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Research Article

The L-Total Graph of an L-Module

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Let L be a complete lattice. We introduce and investigate the L-total graph of an L-module over an L-commutative ring. The main purpose of this paper is to extend the definition and results given in (Anderson and Badawi, 2008) to more generalize the L-total graph of an L-module case.

1. Introduction

It was Beck (see [1]) who first introduced the notion of a zero-divisor graph for commutative rings. This notion was later redefined by Anderson and Livingston in [2]. Since then, there has been a lot of interest in this subject, and various papers were published establishing different properties of these graphs as well as relations between graphs of various extensions (see [2–5]). Let R be a commutative ring with $Z(R)$being its set of zero-divisors elements. The total graph of R, denoted by $T(\Gamma(R))$, is the (undirected) graph with all elements of R as vertices, and, for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$. The total graph of a commutative ring has been introduced and studied by Anderson and Badawi in [3]. In [6], the notion of the total torsion element graph of a module over a commutative ring is introduced.

In [7], Zadeh introduced the concept of fuzzy set, which is a very useful tool to describe the situation in which the data is imprecise or vague. Many researchers used this concept to generalize some notions of algebra. Goguen in [8] generalized the notion of fuzzy subset of X to that of an L-subset, namely, a function from X to a lattice L. In [9], Rosenfeld considered the fuzzification of algebraic structures. Liu [10] introduced and examined the notion of a fuzzy ideal of a ring. Since then several authors have obtained interesting results on L-ideals of a ring R and L-modules (see [11, 12]). Also, L-zero-divisor graph of an L-commutative ring has been introduced and studied in [13].

In the present paper we introduce a new class of graphs, called the L-total torsion element graph of a L-module (see Definition 2.2), and we completely characterize the structure of this graph. The total torsion element graph of a module over a commutative ring and the L-total torsion element graph of a L-module over a L-commutative ring are different concepts. Some of our results are analogous to the results given in [6]. The corresponding results are obtained by modification, and here we give a complete description of the L-total torsion element graph of an L-module.

For the sake of completeness, we state some definitions and notation used throughout. For a graph Γ, by $E(\Gamma)$ and $V(\Gamma)$, we denote the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two distinct vertices. The distance between two distinct vertices a and b, denoted by $d(a,b)$, is the length of the shortest path connecting them (if such a path does not exist, then $d(a,a) = 0$ and $d(a,b) = \infty$). The diameter of a graph Γ, denoted by $\operatorname{diam}(\Gamma)$, is equal to $\sup\{d(a,b) : a,b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph Γ, denoted $\operatorname{gr}(\Gamma)$, is the length of the shortest cycle in Γ, provided Γ contains a cycle; otherwise, $\operatorname{gr}(\Gamma) = \infty$. We denote the complete graph on n vertices by K^n and the complete bipartite graph on m and n vertices by $K^{m,n}$ (we allow m and n to be infinite cardinals). We will sometimes call a $K^{1,m}$ a star graph. We say that two (induced) subgraphs Γ_1 and Γ_2 of Γ are disjoint if Γ_1 and Γ_2 have no common vertices and no vertex of Γ_1 (resp., Γ_2) is adjacent (in Γ) to any vertex not in Γ_1 (resp., Γ_2).

Let R be a commutative ring, and L stands for a complete lattice with least element 0 and greatest element 1. By an L-subset μ of a nonempty set X, we mean a function μ from X to L. If $L = [0,1]$, then μ is called a fuzzy subset of X. L^X denotes the set of all L-subsets of X. We recall some definitions and lemmas from the book [12], which we need for development of our paper.

Definition 1.1. An L-ring is a function $\mu : R \to L$, where $(R, +, .)$ is a ring, which satisfies the following.

(1) $\mu \neq 0$.

(2) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$ for every x, y in R.

(3) $\mu(xy) \geq \mu(x) \vee \mu(y)$ for every x, y in R.

Definition 1.2. Let $\mu \in L^R$. Then μ is called an L-ideal of R if for every $x, y \in R$ the following conditions are satisfied.

(1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$.

(2) $\mu(xy) \geq \mu(x) \vee \mu(y)$.

The set of all L-ideals of R is denoted by $LI(R)$.

Definition 1.3. Assume that M is an R-module, and let $\mu \in L^M$. Then μ is called an L-fuzzy R-module of M if for all $x, y \in M$ and for all $r \in R$ the following conditions are satisfied.

