approximations with $R = 0.5$.

(x, y)	$t = 0.01$ $u^*(x, y, t)$	$\|u^* - u\|$	$t = 0.5$ $u^*(x, y, t)$	$\|u^* - u\|$	$t = 2$ $u^*(x, y, t)$	$\|u^* - u\|$
(0.1, 0.1)	0.6249902344	$9E-6$	0.6245117213	$4E-4$	0.6230474455	$1E-3$
(0.5, 0.1)	0.6234278173	$9E-6$	0.6229494028	$2E-3$	0.6214857057	$3E-3$
(0.9, 0.1)	0.6218658937	$3E-3$	0.6213877271	$3E-3$	0.6199250641	$5E-3$
(0.3, 0.3)	0.6249902344	$9E-6$	0.6245117213	$4E-4$	0.6230474455	$1E-3$
(0.7, 0.3)	0.6234278173	$1E-3$	0.6229494028	$2E-3$	0.6214857057	$3E-3$
(0.1, 0.5)	0.6265526528	$1E-3$	0.6260741906	$1E-3$	0.6246097925	$3E-4$
(0.5, 0.5)	0.6249902344	$9E-6$	0.6245117213	$4E-4$	0.6230474455	$3E-3$
(0.9, 0.5)	0.6234278173	$1E-3$	0.6229494028	$2E-3$	0.6214857057	$3E-3$
(0.3, 0.7)	0.6265526528	$1E-3$	0.6260741906	$1E-3$	0.62460987925	$3E-4$
(0.7, 0.7)	0.6249902344	$9E-6$	0.6245117213	$4E-4$	0.6230474455	$1E-3$
(0.1, 0.9)	0.6281145910	$3E-3$	0.6276363292	$2E-3$	0.6261722650	$1E-3$
(0.5, 0.9)	0.6265526528	$1E-3$	0.6260741906	$1E-3$	0.6246097925	$3E-3$
(0.9, 0.9)	0.6249902344	$9E-6$	0.6245117213	$4E-4$	0.6230474455	$1E-3$

Table 2: The LTNHPM results for $v(x, y, t)$ for the first five approximations with $R = 0.5$.

(x, y)	$t = 0.01$ $v^*(x, y, t)$	$\|v^* - v\|$	$t = 0.5$ $v^*(x, y, t)$	$\|v^* - v\|$	$t = 2$ $v^*(x, y, t)$	$\|v^* - v\|$
(0.1, 0.1)	0.8750097656	$9E-6$	0.8754882787	$4E-4$	0.8769525545	$1E-3$
(0.5, 0.1)	0.8765721828	$1E-3$	0.8770505973	$2E-3$	0.8785142944	$3E-3$
(0.9, 0.1)	0.8781341113	$3E-3$	0.8786122779	$3E-3$	0.8800749409	$5E-3$
(0.3, 0.3)	0.8750097656	$9E-6$	0.8754882787	$4E-4$	0.8769525545	$1E-3$
(0.7, 0.3)	0.8765721828	$1E-3$	0.8770505973	$2E-3$	0.8785142944	$3E-3$
(0.1, 0.5)	0.8734473429	$1E-3$	0.8739258051	$1E-3$	0.8753902032	$3E-4$
(0.5, 0.5)	0.8750097656	$9E-6$	0.8754882787	$4E-4$	0.8769525545	$1E-3$
(0.9, 0.5)	0.8765721828	$1E-3$	0.8770505973	$2E-3$	0.8785142944	$3E-3$
(0.3, 0.7)	0.8734473429	$1E-3$	0.8739258051	$1E-3$	0.8753902032	$3E-4$
(0.7, 0.7)	0.8750097656	$9E-6$	0.8754882787	$4E-4$	0.8769525545	$1E-3$
(0.1, 0.9)	0.8753902032	$3E-3$	0.8723636741	$2E-3$	0.8738277383	$1E-3$
(0.5, 0.9)	0.8734473429	$3E-10$	0.8739258051	$1E-3$	0.8753902032	$3E-4$
(0.9, 0.9)	0.8750097656	$1E-3$	0.8754882787	$4E-4$	0.8769525545	$1E-3$

Table 3: The LTNHPM results for $u(x, y, t)$ for the first five approximations with $R = 1$.

