

## Research Article

# $\Gamma$ -Extension of Binary Matroids

**Habib Azanchiler**

*Department of Mathematics, Faculty of Sciences, University of Urmia, P.O. Box 57135, Urmia, Iran*

Correspondence should be addressed to Habib Azanchiler, h.Azanchiler@urmia.ac.ir

Received 13 August 2011; Accepted 22 September 2011

Academic Editors: Y. Hou and T. Prellberg

Copyright © 2011 Habib Azanchiler. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We extend the notion of a point-addition operation from graphs to binary matroids. This operation can be expressed in terms of element-addition operation and splitting operation. We consider a special case of this construction and study its properties. We call the resulting matroid of this special case a  $\Gamma$ -extension of the given matroid. We characterize circuits and bases of the resulting matroids and explore the effect of this operation on the connectivity of the matroids.

## 1. Introduction

Slater [1] defined few operations for graphs which preserve connectedness of graphs. One such operation is a point-addition (vertex-addition) operation. This operation is defined in the following way. Let  $G$  be a graph and  $V(G)$  be the set of vertices of  $G$ . Let  $H$  be the graph obtained from  $G$  by adding a new vertex  $v$  adjacent to vertices  $v_1, v_2, \dots, v_n$  of  $G$ . The graph  $H$  is said to be obtained from  $G$  by point-addition operation. Letting  $X = \{v_1, v_2, \dots, v_n\}$ , for convenience, we denote the graph  $H$  by  $G^X$ . Thus,  $V(G^X) = V(G) \cup \{v\}$  and  $E(G^X) = E(G) \cup \{vv_1, vv_2, \dots, vv_n\}$ .

Point-addition operation has several applications in graph theory. For example, Slater classified  $n$ -connected graphs using point-addition operation along with some other operations [2].

If  $|V(G)| = n$ , then the new vertex  $v$  can be joined to at most  $n$  vertices of the graph. That means, we can add at most  $n$  edges in the original graph.

*Definition 1.1.* Let  $M = M[A]$  be a binary matroid of rank  $r$  on a set  $S$ . Let  $A''$  be the matrix obtained from  $A$  by the following way.

- (1) Adjoin  $k$  columns to  $A$  ( $k \leq r$ ) with labels say  $\gamma_1, \gamma_2, \dots, \gamma_k$ . Let the resulting matrix be denoted by  $A'$ .

- (2) Adjoin a new row to  $A'$  with entries zero except in the columns corresponding to  $\gamma_1, \gamma_2, \dots, \gamma_k$ , where it takes the value 1.

Let  $M''$  be the vector matroid of the matrix  $A''$ . We say that  $M''$  is obtained from  $M$  by the point-addition operation. We call the matroid  $M''$  point-addition matroid or  $\Gamma$ -extension of  $M$ . Let us denote by  $\Gamma$ , the set of columns  $\gamma_1, \gamma_2, \dots, \gamma_k$  which are adjoined to  $A$  in the first step. That is,  $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ . Then, second step consists of splitting the matroid  $M[A']$  with respect to the set  $\Gamma$  (see [3, 4]).

In fact, the matroid  $M''$  is obtained by elements addition and generalized splitting operation [5]. As an immediate consequence of the definition, we have the following result.

Let  $v_i$  and  $v_j$  be two vertices of  $G$ . Then, the addition of an edge  $v_i v_j$ , results in the smallest supergraph of  $G$  containing edge  $v_i v_j$ .

**Proposition 1.2.** *Let  $M = M(G)$  be a cycle matroid of rank  $r$ . Let  $G'$  be the graph obtained from  $G$  by adding adjacent edges  $\gamma_1, \gamma_2, \dots, \gamma_n$  ( $n \leq r$ ) to  $G$ . Let  $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ . Then, the point-addition matroid  $M''$  is graphic and  $M'' = M(G'_\Gamma)$ .*

*Proof.* Let  $A$  be representation matrix of  $M$  over  $GF(2)$ . Let the matrix  $A'$  be obtained from  $A$  by adding column vectors say,  $\gamma_1, \gamma_2, \dots, \gamma_n$ . Suppose that  $A''$  is obtained from  $A'$  by adding a new row where entries are zero, except in the columns corresponding to  $\gamma_1, \gamma_2, \dots, \gamma_n$ , where it takes the value 1. Thus  $M'' = M[A'']$  is a binary matroid with ground set  $E(G) \cup \Gamma$ . Since  $\gamma_1, \gamma_2, \dots, \gamma_n$  are adjacent edges in  $G'$ , the splitting of  $M(G')$  with respect to  $\Gamma$  is graphic (see [5]), and we have  $M(G')_\Gamma = M(G'_\Gamma)$ , where  $G'_\Gamma$  is the graph obtained from  $G$  by splitting operation with respect to  $\Gamma$ . It follows that  $M'' = [M(G)]'' = M(G'_\Gamma)$ .  $\square$

We assume that the reader is familiar with elementary notions in matroid theory, including minors, binary, and connectivity. For an excellent introduction to the subject, read Oxley [6].