(1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$.

(2) $\mu(rx) \geq \mu(x)$.

(3) $\mu(0_M) = \mu(1)$.

The set of all L-fuzzy R-modules of M is denoted by $L(M)$.

Lemma 1.4. *Let M be a module over a ring R, and $\mu \in L(M)$. Then $\mu(m) \leq \mu(0_M)$ for every $m \in M$.*

2. $T(\mu)$ Is a Submodule of M

Let M be a module over a commutative ring R, and let $\mu \in L(M)$. The structure of the L-total torsion element graph $T(\Gamma(\mu))$ may be completely described in those cases when μ-torsion elements form a submodule of M. We begin with the key definition of this paper.

Definition 2.1. Let M be a module over a commutative ring R, and let $\mu \in L(M)$. A μ-torsion element is an element $m \in M$ with $\mu(m) \neq \mu(0_M)$ for which there exists a nonzero element r of R such that $\mu(rm) = \mu(0_M)$.

The set of μ-torsion elements in M will be denoted by $T(\mu)$.

Definition 2.2. Let M be a module over a ring R, and let $\mu \in L(M)$. We define the L-total torsion element graph of an L-module $T(\Gamma(\mu))$ as follows: $V(T(\Gamma(\mu))) = M$, $E(T(\Gamma(\mu))) = \{\{x, y\} : x + y \in T(\mu)\}$.

Notation 1. For the μ-torsion element graph $T(\Gamma(\mu))$, we denote the diameter, the girth, and the distance between two distinct vertices a and b, by $\mathrm{diam}(T(\Gamma(\mu)))$, $\mathrm{gr}(T(\Gamma(\mu)))$, and $d_\mu(a, b)$, respectively.

Remark 2.3. Let M be a module over a ring R, and let $\mu \in L(M)$. Clearly, if μ is a nonzero constant, then $T(\Gamma(\mu)) = \emptyset$. So throughout this paper, we will assume, unless otherwise stated, that μ is not a nonzero constant. Thus, there is a nonzero element y of M such that $\mu(y) \neq \mu(0_M)$.

We will use $\mathrm{Tof}(\mu)$ to denote the set of elements of M that are not μ-torsion elements. Let $\mathrm{Tof}(\Gamma(\mu))$ be the (induced) subgraph of $T(\Gamma(\mu))$ with vertices $\mathrm{Tof}(\mu)$, and let $\mathrm{Tor}(\Gamma(\mu))$ be the (induced) subgraph of $T(\Gamma(\mu))$ with vertices $T(\mu)$.

Definition 2.4. Let M be a module over a ring R, and $\mu \in L(M)$. One defines the set $\mathrm{ann}_\mu(M)$ by $\mathrm{ann}_\mu(M) = \{r \in R : \mu(rM) = \{\mu(0_M)\}\}$, the μ-annihilator of M.

Lemma 2.5. *Let M be a module over a ring R, and let $\mu \in L(M)$. Then $\mathrm{ann}_\mu(M)$ is an L-ideal of R.*

Proof. Let $r, s \in \mathrm{ann}_\mu(M)$ and $t \in R$. If $m \in M$, then we have $\mu((r - s)m) \geq \mu(rm) \wedge \mu(-sm) = \mu(0_M) \wedge \mu(0_M) = \mu(0_M)$ and $\mu(trm) = \mu(t(rm)) \geq \mu(rm) = \mu(0_M)$. It then follows from Lemma 1.4 that $\mu((r - s)m) = \mu(0_M)$; hence $r - s \in \mathrm{ann}_\mu(M)$. Similarly, $rt \in \mathrm{ann}_\mu(M)$. □

Theorem 2.6. *Let M be a module over a ring R and let $\mu \in L(M)$. Then the L-torsion element graph $T(\Gamma(\mu))$ is complete if and only if $T(\mu) = M$.*

Proof. If $T(\mu) = M$, then for any vertices $m, m' \in M$, one has $m + m' \in T(\mu)$; hence they are adjacent in $T(\Gamma(\mu))$. On the other hand, if $T(\Gamma(\mu))$ is complete, then every vertex is adjacent to 0. Thus,$m = m + 0 \in T(\mu)$ for every $m \in M$. This completes the proof. □

Theorem 2.7. *Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is a submodule of M. Then one has the following.*

(i) $\mathrm{Tor}(\Gamma(\mu))$ *is a complete (induced) subgraph of* $T(\Gamma(\mu))$ *and* $\mathrm{Tor}(\Gamma(\mu))$ *is disjoint from* $\mathrm{Tof}(\Gamma(\mu))$.