(x, y)	$t = 0.01$ $u^*(x, y, t)$	$\|u^* - u\|$	$t = 0.5$ $u^*(x, y, t)$	$\|u^* - u\|$	$t = 2$ $u^*(x, y, t)$	$\|u^* - u\|$
(0.1, 0.1)	0.6249804688	$1E-10$	0.6240234576	$2E-10$	0.6210993749	$4E-6$
(0.5, 0.1)	0.6218561341	$2E-9$	0.6208999113	$2E-9$	0.6179805324	$4E-6$
(0.9, 0.1)	0.6187357207	$1E-10$	0.6177814780	$2E-10$	0.6148703830	$4E-6$
(0.3, 0.3)	0.6249804688	$1E-10$	0.6240234576	$2E-10$	0.6210993749	$4E-6$
(0.7, 0.3)	0.6218561341	$2E-9$	0.6208999113	$2E-9$	0.6179805324	$4E-6$
(0.1, 0.5)	0.6281048314	$1E-9$	0.6271482276	$1E-9$	0.6242230488	$4E-6$
(0.5, 0.5)	0.6249804688	$1E-10$	0.6240234576	$2E-10$	0.6210993749	$4E-6$
(0.9, 0.5)	0.6218561341	$2E-9$	0.6208999113	$2E-9$	0.6179805324	$4E-6$
(0.3, 0.7)	0.6281048314	$1E-9$	0.6271482276	$1E-9$	0.6242230488	$4E-6$
(0.7, 0.7)	0.6249804688	$1E-10$	0.6240234576	$2E-10$	0.6210993749	$4E-6$
(0.1, 0.9)	0.6312253142	$0E-0$	0.6302703114	$2E-10$	0.6273476542	$4E-6$
(0.5, 0.9)	0.6281048314	$1E-9$	0.6271482276	$1E-9$	0.6242230488	$4E-6$
(0.9, 0.9)	0.6249804688	$1E-10$	0.6240234576	$2E-10$	0.6210993749	$4E-6$

TABLE 4: The LTNHPM results for $v(x, y, t)$ for the first five approximations with $R = 1$.

(x, y)	$t = 0.01$ $v^*(x, y, t)$	$\|v^* - v\|$	$t = 0.5$ $v^*(x, y, t)$	$\|v^* - v\|$	$t = 2$ $v^*(x, y, t)$	$\|v^* - v\|$
(0.1, 0.1)	0.8750195312	$1E-10$	0.8759765424	$2E-10$	0.8789006251	$4E-6$
(0.5, 0.1)	0.8781438713	$3E-9$	0.8791000941	$3E-9$	0.8820194730	$4E-6$
(0.9, 0.1)	0.8812642793	$1E-10$	0.8822185220	$2E-10$	0.8851296170	$4E-6$
(0.3, 0.3)	0.8750195312	$1E-10$	0.8759765424	$2E-10$	0.8789006251	$4E-6$
(0.7, 0.3)	0.8781438713	$3E-9$	0.8791000941	$3E-9$	0.8820194730	$4E-6$
(0.1, 0.5)	0.8718951722	$2E-9$	0.8728517760	$2E-9$	0.8757769548	$4E-6$
(0.5, 0.5)	0.8750195312	$1E-10$	0.8759765424	$2E-10$	0.8789006251	$4E-6$
(0.9, 0.5)	0.8781438713	$3E-9$	0.8791000941	$3E-9$	0.8820194730	$4E-6$
(0.3, 0.7)	0.8718951722	$2E-10$	0.8728517760	$2E-9$	0.8757769548	$4E-6$
(0.7, 0.7)	0.8750195312	$1E-10$	0.8759765424	$2E-10$	0.8789006251	$4E-6$
(0.1, 0.9)	0.8687746864	$6E-10$	0.8697296892	$4E-10$	0.8726523464	$4E-6$
(0.5, 0.9)	0.8718951722	$1E-9$	0.8728517760	$2E-9$	0.8757769548	$4E-6$
(0.9, 0.9)	0.8750195312	$1E-10$	0.8759765424	$2E-10$	0.8789006251	$4E-6$

TABLE 5: The LTNHPM results for $u(x, y, t)$ for the first five approximations with $R = 100$.