## 2. $\Gamma$ -Extension of a Binary Matroid

If a matroid  $M$  is obtained from a matroid  $N$  by adding a nonempty subset  $T$  of  $S(N)$ , then  $N$  is called an extension of  $M$ . In particular, if  $|T| = 1$ , then  $N$  is a single-element extension of  $M$  (see [6]). Another term, that is sometimes used instead of single-element extension, is addition (see [7]).

Now we consider a special case of the operation that is introduced in the first section.

*Definition 2.1.* Let  $M = M[A]$  be a binary matroid of rank  $r$  on a set  $S$ , and let  $A = [I_r \mid J]$  be the standard representation of  $M$  over  $GF(2)$ . Let  $B$  be a base of  $M$ , and let  $X = \{e_{i_1}, e_{i_2}, \dots, e_{i_m}\}$  be a subset of  $B$ . We obtain the matrix  $A^X$  by the following way.

- (1) Obtain a matrix  $A_1$  from  $A$  by adjoining  $m$  ( $m \leq r$ ) columns say  $\gamma_{i_1}, \gamma_{i_2}, \dots, \gamma_{i_m}$  to  $A$ , parallel to  $e_{i_1}, e_{i_2}, \dots, e_{i_m}$ , respectively.
- (2) Split the matrix  $A_1$  with respect to the set  $\Gamma$ , where  $\Gamma = \{\gamma_{i_1}, \gamma_{i_2}, \dots, \gamma_{i_m}\}$ . Denote the resulting matrix by  $A^X$ .

Let  $M^X$  be the vector matroid of the matrix  $A^X$ . We say that  $M^X$  is  $\Gamma$ -extension of  $M$ . Note that  $M^X$  is a binary matroid with ground set  $S \cup \Gamma$ , where  $S \cap \Gamma = \emptyset$ , and  $|X| = |\Gamma|$ . The

transition from  $M$  to  $M^X$  is called  $\Gamma$ -extension operation on  $M$ . In particular, if  $|X| = \ell$ , it is called  $\ell$ - $\Gamma$ -extension operation, and, for  $|X| = 1$ , we call it single- $\Gamma$ -extension operation.

The next example illustrates this construction for the dual of Fano matroid.

*Example 2.2.* Let  $M = F_7^*$  be the dual of the Fano matroid  $F_7$ , and let  $S = \{1, 2, 3, 4, 5, 6, 7\}$  be the ground set of  $M$ . The matrix  $A$  that represents  $M$  over  $GF(2)$  is given by.

$$A = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} & \end{matrix} \cdot \quad (2.1)$$

Consider the set  $X = \{1, 3, 4\}$  contained in the base of  $M$ . Then, the corresponding matrix  $A^X$  is given by

$$A^X = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \gamma_1 & \gamma_3 & \gamma_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} & \end{matrix} \cdot \quad (2.2)$$

The vector matroid of  $A^X$  is the matroid  $(F_7^*)^X$ .

**Corollary 2.3.** Let  $M = M[A]$  be a binary matroid on  $S$ . Let  $X$  be a subset of a base of  $M$ , and  $M^X$  be the  $\Gamma$ -extension of  $M$  on the set  $S \cap \Gamma$ . Then,  $M^X \setminus \Gamma = M$ , that is,  $M^X$  is an extension of  $M$ .

**Corollary 2.4.** Let  $r$  and  $r'$  be the rank functions of the matroids  $M$  and  $M^X$ , respectively. Then  $r'(M^X) = r(M) + 1$ .

With the help of Lemma 2.5, we characterize the circuits of the matroid  $M^X$ .

**Lemma 2.5.** (1) Every circuit of  $M$  is a circuit of  $M^X$ .

(2) Every circuit of  $M^X$  contains at least one element of  $S$ .

(3) Every circuit of  $M^X$  contains even number of elements of  $\Gamma$ .

The proof follows from the construction of the matrix  $A^X$ .