(ii) *If* $\mathrm{ann}_\mu\ (M) \neq 0$, *then* $T(\Gamma(\mu))$ *is a complete graph.*

Proof. (i) $\mathrm{Tor}(\Gamma(\mu))$ is complete directly from the definition. Finally, if $m \in T(\mu)$ and $m' \in \mathrm{Tof}(\mu)$ were adjacent, then $m + m' \in T(\mu)$; so this, since $T(\mu)$ is a submodule, would lead to the contradiction $m' \in T(\mu)$.

(ii) Let $m \in M$. we may assume that $\mu(m) \neq \mu(0_M)$. By assumption, there exists $0 \neq s \in R$ with $\mu(sM) = \mu(0_M)$, so $\mu(sm) = \mu(0_M)$. Thus $m \in T(\mu)$, and; therefore, $T(\Gamma(\mu))$ is a complete graph by Theorem 2.6. □

Theorem 2.8. *Let M be a module over a ring R, and let $\mu \in L(M)$. Then $T(\Gamma(\mu))$ is totally disconnected if and only if R has characteristic 2 and $T(\mu) = \{0_M\}$.*

Proof. If $T(\mu) = \{0_M\}$, then the vertices m_1 and m_2 are adjacent if and only if $m_1 = -m_2$. Then $T(\Gamma(\mu))$ is a disconnected graph, and its only edges are those that connect vertices m_i and $-m_i$ (we do not need a priori assumption that R has characteristic 2). Conversely, assume that $T(\Gamma(\mu))$ is totally disconnected. Then $0 + m \notin T(\mu)$ for every nonzero element m of M. Thus, $T(\mu) = \{0_M\}$. Further, since $m + (-m) = 0$, we have $m = -m$ (so $\mu(2m) = \mu(0_M)$) for every $m \in M$ with $\mu(m) \neq \mu(0_M)$ by the total disconnectedness of the graph $T(\Gamma(\mu))$. As $T(\mu) = \{0_M\}$, it follows that $2 = 1_R + 1_R = 0$. Thus, $\mathrm{char}(R) = 2$. □

Proposition 2.9. *Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is a submodule of M. If $m \in \mathrm{Tof}(\mu)$, then $2m \in T(\mu)$ if and only if $2 \in Z(R)$.*

Proof. First suppose that $2m \in T(\mu)$. Since $m \notin T(\mu)$, we get that $\mu(m) \neq \mu(0_M)$, and, for all $r \in R, \mu(rm) = \mu(0_M)$ implies that $r = 0$. Since $2m \in T(\mu)$, there is a nonzero element $c \in R$ such that $\mu(c(2m)) = \mu((2c)m) = \mu(0_M)$, and, since $m \notin T(\mu)$, one must have $2c = 0$; hence, $2 \in Z(R)$. Conversely, assume that $2 \in Z(R)$. Then there exists $0 \neq d \in R$ with $2d = 0$. Since $\mu(0_M) = \mu((2d)m) = \mu(d(2m))$, we have $2m \in T(\mu)$. □

Theorem 2.10. *Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of M. Then $T(\Gamma(\mu))$ is disconnected.*

Proof. If $T(\mu) = \{0_M\}$, then $T(\Gamma(\mu))$ is disconnected by Theorem 2.8. If $T(\mu) \neq \{0_M\}$, then the subgraphs of $\mathrm{Tor}(\Gamma(\mu))$ and $\mathrm{Tof}(\Gamma(\mu))$ are disjoint by Theorem 2.7 (i), as required. □

Theorem 2.11. *Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of M. Suppose $|T(\mu)| = \alpha$ and $|M/T(\mu)| = \beta$. Then one has the following.*

(i) *If $2 \in Z(R)$, then $T(\Gamma(\mu))$ is a union of β disjoint complete graphs K^α.*

(ii) *If $2 \notin Z(R)$, then $T(\Gamma(\mu))$ is a union of $(\beta - 1)/2$ disjoint bipartite graphs $K^{\alpha,\alpha}$ and one complete graph K^α.*

Proof. (i) Assume that $2 \in Z(R)$ and let $m, m' \in \mathrm{Tof}(\mu)$ be such that $m + T(\mu) \neq m' + T(\mu)$. The elements $m + t$, $m + t'$ from the same coset $m + T(\mu)$ are adjacent if and only if $2m \in T(\mu)$, so $2 \in Z(R)$, according to the Proposition 2.9. Then $m + t$ and $m' + t'$ are not adjacent (otherwise, we would have $m - m' = m + m' - 2m' \in T(\mu)$), and; therefore, $m + T(\mu) = m' + T(\mu)$. Since every coset has cardinality α, we conclude that $T(\Gamma(\mu))$ is the disjoint union of β complete graph K^{α}.