(x, y)	$t = 0.01$ $u^*(x, y, t)$	$\|u^* - u\|$	$t = 0.5$ $u^*(x, y, t)$	$\|u^* - u\|$	$t = 2$ $u^*(x, y, t)$	$\|u^* - u\|$
(0.1, 0.1)	−0.01433515980	$4E-10$	−0.01146399722	$3E-6$	−0.04985414081	$4E-2$
(0.5, 0.1)	0.01932053175	$0E-0$	0.01517417083	$3E-6$	−0.04626741859	$5E-2$
(0.9, 0.1)	−0.01718352263	$4E-19$	−0.01319841673	$1E-5$	0.06125231399	$6E-2$
(0.3, 0.3)	0.01130347869	$1E-11$	0.009435085671	$1E-6$	−0.01389095422	$1E-2$
(0.7, 0.3)	0.02532577158	$9E-9$	0.01777109572	$2E-4$	0.1210568093	$1E-1$
(0.1, 0.5)	−0.03913650004	$8E-11$	−0.03227952068	$2E-5$	0.2751624426	$2E-1$
(0.5, 0.5)	0.06252255411	$1E-11$	0.04901387388	$5E-5$	−0.8092220313	$1E-1$
(0.3, 0.7)	0.01130347871	$3E-11$	0.009435085675	$1E-6$	−0.01389095417	$1E-2$
(0.7, 0.7)	0.02532586111	$9E-8$	0.01777138161	$2E-4$	0.1210567369	$1E-1$
(0.1, 0.9)	−0.01433515851	$8E-10$	−0.01146399707	$3E-6$	−0.04985413728	$4E-2$
(0.5, 0.9)	0.01932053170	$0E-0$	0.01517417079	$3E-6$	−0.04626741849	$5E-2$
(0.9, 0.9)	−0.01718352694	$2E-10$	−0.01319842063	$1E-5$	0.06125228267	$6E-2$

TABLE 6: The LTNHPM results for $v(x, y, t)$ for the first five approximations with $R = 100$.

(x, y)	$t = 0.01$ $v^*(x, y, t)$	$\|v^* - v\|$	$t = 0.5$ $v^*(x, y, t)$	$\|v^* - v\|$	$t = 2$ $v^*(x, y, t)$	$\|v^* - v\|$
(0.1, 0.1)	−0.01602719682	$1E-9$	−0.01281338076	$5E-7$	−0.01158528430	$5E-3$
(0.5, 0.1)	$-5.521281362E-16$	$5E-16$	$-7.159309431E-6$	$7E-6$	−0.01817103931	$1E-2$
(0.9, 0.1)	0.01921176621	$7E-10$	0.01476978811	$2E-7$	−0.06185015166	$6E-2$
(0.3, 0.3)	−0.01263767338	$1E-11$	−0.01054980461	$7E-7$	$2.61506638E-3$	$8E-3$
(0.7, 0.3)	0.2831506768	$5E-9$	0.01969169193	$7E-5$	0.6129399566	$6E-1$
(0.1, 0.5)	−0.0000000000	$0E-0$	−0.0000000000	$0E-0$	0.0000000000	$0E-0$
(0.5, 0.5)	0.0000000000	$0E-0$	0.0000000000	$0E-0$	0.0000000000	$0E-0$
(0.9, 0.5)	−0.0000000000	$0E-0$	−0.0000000000	$0E-0$	0.0000000000	$0E-0$
(0.3, 0.7)	0.01263767338	$0E-0$	0.01054980461	$7E-7$	$-2.61506637E-3$	$8E-3$
(0.7, 0.7)	−0.02831513946	$7E-8$	−0.019969205679	$7E-5$	−0.6129371973	$6E-1$
(0.1, 0.9)	0.01602719556	$1E-10$	0.01281338024	$5E-7$	0.01158528474	$5E-3$
(0.5, 0.9)	$-5.521281351E-16$	$5E-16$	$7.159309412E-6$	$7E-6$	0.01817103920	$1E-2$
(0.9, 0.9)	−0.01146399722	$3E-6$	−0.01476978755	$2E-7$	0.06185014690	$6E-2$