*Remark 2.6.* Let  $M^X$  be a single- $\Gamma$ -extension of  $M$  (i.e.,  $|X| = 1$ ). Then every circuit of  $M^X$  is a circuit of  $M$  and vice versa. In fact, the added element  $\gamma$  is a coloop in the resulting matroid.

**Theorem 2.7.** Let  $M$  be a binary matroid on  $S$  with representation matrix  $A = [I_r \mid E]$  over  $GF(2)$  and  $X$  be a subset of a base of  $M$ . Then, a subset  $Z$  of  $S \cup \Gamma$  is a circuit of  $M^X$  if and only if one of the following conditions hold:

- (1)  $Z = \{e_i, e_j, \gamma_i, \gamma_j\}$ , where  $i \neq j$ ,  $\gamma_i, \gamma_j \in \Gamma$  and  $e_i, e_j \in X$  for  $1 \leq i, j \leq r$ ,
- (2)  $Z = J \cup D$ , where  $J \subseteq \Gamma$ ,  $|J|$  is an even integer and  $\phi \neq D \subseteq S$  is such that  $D \cup X_J$  is a circuit in  $M$ , where  $X_J = \{e_i \in X \mid \gamma_i \in J\}$ .

*Proof.* If  $Z = \{e_i, e_j, \gamma_i, \gamma_j\}$ , then, by Definition 2.1 of  $A^X$ ,  $Z$  is a circuit of  $M^X$ . Now, let  $Z = J \cup D$  be as stated in (2). If  $|J| = 0$ , then  $J = \phi$ ,  $X_J = \phi$ , and  $Z = D$  is a circuit of  $M$ . Suppose that  $J \subseteq \Gamma$ ,  $J \neq \phi$ , and  $X_J \cup D$  is a circuit of  $M$ . Then clearly,  $Z = D \cup J$  is a circuit of  $M^X$ .

Conversely, let  $Z \subseteq S \cup \Gamma$  be a circuit of  $M^X$ , we have two cases:

- (I)  $Z \cap \Gamma = \phi$ . Then  $J = \phi$ ,  $X_J = \phi$ . Thus,  $Z = D$  is a circuit in  $M$  and the condition (2) in the result holds.
- (II) Let  $Z \cap \Gamma \neq \phi$ , and suppose that  $Z \cap \Gamma = J$ . We have two subcases:
  - (i)  $Z \cap X \neq \phi$ . Then,  $Z \cap X = \{e_i, e_j\} = D = X_J$  and  $J = \{\gamma_i, \gamma_j\}$ . Thus,  $Z = \{e_i, e_j, \gamma_i, \gamma_j\}$  and condition (1) in the result holds.
  - (ii)  $Z \cap X = \phi$ . Take  $Z \cap S = D$ . Then  $D \cup X_J$  is a circuit of  $M$  and  $D \cup J$  is a circuit of  $M^X$ . Thus,  $Z = D \cup J$ , and the condition (2) in the result holds.  $\square$

We characterize the independent sets of  $M^X$  in terms of independent sets of  $M$ . Firstly, we have the following lemma.

**Lemma 2.8.** (1) Every independent set of  $M$  is independent in  $M^X$ .  
(2) Every subset of  $\Gamma$  is independent in  $M^X$ .

The proof is straightforward.

*Remark 2.9.* Let  $M^X$  be a single- $\Gamma$ -extension of  $M$ . Then, every independent set in  $M^X$  is also independent in  $M$  and vice versa.

**Theorem 2.10.** Let  $M$  be a binary matroid on  $S$  and  $M^X$  be the  $\Gamma$ -extension matroid of  $M$  with respect to  $X$ . Let  $\mathcal{O}$  be a collection of independent sets of  $M$ . Then, a subset  $I$  of  $S \cup \Gamma$  is an independent set of  $M^X$  if and only if one of the following conditions hold:

- (1)  $I = I_1 \cup \{\gamma\}$ , where  $\gamma \in \Gamma$  and  $I_1 \in \mathcal{O}$ .
- (2)  $I = I_1 \cup J$ , where  $J \subseteq \Gamma$ ,  $I_1 \in \mathcal{O}$  and  $I_1 \cup X_J$  contains no circuit of  $M$ .