(ii) If $2 \notin Z(R)$, then the elements $m+t$, $m+t'$ from $m+T(\mu)$ are obviously not adjacent. The elements $m + t$, $m' + t'$ from different cosets are adjacent if and only if $m + m' \in T(\mu)$ or $m+T(\mu) = (-m)+T(\mu)$. In this way we obtain that the subgraph spanned by the vertices from $\mathrm{Tof}(\mu)$ is a disjoint union of $(\beta - 1)/2$ ($= \beta$ if β is infinite) disjoint bipartite graph $K^{\alpha,\alpha}$. □

Proposition 2.12. *Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of M. Then one has the following.*

(i) $\mathrm{Tof}(\Gamma(\mu))$ *is complete if and only if either* $|M/T(\mu)| = 2$ *or* $|M/T(\mu)| = |M| = 3$.

(ii) $\mathrm{Tof}(\Gamma(\mu))$ *is connected if and only if either* $|M/T(\mu)| = 2$ *or* $|M/T(\mu)| = 3$.

(iii) $\mathrm{Tof}(\Gamma(\mu))$ *and, hence;* ($\mathrm{Tor}(\Gamma(\mu))$ *and* $T(\Gamma(\mu))$) *is totally disconnected if and only if* $T(\mu) = \{0_M\}$ *and* $2 \in Z(R)$.

Proof. Let $|M/T(\mu)| = \beta$ and $|T(\mu)| = \alpha$.

(i) Let $\mathrm{Tof}(\Gamma(\mu))$ be complete. Then, by Theorem 2.11, $\mathrm{Tof}(\Gamma(\mu))$ is complete if and only if $\mathrm{Tof}(\Gamma(\mu))$ is a single K^{α} or $K^{1,1}$. If $2 \in Z(R)$, then $\beta - 1 = 1$. Thus, $\beta = 2$, and hence $|M/T(\mu)| = 2$. If $2 \notin Z(R)$, then $\alpha = 1$ and $(\beta - 1)/2 = 1$. Thus, $T(\mu) = \{0\}$ and $\beta = 3$; hence, $|M| = |M/T(\mu)| = 3$. The reverse implication may be proved in a similar way as in [6, Theorem 2.6 (1)].

(ii) By theorem 2.11, $\mathrm{Tof}(\Gamma(\mu))$ is connected if and only if $\mathrm{Tof}(\Gamma(\mu))$ is a single K^{α} or $K^{\alpha,\alpha}$. Thus, either $\beta - 1 = 1$ if $2 \in Z(R)$ or $(\beta - 1)/2 = 1$ if $2 \notin Z(R)$; hence, $\beta = 2$ or $\beta = 3$, respectively, as needed. The reverse implication may be proved in a similar way as in [3, Theorem 2.6 (2)].

(iii) $\mathrm{Tof}(\Gamma(\mu))$ is totally disconnected if and only if it is a disjoint union of K^1's. So by Theorem 2.11, $|T(\mu)| = 1$ and $|M/T(\mu)| = 1$, and the proof is complete. □

By the proof of the Proposition 2.12, the next theorem gives a more explicit description of the diameter of $\mathrm{Tof}(\Gamma(\mu))$.

Theorem 2.13. *Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of M. Then one has the following.*

(i) $\mathrm{diam}(\mathrm{Tof}(\Gamma(\mu))) = 0$ *if and only if* $T(\mu) = \{0\}$ *and* $|M| = 2$.

(ii) $\mathrm{diam}(\mathrm{Tof}(\Gamma(\mu))) = 1$ *if and only if either* $T(\mu) \neq \{0_M\}$ *and* $|M/T(\mu)| = 2$ *or* $T(\mu) = \{0\}$ *and* $|M| = 3$.

(iii) $\mathrm{diam}(\mathrm{Tof}(\Gamma(\mu))) = 2$ *if and only if* $T(\mu) \neq \{0_M\}$ *and* $|M/T(\mu)| = 3$.