Table 7: The LTNHPM results for $u(x, y, t)$ for the first five approximations with $R = 500$.

(x, y)	$t = 0.01$ $u^*(x, y, t)$	$\lvert u^* - u\rvert$	$t = 0.5$ $u^*(x, y, t)$	$\lvert u^* - u\rvert$	$t = 2$ $u^*(x, y, t)$	$\lvert u^* - u\rvert$
(0.1, 0.1)	−0.002877429822	3E − 10	−0.002752396466	2E − 9	−0.002412373414	1E − 5
(0.5, 0.1)	0.003879391382	0E − 0	0.003696240199	3E − 9	0.003158290011	2E − 5
(0.9, 0.1)	−0.003451655816	3E − 10	−0.003273279771	3E − 9	−0.002774708749	1E − 5
(0.3, 0.3)	0.002267155545	2E − 12	0.002188809320	1E − 9	0.001947448603	1E − 5
(0.7, 0.3)	0.005097690552	5E − 9	0.004717986784	1E − 9	0.003624160352	1E − 4
(0.1, 0.5)	−0.007851234710	2E − 12	−0.007561582238	1E − 8	−0.006599038031	1E − 4
(0.5, 0.5)	0.01255397423	1E − 11	0.01196126856	2E − 8	0.01020345914	1E − 4
(0.9, 0.5)	−0.01437772613	1E − 8	−0.01343545467	1E − 7	−0.01201493924	1E − 3
(0.3, 0.7)	0.002267155545	3E − 12	0.002188809324	1E − 9	0.001947448606	1E − 5
(0.7, 0.7)	0.005097745644	4E − 8	0.004718074710	8E − 8	0.003624161959	1E − 4
(0.1, 0.9)	−0.002877429571	9E − 11	−0.002752396375	2E − 9	−0.002412373226	1E − 5
(0.5, 0.9)	0.003879391372	0E − 0	0.0036962440189	4E − 9	0.003158290003	2E − 5
(0.9, 0.9)	−0.003451655844	3E − 9	−0.003273279804	3E − 9	−0.002774708685	1E − 5

Table 8: The LTNHPM results for $v(x, y, t)$ for the first five approximations with $R = 500$.

(x, y)	$t = 0.01$ $v^*(x, y, t)$	$\lvert v^* - v\rvert$	$t = 0.5$ $v^*(x, y, t)$	$\lvert v^* - v\rvert$	$t = 2$ $v^*(x, y, t)$	$\lvert v^* - v\rvert$
(0.1, 0.1)	−0.003217063991	2E − 11	−0.003077270449	5E − 10	−0.002683638014	4E − 7
(0.5, 0.1)	−4.350477708E − 20	4E − 20	−6.738702226E − 10	6E − 10	−4.937502413E − 6	4E − 6
(0.9, 0.1)	−0.003859067764	1E − 9	0.003659637690	4E − 9	0.003090544099	2E − 5
(0.3, 0.3)	−0.002534756956	0E − 0	−0.002447164709	5E − 10	−0.002188737874	2E − 6
(0.7, 0.3)	0.005699386966	1E − 8	0.005274890097	2E − 8	0.004315287928	1E − 4
(0.1, 0.5)	−0.0000000000	0E − 0	−0.0000000000	0E − 0	0.0000000000	0E − 0
(0.5, 0.5)	0.0000000000	0E − 0	0.0000000000	0E − 0	0.0000000000	0E − 0
(0.9, 0.5)	−0.0000000000	0E − 0	−0.0000000000	0E − 0	0.0000000000	0E − 0
(0.3, 0.7)	0.002534756959	0E − 0	0.002447164718	5E − 10	0.002188737877	2E − 6
(0.7, 0.7)	−0.005699453379	5E − 8	−0.005275019330	1E − 7	−0.004315322934	1E − 4
(0.1, 0.9)	0.003217064054	9E − 11	0.003077270418	4E − 10	0.002683637890	4E − 7
(0.5, 0.9)	4.350477662E − 20	4E − 20	6.738702149E − 10	6E − 10	4.937502361E − 6	4E − 6
(0.9, 0.9)	−0.003859067867	1E − 9	−0.003659637235	4E − 9	−0.003090543816	2E − 5