*Proof.* If  $I_1 \in \mathcal{O}$ , then clearly  $I_1 \cup \{\gamma\}$  for  $\gamma \in \Gamma$  is an independent set in  $M^X$ . Now, suppose that  $I_1 \cup X_J$  contains no circuit of  $M$ . On the contrary, suppose that  $I_1 \cup J$  contains a circuit say  $C'$  of  $M^X$ . Then  $C' \setminus J \subseteq I_1$  and  $(C' \setminus J) \cup X_J \subseteq I_1 \cup X_J$ . But  $(C' \setminus J) \cup X_J$  is a circuit of  $M$  and is contained in  $I_1 \cup X_J$ , a contradiction.

Conversely, let  $I$  be an independent set in  $M^X$  and  $I \subseteq S \cup \Gamma$ . We have two cases.

- (I) Let  $I \cap \Gamma = \phi$ . Then  $I \subseteq S$  and  $I$  is independent in  $M$ .
- (II) Let  $I \cap \Gamma \neq \phi$ , and let  $I \cap \Gamma = J$ . Then  $J \subseteq \Gamma$  and  $J \subseteq I$ .

We prove that  $I - J \subseteq S$  is an independent set in  $M$ . On the contrary, suppose that  $I - J$  contains a circuit of  $M$ , say  $C$ , then  $C \subseteq I - J$  and  $C \subseteq I$  gives a contradiction.

Letting  $I - J = I_1$ , we have  $I = I_1 \cup J$ . We claim that  $I_1 \cup X_J$  does not contain any circuit of  $M$ . If  $I_1 \cup X_J$  contains a circuit of  $M$ , say  $C'$ , then  $C' \setminus X_J \subseteq I_1$ . Further,  $(C' \setminus X_J) \cup J \subseteq I_1 \cup J$ , and thus  $(C' \setminus X_J) \cup J \subseteq I$  leads to a contradiction as  $(C' \setminus X_J) \cup J$  is a circuit of  $M^X$ . This completes the proof of the theorem.  $\square$

**Corollary 2.11.** *Let  $\mathcal{O}$  and  $\mathcal{O}^X$  denote the collection of independent sets of  $M$  and  $M^X$ , respectively. If  $|X| = 1$ , then  $\mathcal{O}^X = \{I \cup \{\gamma\} \mid I \in \mathcal{O}, \gamma \in \Gamma\}$ .*

**Corollary 2.12.** *A subset  $I$  of  $S$  is independent in  $M$  if and only if  $I \cup \{\gamma\}$  for  $\gamma \in \Gamma$  is independent in  $M^X$ .*

**Corollary 2.13.** *Let  $r$  and  $r'$  be the rank functions of  $M$  and  $M^X$ , respectively. Then  $r'(Z \cup \{\gamma\}) = r(Z) + 1$  for  $Z \subseteq S$ .*

In the next theorem, we characterize the bases of the matroid  $M^X$  in terms of the bases of  $M$ .

**Lemma 2.14.** *Let  $Z \subseteq S$ , then  $Z$  is an independent set in  $M^X$  if and only if  $Z$  is an independent set in  $M$ .*

The proof is straightforward.

**Corollary 2.15.** *Let  $Z$  be a subset of  $S$ . Then  $r'(Z) = r(Z)$  where  $r'$  and  $r$  are rank functions of  $M^X$  and  $M$ , respectively.*

**Theorem 2.16.** *A subset  $B'$  of  $S \cup \Gamma$  is a base for  $M^X$  if and only if  $B' = (B \setminus D) \cup J$ , where  $\phi \neq J \subseteq \Gamma$ ,  $D \subset B$ ,  $|D| = |J| - 1$  and  $(B \setminus D) \cup X_J$  contains no circuit of  $M$ .*

*Proof.* Let  $B$  be a base for  $M$ . Then  $B$  is an independent set in  $M$ , and so  $B \setminus D$  is independent in  $M$ . Let  $B \setminus D = I_1$ . Then, by Theorem 2.10,  $I_1 \cup J$ , where  $J \subseteq \Gamma$ ,  $|D| = |J| - 1$  and  $(B \setminus D) \cup X_J$  contains no circuit of  $M$  and hence is independent in  $M^X$ . Moreover,  $r'((B \setminus D) \cup J) = |(B \setminus D) \cup J| = |B| - |D| + |J| = |B| + 1 = r'(M^X)$ .

We conclude that  $B' = (B \setminus D) \cup J$  is a base for  $M^X$ .