(iv) *Otherwise,* $\mathrm{diam}(\mathrm{Tof}(\Gamma(\mu))) = \infty$.

Proposition 2.14. *Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of M. Then* $\mathrm{gr}(\mathrm{Tof}(\Gamma(\mu))) = 3, 4$ *or* ∞. *In particular,* $\mathrm{gr}(\mathrm{Tof}(\Gamma(\mu))) \leq 4$ *if* $\mathrm{Tof}(\Gamma(\mu))$ *contains a cycle.*

Proof. Let $\mathrm{Tof}(\Gamma(\mu))$ contain a cycle. Then since $\mathrm{Tof}(\Gamma(\mu))$ is disjoint union of either complete or complete bipartite graphs by Theorem 2.11, it must contain either a 3 cycles or a 4 cycles. Thus $\mathrm{gr}(\mathrm{Tof}(\Gamma(\mu))) \leq 4$. □

Theorem 2.15. *Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of M. Then one has the following.*

(i) (a) $\mathrm{gr}(\mathrm{Tof}(\Gamma(\mu))) = 3$ *if and only if* $2 \in Z(R)$ *and* $|T(\mu)| \geq 3$.
(b) $\mathrm{gr}(\mathrm{Tof}(\Gamma(\mu))) = 4$ *if and only if* $2 \notin Z(R)$ *and* $|T(\mu)| \geq 2$.
(c) *Otherwise,* $\mathrm{gr}(\mathrm{Tof}(\Gamma(\mu))) = \infty$.

(ii) (a) $\mathrm{gr}(T(\Gamma(\mu))) = 3$ *if and only if* $|T(\mu)| \geq 3$.
(b) $\mathrm{gr}(T(\Gamma(\mu))) = 4$ *if and only if* $2 \notin Z(R)$ *and* $|T(\mu)| = 2$.
(c) *Otherwise,* $\mathrm{gr}(T(\Gamma(\mu))) = \infty$.

Proof. Apply Theorem 2.11, Proposition 2.14, and Theorem 2.7 (i). □

The previous theorems give a complete description of the structure of the L-total torsion element graph of an L-module M when $T(\mu)$ is a submodule. The question under what conditions $T(\mu)$ is a submodule of M and how is this related to the condition that $Z(R)$ is an ideal in R naturally arises. We prove that the following results holds.

Theorem 2.16. *Let M be a module over a ring R, and let $\mu \in L(M)$. Then one has the following.*

(i) *If $Z(R) = \{0_R\}$, then $T(\mu)$ is a submodule of M.*

(ii) *If $Z(R) = Rc$ is a principal ideal of R with c a nilpotent element of R, then $T(\mu)$ is a submodule of M.*

Proof. (i) Let $m, m' \in T(\mu)$ and $r \in R$. There are nonzero elements $a, b \in R$ such that $\mu(m) \neq \mu(0_M)$, $\mu(m') \neq \mu(0_M)$, and $\mu(am) = \mu(bm') = \mu(0_M)$ with $ab \neq 0$ (since R is an integral domain). It follows that $\mu(ab(m+m')) \geq \mu(abm) \wedge \mu(abm') = \mu(0_M) \wedge \mu(0_M) = \mu(0_M)$; hence, $\mu(ab(m+m')) = \mu(0_M)$ by Lemma 1.4. Thus, $m + m' \in T(\mu)$. Similarly, $rm \in T(\mu)$, and this completes the proof.

(ii) Assume that $T(\mu)$ is not a submodule of M. Then there are elements $m, m' \in T(\mu)$ such that $m + m' \notin T(\mu)$. By assumption, there exist nonzero elements $r, s \in R$ such that $\mu(rm) = \mu(0_M) = \mu(sm') = \mu(0_M)$, where $\mu(m) \neq \mu(0_M)$ and $\mu(m') \neq \mu(0_M)$. Then $\mu(rs(m + m')) = \mu(0_M)$ and $m + m' \notin T(\mu)$, so we must have $rs = 0$, and; thus, $r, s \in Z(R)$. Since c is nilpotent, we have $r = r_1 c^t$ and $s = s_1 c^u$, for some $r_1, s_1 \notin Z(R)$. We may assume that $t \geq u$. Then for the nonzero element $s_1 r$ of R we have $\mu(s_1 r(m + m')) = \mu(0_M)$ which is contrary to the assumption that $m + m' \notin T(\mu)$. □

Example 2.17. Assume that $R = \mathbb{Z}$ is the ring integers, and let $M = R$. We define the mapping $\mu : M \to [0, 1]$ by

$$\mu(m) = \begin{cases} \dfrac{1}{2} & \text{if } x \in 2\mathbb{Z}, \\[2mm] \dfrac{1}{5} & \text{otherwise}. \end{cases} \tag{2.1}$$

Then $\mu \in L(M)$ and $T(\mu) = M$. Thus, $T(\Gamma(\mu))$ is a complete graph by Theorem 2.6.