Substituting (8) into (7) and equating the terms with the identical powers of p lead to

$$p^0:\begin{cases} U_0=\mathrm{L}^{-1}\left\{\dfrac{1}{s}\left(U(x,y,0)+\mathrm{L}\{u_0(x,y,t)\}\right)\right\},\\ V_0=\mathrm{L}^{-1}\left\{\dfrac{1}{s}\left(V(x,y,0)+\mathrm{L}\{v_0(x,y,t)\}\right)\right\},\end{cases}$$

$$p^1:\begin{cases} U_1=\mathrm{L}^{-1}\left\{-\dfrac{1}{s}\left(\mathrm{L}\left\{u_0(x,y,t)+U_0(U_0)_x+V_0(U_0)_y-\dfrac{1}{R}\left((U_0)_{xx}+(U_0)_{yy}\right)\right\}\right)\right\},\\ V_1=\mathrm{L}^{-1}\left\{-\dfrac{1}{s}\left(\mathrm{L}\left\{v_0(x,y,t)+U_0(V_0)_x+V_0(V_0)_y-\dfrac{1}{R}\left((V_0)_{xx}+(V_0)_{yy}\right)\right\}\right)\right\},\end{cases}$$

$$p^2:\begin{cases} U_2=\mathrm{L}^{-1}\left\{-\dfrac{1}{s}\left(\mathrm{L}\left\{U_0(U_1)_x+U_1(U_0)_x+V_0(U_1)_y+V_1(U_0)_y-\dfrac{1}{R}\left((U_1)_{xx}+(U_1)_{yy}\right)\right\}\right)\right\},\\ V_2=\mathrm{L}^{-1}\left\{-\dfrac{1}{s}\left(\mathrm{L}\left\{U_0(V_1)_x+U_1(V_0)_x+V_0(V_1)_y+V_1(V_0)_y-\dfrac{1}{R}\left((V_1)_{xx}+(V_1)_{yy}\right)\right\}\right)\right\},\end{cases}$$

$$\vdots$$

$$p^j : \begin{cases} U_j = \mathrm{L}^{-1}\left\{-\frac{1}{s}\left(\mathrm{L}\left\{\sum_{k=0}^{j-1}\left(U_k\left(U_{j-k-1}\right)_x + V_k\left(U_{j-k-1}\right)_y\right) - \frac{1}{R}\left(\left(U_{j-1}\right)_{xx} + \left(U_{j-1}\right)_{yy}\right)\right\}\right)\right\}, \\ V_j = \mathrm{L}^{-1}\left\{-\frac{1}{s}\left(\mathrm{L}\left\{\sum_{k=0}^{j-1}\left(U_k\left(V_{j-k-1}\right)_x + V_k\left(V_{j-k-1}\right)_y\right) - \frac{1}{R}\left(\left(V_{j-1}\right)_{xx} + \left(V_{j-1}\right)_{yy}\right)\right\}\right)\right\}, \end{cases}$$
$$\vdots \tag{9}$$