Conversely, let  $B' \subseteq S \cup \Gamma$  be a base for  $M^X$ . Firstly, we show that  $B' \cap \Gamma \neq \phi$ . On the contrary, suppose that  $B' \cap \Gamma = \phi$ . Then  $B' \subseteq S$  and is independent in  $M^X$ . So by Lemma 2.14,  $B'$  is independent in  $M$ . Also by Corollary 2.15,

$$r'(B') = r(B) = |B'| = r(M) + 1. \quad (2.3)$$

This shows that  $B' \subseteq S$  and  $r(B') > r(M)$ ; a contradiction.

Now, let  $B' \cap \Gamma = J$ . Then  $B' - J \subseteq S$  is independent in  $M^X$  as well as in  $M$ . It can be extended to form the base of  $M$ . Let  $D \subseteq S - (B' - J)$  be such that  $(B' - J) \cup D = B$  is a base for  $M$ . Then  $B' = (B \setminus D) \cup J$ . We claim that  $|D| = |J| - 1$ . Now,

$$|B| = |(B' - J) \cup D| = |B'| - |J| + |D| = r(M) + 1 - |J| + |D|. \quad (2.4)$$

Since  $|B| = r(M)$ , we conclude that  $1 - |J| + |D| = 0$ , that is,  $|D| = |J| - 1$ . Finally, we show that  $(B \setminus D) \cup X_J$  contains no circuit of  $M$ . On the contrary, suppose that  $(B \setminus D) \cup X_J$  contains a

circuit, say  $C$  of  $M$ . Then  $C \setminus X_J \subseteq B \setminus D$  and  $(C \setminus X_J) \cup J \subseteq (B \setminus D) \cup J$ . Thus,  $(C \setminus X_J) \cup J \subseteq B'$  leads to a contradiction. This completes the proof of theorem.  $\square$

**Corollary 2.17.** *Every base of  $M^X$  contains at least one element of  $\Gamma$ .*

### 3. Connectivity of $M^X$

Let  $M$  be a binary matroid on a set  $S$  and  $A$  be the representation matrix of  $M$  over  $GF(2)$ . If  $M$  is bridgeless, then  $\Gamma$ -extension of  $M$  with respect to a singleton subset  $X$  of  $B$  yields a disconnected matroid.

**Lemma 3.1.** *Let  $e_i \in S$  be a coloop in a matroid  $M$  and  $e_i \notin X$ . Then  $e_i$  is a coloop in  $M^X$ .*

The proof is straightforward.

**Corollary 3.2.** *Suppose that no element of  $S - X$  is a coloop of  $M$ . Then  $M^X$  has no coloops.*

**Theorem 3.3.** *Let  $|X| \geq 2$ . If  $M$  is connected matroid, then, so is  $M^X$ .*

*Proof.* Assume that  $M$  is connected. We show that for every pair of elements  $u, v \in S \cup \Gamma$  there is a circuit of  $M^X$  containing  $u$  and  $v$ . We have three cases.

- (1) Let  $u, v \in S$ . By hypothesis,  $M$  is connected. So there is a circuit of  $M$  say  $C$ , containing  $u$  and  $v$ . Since  $C$  is a circuit in  $M^X$ , we are through.
- (2) Let  $u, v \in \Gamma$  and let  $u = \gamma_i$  and  $v = \gamma_j$ . Then the 4-circuit  $\{e_i, e_j, \gamma_i, \gamma_j\}$  in  $M^X$  contains  $u$  and  $v$ .
- (3) Let  $u \in S$  and  $v \in \Gamma$ . By assumption  $|X| \geq 2$ . So there is an element say  $w \in \Gamma$ . Let  $C_1$  be a circuit of  $M^X$  containing  $v$  and  $w$ . By Lemma 2.5,  $C_1$  contains at least one element of  $S$ , say  $e$ . Now  $u, e \in S$ , and  $M$  is connected, so there is a circuit of  $M$ , say  $C_2$  which contains  $u$  and  $e$ . Thus  $C_1 \cap C_2 \neq \emptyset$  and  $u \in C_1, v \in C_2$ . Then there is a circuit in  $M^X$ , say  $C_3$ , such that  $C_3 \subseteq C_1 \cup C_2$  and  $u, v \in C_3$ . This completes the proof of the theorem.  $\square$

*Remark 3.4.* Converse of the above theorem is not true.