Example 2.18. Let $M_1 = R_1 = Z_8$ denote the ring of integers modulo 8 and $M_2 = R_2 = Z_{25}$ the ring of integers modulo 25. We define the mappings $\mu_1 : M_1 \to [0,1]$ by

$$\mu_1(x) = \begin{cases} 1 & \text{if } x = \overline{0}, \\ \dfrac{1}{2} & \text{otherwise} \end{cases} \tag{2.2}$$

and $\mu_2 : M_2 \to [0,1]$ by

$$\mu_2(m) = \begin{cases} 1 & \text{if } x = \overline{0}, \\ \dfrac{1}{3} & \text{otherwise.} \end{cases} \tag{2.3}$$

Then, for each i $(1 \leq i \leq 2)$, $\mu_i \in L(M_i)$, $T(\mu_1) = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$, and $T(\mu_2) = \{\overline{0}, \overline{5}, \overline{10}, \overline{15}, \overline{20}\}$. An inspection will show that $T(\mu_1)$ and $T(\mu_2)$ are submodules of M_1 and M_2, respectively. Therefore, by Theorem 2.11, we have the following results.

(1) Since $2 \in Z(R_1)$, we conclude that $T(\Gamma(\mu_1))$ is a union of 2 disjoint K^4.

(2) Since $2 \notin Z(R_2)$, we conclude that $T(\Gamma(\mu_2))$ is a disjoint union of 2 complete graph K^5 and 5 bipartite $K^{5,5}$.

3. $T(\mu)$ Is Not a Submodule of M

We continue to use the notation already established, so M is a module over a commutative ring R and $\mu \in L(M)$. In this section, we study the L-torsion element graph $T(\Gamma(\mu))$ when $T(\mu)$ is not a submodule of M.

Lemma 3.1. *Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of M. Then there are distinct $m, m' \in T(\mu)^*$ such that $m + m' \in \mathrm{Tof}(\mu)$.*

Proof. It suffices to show that $T(\mu)$ is always closed under scalar multiplication of its elements by elements of R. Let $m \in T(\mu)$ and $r \in R$. There is a nonzero element $s \in R$ with $\mu(sm) = \mu(0_M)$ such that $\mu(m) \neq \mu(0_M)$, so $\mu(s(rm)) = \mu(r(sm)) \geq \mu(sm) = \mu(0_M)$; hence, $\mu(s(rm)) = \mu(0_M)$ by Lemma 1.4, as required. □

Theorem 3.2. *Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of M. Then one has the following.*

(i) $\mathrm{Tor}(\Gamma(\mu))$ *is connected with* $\mathrm{diam}(\mathrm{Tor}(\Gamma(\mu))) = 2$.

(ii) *Some vertex of* $\mathrm{Tor}(\Gamma(\mu))$ *is adjacent to a vertex of* $\mathrm{Tof}(\Gamma(\mu))$. *In particular, the subgraphs* $\mathrm{Tor}(\Gamma(\mu))$ *and* $\mathrm{Tof}(\Gamma(\mu))$ *of* $T(\Gamma(\mu))$ *are not disjoint.*

(iii) *If* $\mathrm{Tof}(\Gamma(\mu))$ *is connected, then* $T(\Gamma(\mu))$ *is connected.*

Proof. (i) Let $x \in T(\mu)^*$. Then x is adjacent to 0. Thus, $x - 0 - y$ is a path in $\mathrm{Tor}(\Gamma(\mu))$ of length two between any two distinct $x, y \in T(\mu)^*$. Moreover, there exist nonadjacent $x, y \in T(\mu)^*$ by Lemma 3.1; thus, $\mathrm{diam}(\mathrm{Tor}(\Gamma(\mu))) = 2$.