Suppose that the initial approximation has the form $U(x, y, 0) = u_0(x, y, t) = f(x, y)$ and $V(x, y, 0) = v_0(x, y, t) = g(x, y)$; therefore the exact solution may be obtained as follows

$$\begin{aligned} u(x, y, t) &= \lim_{p\to 1} U(x, y, t) = U_0(x, y, t) + U_1(x, y, t) + \cdots, \\ v(x, y, t) &= \lim_{p\to 1} V(x, y, t) = V_0(x, y, t) + V_1(x, y, t) + \cdots. \end{aligned} \tag{10}$$

3. Examples

Example 1. Consider the following homogeneous form of a coupled Burgers equation [13]:

$$\begin{aligned} u_t + uu_x + vu_y &= \frac{1}{R}\left(u_{xx} + u_{yy}\right), \\ v_t + uv_x + vv_y &= \frac{1}{R}\left(v_{xx} + v_{yy}\right), \end{aligned} \tag{11}$$

subject to the initial condition

$$\begin{aligned} u(x, y, 0) &= x + y, \\ v(x, y, 0) &= x - y. \end{aligned} \tag{12}$$

The exact solution of this equation is $u(x, y, t) = (x + y - 2xt)/(1 - 2t^2)$ and $v(x, y, t) = (x - y - 2yt)/(1 - 2t)^2$.

Starting with $U(x, y, 0) = u_0 = x + y$, $V(x, y, 0) = v_0 = x - y$ and using (9), we obtain

$$\begin{aligned} U_0 &= \mathrm{L}^{-1}\left\{\frac{1}{s}(x + y + \mathrm{L}\{x + y\})\right\} = (x + y)(1 + t), \\ V_0 &= \mathrm{L}^{-1}\left\{\frac{1}{s}(x - y + \mathrm{L}\{x - y\})\right\} = (x - y)(1 + t), \\ U_1 &= \mathrm{L}^{-1}\left\{-\frac{1}{s}\left(\mathrm{L}\left\{x + y + U_0(U_0)_x + V_0(U_0)_y - \frac{1}{R}\left((U_0)_{xx} + (U_0)_{yy}\right)\right\}\right)\right\} \\ &= -(3x + y)t - 2xt^2 - \frac{2}{3}xt^3, \\ V_1 &= \mathrm{L}^{-1}\left\{-\frac{1}{s}\left(\mathrm{L}\left\{x - y + U_0(V_0)_x + V_0(V_0)_y - \frac{1}{R}\left((V_0)_{xx} + (V_0)_{yy}\right)\right\}\right)\right\} \\ &= -(x + y)t - 2yt^2 - \frac{2}{3}yt^3, \\ U_2 &= \mathrm{L}^{-1}\left\{-\frac{1}{s}\left(\mathrm{L}\left\{U_0(U_1)_x + U_1(U_0)_x + V_0(U_1)_y + V_1(U_0)_y - \frac{1}{R}\left((U_1)_{xx} + (U_1)_{yy}\right)\right\}\right)\right\} \\ &= (4x + 2y)t^2 + \left(4x + \frac{8}{3}y\right)t^3 + \left(\frac{4}{3}x + \frac{4}{3}y\right)t^4 + \left(\frac{4}{15}x + \frac{4}{15}y\right)t^5, \\ V_2 &= \mathrm{L}^{-1}\left\{-\frac{1}{s}\left(\mathrm{L}\left\{U_0(V_1)_x + U_1(V_0)_x + V_0(V_1)_y + V_1(V_0)_y - \frac{1}{R}\left((V_1)_{xx} + (V_1)_{yy}\right)\right\}\right)\right\} \\ &= 2xt^2 + \left(\frac{8}{3}x - \frac{4}{3}y\right)t^3 + \left(\frac{4}{3}x - \frac{4}{3}y\right)t^4 + \left(\frac{4}{15}x - \frac{4}{15}y\right)t^5, \end{aligned}$$