**Theorem 3.5.** *Let  $M^X$  be a 3- $\Gamma$ -extension matroid of  $M$  and  $\Gamma = \{\gamma_i, \gamma_j, \gamma_k\}$ . If  $M$  is a 3-connected matroid on  $S$ , then  $M^X$  is 3-connected.*

*Proof.* On the contrary, suppose that  $M^X$  is not 3-connected, then  $M^X$  has a 1-separated or 2-separated partition. Let  $(A, B)$  be a 2-separated partition of  $S \cup \Gamma$ . That means,  $\min\{|A|, |B|\} \geq 2$  and

$$r'(A) + r'(B) - r'(M^X) \leq 1. \quad (*)$$

We consider three cases.

- (i) Let  $A = \Gamma$  and  $B = S$ . By Lemma 2.8,  $A$  is independent in  $M^X$ , so

$$r'(A) = |A| = |\Gamma| = 3. \quad (3.1)$$

Also, by Lemma 2.14,

$$r'(B) = r'(S) = r(S) = r(M). \quad (3.2)$$

Thus,

$$r'(A) + r'(B) - r'(M^X) = 3 + r(M) - r(M) - 1 = 2. \quad (3.3)$$

This is a contradiction to (\*).

(ii) Let  $A = \{\gamma_i, \gamma_j\}$  and  $B = S \cup \{\gamma_k\}$ .

By Lemma 2.14,  $r'(B) = r(M) + 1$ . So

$$r'(A) + r'(B) - r'(M^X) = 2 + r(M) + 1 - r(M) - 1 = 2 \quad (3.4)$$

gives a contradiction to (\*).

(iii) Let  $A = S_1 \cup \{\gamma_i\}$  and  $B = S_2 \cup \{\gamma_j, \gamma_k\}$ , where  $S_1, S_2 \subseteq S$ , and  $S_1 \neq \emptyset, S_2 \neq \emptyset$ . Then  $\min\{|S_1|, |S_2|\} \geq 1$  and  $r(S_1) + r(S_2) - r(M) \leq r'(A) - 1 + r'(B) - 1 - r'(M^X) + 1 \leq 0$ . Moreover,

$$r(S_1) + r(S_2) \geq r(M). \quad (3.5)$$

Thus,  $r(S_1) + r(S_2) - r(M) = 0$ , and we conclude that  $(S_1, S_2)$  is a 1-separated partition for  $M$ .

This is a contradiction to the fact that  $M$  is 3-connected. By the same argument, we can show that  $M^X$  does not have 1-separated partition.  $\square$

In the last theorem, the condition that  $|X| = 3$  is necessary. Consider the following example.

*Example 3.6.*  $M(K_4)$  is a 3-connected matroid. Let a representation matrix of  $M(K_4)$  be

$$A = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} & \end{matrix}. \quad (3.6)$$

Let  $X \subseteq B$  and  $X = \{1, 2\}$ ,  $\Gamma = \{\gamma_1, \gamma_2\}$ . Then

$$A^X = \begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 & \gamma_1 & \gamma_2 \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{matrix} \quad (3.7)$$

By row operations on  $A^X$ , we can show that

$$A^X = \left[ I_4 \left| \begin{array}{cccc} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right. \right] \quad (3.8)$$

By  $[\ ]$ ,  $M^X = M[A^X]$  is not 3-connected.

If  $|X| = 1$ , then  $M^X$  has a coloop, and it is not 3-connected.

In general, we state the following result whose proof is immediate.

**Corollary 3.7.** *Let  $M$  be a  $n$ -connected binary matroid and  $|X| < n$ . Then  $M^X$  is not  $n$ -connected.*

## References

- [1] P. J. Slater, "A classification of 4-connected graphs," *Journal of Combinatorial Theory. Series B*, vol. 17, pp. 281–298, 1974.
- [2] P. J. Slater, "Soldering and point splitting," *Journal of Combinatorial Theory. Series B*, vol. 24, no. 3, pp. 338–343, 1978.
- [3] T. T. Raghunathan, M. M. Shikare, and B. N. Waphare, "Splitting in a binary matroid," *Discrete Mathematics*, vol. 184, no. 1–3, pp. 267–271, 1998.
- [4] M. M. Shikare, "Splitting operation and connectedness in binary matroids," *Indian Journal of Pure and Applied Mathematics*, vol. 31, no. 12, pp. 1691–1697, 2000.
- [5] M. M. Shikare, G. Azadi, and B. N. Waphare, "Generalized splitting operation and its applications to binary matroids," *Preprint*.
- [6] J. G. Oxley, *Matroid Theory*, Oxford Science Publications, The Clarendon Press/Oxford University Press, New York, NY, USA, 1992.
- [7] K. Truemper, *Matroid Decomposition*, Academic Press, Boston, Mass, USA, 1992.