(ii) By Lemma 3.1, there exist distinct $x, y \in T(\mu)^*$ such that $x + y \in \mathrm{Tof}(\mu)$. Then $-x \in T(\mu)$ and $x + y \in \mathrm{Tof}(\mu)$ are adjacent vertices in $T(\Gamma(\mu))$ since $-x + (x + y) = y \in T(\mu)$. Finally, the "in particular" statement follows from Lemma 3.1.

(iii) By part (i) above, it suffices to show that there is a path from x to y in $T(\Gamma(\mu))$ for any $x \in T(\mu)$ and $y \in \mathrm{Tof}(\mu)$. By part (ii) above, there exist adjacent vertices c and d in $\mathrm{Tor}(\Gamma(\mu))$ and $\mathrm{Tof}(\Gamma(\mu))$, respectively. Since $\mathrm{Tor}(\Gamma(\mu))$ is connected, there is a path from x to c in $\mathrm{Tor}(\Gamma(\mu))$, and, since $\mathrm{Tof}(\Gamma(\mu))$ is connected, there is a path from d to y in $\mathrm{Tof}(\Gamma(\mu))$. Then there is a path from x to y in $T(\Gamma(\mu))$ since c and d are adjacent in $T(\Gamma(\mu))$. Thus, $T(\Gamma(\mu))$ is connected. □

Proposition 3.3. *Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of M. If the identity of the ring R is a sum of n zero divisors, then every element of the M is the sum of at most n μ-torsion elements.*

Proof. Let $x \in M$ and $r \in Z(R)$. We may assume that $\mu(x) \neq \mu(0_M)$. Then there is a nonzero element $b \in R$ such that $rb = 0$, so $\mu(b(rx)) = \mu((rb)x) = \mu(0_M)$ with $\mu(rx) \neq \mu(0_M)$. Therefore, if $x \in M$ and $r \in R$, then $rx \in T(\mu)$, so, for all $x \in M$, $1 = c_1 + \cdots + c_n$ implies that $x = c_1 x + \cdots + c_n x$, as needed. □

Theorem 3.4. *Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of M. Then $T(\Gamma(\mu))$ is connected if and only if M is generated by its μ-torsion elements.*

Proof. Let us first prove that the connectedness of the graph $T(\Gamma(\mu))$ implies that the module M is generated by its μ-torsion elements. Suppose that this is not true. Then there exists $x \in M$ which does not have a representation of the form $x = x_1 + \cdots + x_n$, where $x_i \in T(\mu)$. Moreover, $x \neq 0$ since $0 \in T(\mu)$. We show that there does not exist a path from 0 to x in $T(\Gamma(\mu))$. If $0 - y_1 - y_2 - \cdots - y_m - x$ is a path in $T(\Gamma(\mu))$, $y_1, y_1 + y_2, \ldots, y_{m-1} + y_m, y_m + x$ are μ-torsion elements and x may be represented as $x = (y_m + x) - (y_{m-1} + y_m) + \cdots + (-1)^{m-1}(y_1 + y_2) + (-1)^m y_1$. This contradicts the assumption that x is not a sum of μ-torsion elements. The reverse implication may be proved in a similar way as in [6, Theorem 3.2]. □

We give here with an interesting result linking the L-torsion element graph $T(\Gamma(\mu))$ to the total graph of a commutative ring $T(\Gamma(R))$.

Theorem 3.5. *Let M be a module over a ring R, and let $\mu \in L(M)$. If $T(\Gamma(R))$ is connected, then $T(\Gamma(\mu))$ is a connected graph. In particular, $d_\mu(0, x) \leq d(0, 1)$ for every $x \in M$.*

Proof. Note that, if $x \in M$ and $r \in Z(R)$, then $rm \in T(\mu)$ (see Proposition 3.3). Now suppose that $T(\Gamma(R))$ is connected, and let $x \in M$. Let $0 - s_1 - s_2 - \cdots - s_n - 1$ be a path from 0 to 1 in $T(\Gamma(R))$. Then $s_1, s_1 + s_2, \ldots, s_n + 1 \in Z(R)$; hence, $0_M - s_1 x - \cdots - s_n x - x$ is a path from 0_M to x. As all vertices may be connected via 0_M, $T(\Gamma(\mu))$ is connected. □

Theorem 3.6. *Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of M. If every element of M is a sum of at most n μ-torsion elements, then* $\mathrm{diam}(T(\Gamma(\mu))) \leq n$. *If n is the smallest such number, then* $\mathrm{diam}(T(\Gamma(\mu))) = n$.