$$
\begin{aligned}
U_3 &= \mathrm{L}^{-1}\left\{-\frac{1}{s}\left(\mathrm{L}\left\{U_0(U_2)_x+U_1(U_1)_x+U_2(U_0)_x+V_0(U_2)_y+V_1(U_1)_y+V_2(U_0)_y-\frac{1}{R}\left((U_2)_{xx}+(U_2)_{yy}\right)\right\}\right)\right\}\\
&= -\left(\frac{22}{3}x+\frac{8}{3}y\right)t^3-\left(\frac{28}{3}x+\frac{8}{3}y\right)t^4-\left(\frac{16}{3}x+\frac{4}{5}y\right)t^5-\frac{68}{45}xt^6-\frac{68}{315}xt^7,\\
V_3 &= \mathrm{L}^{-1}\left\{-\frac{1}{s}\left(\mathrm{L}\left\{U_0(V_2)_x+U_1(V_1)_x+U_2(V_0)_x+V_0(V_2)_y+V_1(V_1)_y+V_2(V_0)_y-\frac{1}{R}\left((V_2)_{xx}+(V_2)_{yy}\right)\right\}\right)\right\}\\
&= -\left(\frac{8}{3}x+2y\right)t^3-\left(\frac{8}{3}x+4y\right)t^4-\left(\frac{4}{5}x+\frac{56}{15}y\right)t^5-\frac{68}{45}yt^6-\frac{68}{315}yt^7\\
&\quad\vdots
\end{aligned}
\tag{13}
$$

Therefore we gain the solution of (11) as

$$
\begin{aligned}
u(x,y,t) &= U_0(x,y,t)+U_1(x,y,t)+U_3(x,y,t)+\cdots\\
&= x+y-2xt+2xt^2+2yt^2-4xt^3\\
&\quad +4xt^4+4yt^4-8xt^5+\cdots\\
&= x(1+2t^2+4t^4+\cdots)+y(1+2t^2+4t^4+\cdots)\\
&\quad -2xt(1+2t^2+4t^4+\cdots)\\
&= \frac{x+y-2xt}{1-2t^2},\\
v(x,y,t) &= V_0(x,y,t)+V_1(x,y,t)+V_3(x,y,t)+\cdots\\
&= x-y-2yt+2xt^2-2yt^2-4yt^3\\
&\quad +4xt^4-4yt^4-8yt^5+\cdots\\
&= x(1+2t^2+4t^4+\cdots)\\
&\quad -y(1+2t^2+4t^4+\cdots)\\
&\quad -2yt(1+2t^2+4t^4+\cdots)\\
&= \frac{x-y-2yt}{1-2t^2}
\end{aligned}
\tag{14}
$$

which is exact solution.

Example 2. Let us consider system of Burgers' equations (1), with the following initial conditions [14]:

$$
\begin{aligned}
u(x,y,0) &= \frac{3}{4}-\frac{1}{4[1+\exp(y-x)R/8]},\\
v(x,y,0) &= \frac{3}{4}+\frac{1}{4[1+\exp(y-x)R/8]},
\end{aligned}
\tag{15}
$$

for which exact solutions are

$$
\begin{aligned}
u(x,y,t) &= \frac{3}{4}-\frac{1}{4[1+\exp(4y-4x-t)R/32]},\\
v(x,y,t) &= \frac{3}{4}+\frac{1}{4[1+\exp(4y-4x-t)R/32]}.
\end{aligned}
\tag{16}
$$

To solve system (1) by LTNHPM, following the same procedure discussed in Section 2 and Example 1, we obtain the iterative relations (9); in this example we take initial approximations (15). The accuracy of LTNHPM for the system of two-dimensional Burgers' equation agrees good with the exact solution, and absolute errors are very small for the present choice of x, y, and t. These results are listed in Tables 1, 2, 3, and 4 for $R = 0.5$ and $R = 1$.