Proof. We first show that, by assumption, $d_\mu(0, x) \leq n$ for every nonzero element x of M. Assume that $x = x_1 + \cdots + x_n$, where $x_i \in T(\mu)$. Set $y_i = (-1)^{n+i}(x_1 + \cdots + x_n)$ for $i = 1, \ldots, n$. Then $0 - y_1 - y_2 - \cdots - y_n = x$ is a path from 0 to x of length n in $T(\Gamma(\mu))$. Let u and w be distinct elements in M. We show that $d_\mu(u,w) \leq n$. If $(u-w) - z_1 - z_2 - \cdots - z_{n-1}$ is a path from 0 to $u-w$ and $u+w-s_1-s_2-\cdots-s_{n-1}$ is a path from 0 to $u+w$, then, from the previous discussion, the lengths of both paths are at most n. Depending on the fact whether n is even or odd, we obtain the paths

$$u - (z_1 - w) - (z_2 + w) - \cdots - (z_{n-1} - w) - w \tag{3.1}$$

or $u-(s_1+w)-(s_2-w)-\cdots-(s_{n-1}-w)-w$ from u to w of length n. Assume that n is the smallest such number, and let $a = a_1 + a_2 + \cdots + a_n$ be the shortest representation of the elements x as a sum of μ-torsion elements. From the previous discussion, we have $d_\mu(0,x) \leq n$. Suppose that $d_\mu(0,x) = k \leq n$, and let $0 - t_1 - t_2 - \cdots - t_{k-1} - x$ be a path in $T(\Gamma(\mu))$. It means, a presentation of the element x as a sum of $k < n$ μ-torsion elements (see the proof of Theorem 3.4), which is a contradiction. This completes the proof. □

Corollary 3.7. *Let M be a module over a ring R, and let $\mu \in L(M)$ such that $Z(R)$ is not an ideal of R and $< Z(R) >= R$. If* $\operatorname{diam} T((\Gamma(R))) = n$, *then* $\operatorname{diam} T((\Gamma(\mu))) \leq n$. *In particular, if R is finite, then* $\operatorname{diam} T((\Gamma(\mu))) \leq 2$.

Proof. This follows from Proposition 3.3 and Theorem 3.6. Finally, if R is a finite ring such that $Z(R)$ is not an ideal of R, then $\operatorname{diam} T((\Gamma(R))) = 2$ by [3, Theorem 3.4], as required. □

By Lemma 3.1, the following theorem may be proved in a similar way as in [6, Theorem 3.5].

Theorem 3.8. *Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of M. Then one has the following.*

(i) *Either* $\operatorname{gr}(\operatorname{Tor}(\Gamma(\mu))) = 3$ *or* $\operatorname{gr}(\operatorname{Tor}(\Gamma(\mu))) = \infty$.

(ii) $\operatorname{gr}(T(\Gamma(\mu))) = 3$ *if and only if* $\operatorname{gr}(\operatorname{Tor}(\Gamma(\mu))) = 3$.

(iii) *If* $\operatorname{gr}(T(\Gamma(\mu))) = 4$, *then* $\operatorname{gr}(\operatorname{Tor}(\Gamma(\mu))) = \infty$.

(iv) *If* $\operatorname{Char}(R) \neq 2$, *then* $\operatorname{gr}(\operatorname{Tof}(\Gamma(\mu))) = 3, 4$ *or* ∞.

Example 3.9. Let $M = R = Z_6$ denote the ring of integers modulo 6. We define the mapping $\mu : M \to [0,1]$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \overline{0}, \\ \dfrac{1}{4} & \text{otherwise.} \end{cases} \tag{3.2}$$

Then $\mu \in L(M)$ and $T(\mu) = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}\}$. Now one can easily show that $T(\mu)$ is not a submodule of M and $\operatorname{Tof}(\mu) = \{\overline{1}, \overline{5}\}$. Clearly, $\operatorname{Tor}(\Gamma(\mu))$ is connected with $\operatorname{diam}(\operatorname{Tor}(\Gamma(\mu))) = 2$. Moreover, since $\overline{1} + \overline{3} \in T(\mu)$, we conclude that the subgraphs $\operatorname{Tof}(\Gamma(\mu))$ and $\operatorname{Tor}(\Gamma(\mu))$ of $T(\Gamma(\mu))$ are not disjoint. Furthermore, $T(\Gamma(\mu))$ is connected since $\operatorname{Tof}(\Gamma(\mu))$ is connected.

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