Example 3. Let us consider system of Burgers' equations (8), with the following initial conditions [14]:

$$
\begin{aligned}
u(x,y,0) &= -\frac{4\pi\cos(2\pi x)\sin(\pi y)}{R(2+\sin(2\pi x)\sin(\pi y))},\\
v(x,y,0) &= -\frac{2\pi\sin(2\pi x)\cos(\pi y)}{R(2+\sin(2\pi x)\sin(\pi y))},
\end{aligned}
\tag{17}
$$

for which exact solutions are

$$
\begin{aligned}
u(x,y,t) &= -\frac{4\pi\exp(-5\pi^2 t/R)\cos(2\pi x)\sin(\pi y)}{R(2+\exp(-5\pi^2 t/R)\sin(2\pi x)\sin(\pi y))},\\
v(x,y,t) &= -\frac{2\pi\exp(-5\pi^2 t/R)\sin(2\pi x)\cos(\pi y)}{R(2+\exp(-5\pi^2 t/R)\sin(2\pi x)\sin(\pi y))}.
\end{aligned}
\tag{18}
$$

To solve system (1) by LTNHPM, following the same procedure discussed in Section 2 and Example 1, we obtain the iterative relations (9); in this example we take initial approximations (17). The accuracy of LTNHPM for the system of two-dimensional Burgers' equation agrees good with the exact solution, and absolute errors are very small for the present choice of x, y, and t. These results are listed in Tables 5, 6, 7, and 8 for $R = 100$ and $R = 500$.

4. Conclusions

In this work, we considered a new hybrid of Laplace transform method and homotopy perturbation method (LTNHPM) for solving system of two-dimensional Burgers' equation. Using this method we obtained new efficient relations to solve these systems. New method is a powerful straightforward method. The LTNHPM is apt to be utilized as an alternative approach to current techniques being employed to a wide variety of mathematical problems.

References

[1] J. M. Burgers, "Application of a model system to illustrate some points of the statistical theory of free turbulence," *The Royal Netherlands Academy of Arts and Sciences*, vol. 43, pp. 2–12, 1940.

[2] E. N. Aksan, "Quadratic B-spline finite element method for numerical solution of the Burgers' equation," *Applied Mathematics and Computation*, vol. 174, no. 2, pp. 884–896, 2006.

[3] S. Kutluay and A. Esen, "A lumped galerkin method for solving the burgers equation," *International Journal of Computer Mathematics*, vol. 81, no. 11, pp. 1433–1444, 2004.

[4] S. Abbasbandy and M. T. Darvishi, "A numerical solution of Burgers' equation by modified Adomian method," *Applied Mathematics and Computation*, vol. 163, no. 3, pp. 1265–1272, 2005.

[5] Sirendaoreji, "Exact solutions of the two-dimensional Burgers equation," *Journal of Physics A*, vol. 32, no. 39, pp. 6897–6900, 1999.

[6] H. Bateman, "Some recent researches on the motion of fluids," *Monthly Weather Review*, vol. 43, pp. 163–170, 1915.

[7] E. Hopf, "The partial differential equation ut C uux D uxx," *Communications on Pure and Applied Mathematics*, vol. 3, pp. 201–230, 1950.

[8] J. D. Cole, "On a quasilinear parabolic equation occurring in aerodynamics," *Quarterly of Applied Mathematics*, vol. 9, pp. 225–236, 1951.

[9] E. R. Benton and G. W. Platzman, "A table of solutions of the one-dimensional Burgers' equation," *Quarterly of Applied Mathematics*, vol. 30, pp. 195–212, 1972.

[10] V. I. Karpman, *Nonlinear Waves in Dispersive Media*, Pergamon Press, Oxford, UK, 1975.

[11] J. Burgers, *Advances in Applied Mechanics*, Academic Press, New York, NY, USA, 1948.

[12] H. Aminikhah and M. Hemmatnezhad, "An efficient method for quadratic Riccati differential equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 4, pp. 835–839, 2010.

[13] J. Biazar and H. Ghazvini, "Exact solutions for nonlinear burgers' equation by homotopy perturbation method," *Numerical Methods for Partial Differential Equations*, vol. 25, no. 4, pp. 833–842, 2009.

[14] M. Tamsir and V. K. Srivastava, "A semi-implicit finite-difference approach for two-dimensional coupled Burgers' equations," *International Journal of Scientific & Engineering Research*, vol. 2, pp. 1–6, 2011.