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Research Article

Pairwise Balanced Design of Order $6n + 4$ and 2-Fold System of Order $3n + 2$

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In the Steiner triple system, Bose (1939) constructed the $2-(v, 3, 1)$ design for $v = 6n+3$ and later on Skolem (1958) constructed the same for $v = 6n + 1$. In the literature we found a pairwise balanced design (PBD) for $v = 6n + 5$. We also found the 2-fold triple system of the orders $3n$ and $3n + 1$. In this paper, we construct a PBD for $v = 6n + 4$ and a 2-fold system of the order $3n + 2$. The second construction completes the 2-fold system for all $n \in N$.

1. Introduction

A Latin square of order n is an $n \times n$ array, each cell of which contains exactly one of the symbols in $\{1, 2, \ldots, n\}$, such that each row and each column of the array contain each of the symbols in $\{1, 2, \ldots, n\}$ exactly once. A latin square is said to be idempotent if cell (i, i) contains symbol i for $1 \leq i \leq n$. A latin square of order $2n$ is said to be half-idempotent if for $1 \leq i \leq n$ cells (i, i) and $(n + i, n + i)$ contain the symbol i. A latin square is said to be commutative if cells (i, j) and (j, i) contain the same symbol, for all $1 \leq i, j \leq n$.

A quasi-group of order n is a pair $(Q, \circ)$, where Q is a set of size n and "$\circ$" is a binary operation on Q such that for every pair of elements $a, b \in Q$ the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions. As far as we are concerned a quasi-group is just a latin square with a headline and a sideline.

A $t-(v, k, \lambda)$ design is an ordered pair (X, B) where X is a v set of points and B, called a block set, is of k subsets of X with the property that every t subset of X is contained in exactly λ blocks [1]. A $t - (v, k, 1)$ design is defined as a Steiner system and denoted by $S(t, k, v)$. A Steiner triple system (STS) $2 - (v, 3, 1)$ is an ordered pair (S, B), where S is a finite set of points or symbols, and B is a set of 3-element subsets of S called triples, such that each pair of distinct elements of S occurs together in exactly one triple of B. The order of a Steiner triple system (S, B) is the size of the set S, denoted by $|S| = v$.

Theorem 1.1 (see [2, Theorem 1.1.3]). *A Steiner triple system of order v exists if and only if $v \equiv 1$ or $3 \pmod 6$.*

The Bose Construction ($v \equiv 3 \pmod 6$), see [2, 3]

Let $v = 6n+3$ and let $(Q, \circ)$ be an idempotent commutative quasi-group of order $2n+1$, where $Q = \{1, 2, \dots, 2n+1\}$. Let $S = Q \times \{1, 2, 3\})$, and define B to contain the following two types of triples.

Type 1. For $1 \leq i \leq 2n+1$, $\{(i,1), (i,2), (i,3)\} \in B$ (see Figure 2).

Type 2. For $1 \leq i \leq 2n+1$, $\{(i,1), (j,1), (i \circ j, 2)\}, \{(i,2), (j,2), (i \circ j, 3)\}, \{(i,3), (j,3), (i \circ j, 1)\} \in B$ (see Figure 2).

Then (S, B) is a Steiner triple system of order $6n+3$.

The Skolem Construction ($v \equiv 1 \pmod 6$), see [2, 4]

Let $v = 6n+1$ and let $(Q, \circ)$ be a half idempotent commutative quasi-group of order $2n$, where $Q = \{1, 2, \dots, 2n\}$. Let $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$, and define B as follows.

Type 1. For $1 \leq i \leq n$, $\{(i,1), (i,2), (i,3)\} \in B$.

Type 2. For $1 \leq i \leq n$, $\{\infty, (n+i,1), (i,2)\}, \{\infty, (n+i,2), (i,3)\}, \{\infty, (n+i,3), (i,1)\} \in B$.

Type 3. For $1 \leq i \leq 2n$, $\{(i,1), (j,1), (i \circ j, 2)\}, \{(i,2), (j,2), (i \circ j, 3)\}, \{(i,3), (j,3), (i \circ j, 1)\} \in B$ (see Figure 3).

Then (S, B) is a Steiner triple system of order $6n+1$.

For t a positive integer, a t-wise balanced design D is an ordered pair (S, B), where S is a finite nonempty set (of points) and B is a finite nonempty multiset of subsets of S (called blocks), such that every t subset of S is contained in a constant number $\lambda > 0$ of blocks. If $v = |S|$ and K is the set of sizes of the blocks, then we call D a $t-(v, K, \lambda)$ design. If all blocks of D have the same size k (i.e., $K = \{k\}$), then D is called a t design or a $t-(v, k, \lambda)$ design. A pairwise balanced design of order v with block sizes from K is a pair (S, B), B is a family of subsets (blocks) of S that satisfy (1) if $B_i \in B$, then $|B_i| \in K$ and (2) every pair of distinct elements of S occurs in exactly λ blocks of B.

The $6n+5$ Construction (see [2])

Let $(Q, \circ)$ be an idempotent commutative quasi-group of order $2n+1$, where $Q = \{1, 2, \dots, 2n+1\}$ and let α be the permutation $(1)(2, 3, \dots, 2n+1)$. Let $S = \{\infty_1, \infty_2\} \cup (Q \times \{1, 2, 3\}$ and let B contain the following blocks.

Type 1. $\{\infty_1, \infty_2, (1,1), (1,2), (1,3)\}$.

Type 2. $\{\infty_1, (2i,1), (2i,2)\}$, $\{\infty_1, (2i,3), ((2i)\alpha,1)\}$, $\{\infty_1, ((2i)\alpha,2), ((2i)\alpha,3)\}$, $\{\infty_2, (2i,1), ((2i)\alpha^{-1},3)\}$, for $1 \le i \le n$.

Type 3. $\{(i,1), (j,1), (i \circ j,2)\}$, $\{(i,2), (j,2), (i \circ j,3)\}$, $\{(i,3), (j,3), ((i \circ j)\alpha,1)\}$ for $1 \le i \le 2n+1$.

Then (S, B) is a PBD$(6n+5)$ with exactly one block of size 5 and the rest of size 3.

A λ-fold triple system is a pair of (S, B), where S is a finite set and B is a collection of 3-element subsets of S called triples such that each pair of distinct elements of S belongs to exactly λ triples of B.

Theorem 1.2 (see [2, Theorem 2.3.7]). *The spectrum of* 2*-fold triple systems is precisely the set of all* $v \equiv 0$ *or* $1 \pmod 3$.

$3n$ Construction (see [2])

Let $(Q, \circ)$ be an idempotent (not necessarily commutative) quasi-group, where $Q = \{1, 2, \ldots, n\}$. Let $S = Q \times \{1, 2, 3\}$. We denote B which contains the following two types of triples.

Type 1. $\{(x,1), (x,2), (x,3)\}$ occurs exactly twice in B for all $x \in Q$.

Type 2. For $x \neq y$, the six triples, $\{(x,1), (y,1), (x \circ y,2)\}$, $\{(y,1), (x,1), (y \circ x,2)\}$, $\{(x,2), (y,2), (x \circ y,3)\}$, $\{(y,2), (x,2), (y \circ x,3)\}$, $\{(x,3), (y,3), (x \circ y,1)\}$, $\{(y,3), (x,3), (y \circ x,1)\} \in B$.

Then (S, B) is a 2-fold triple system of order $3n$.

$3n+1$ Construction (see [2, 4])

Let $(Q, \circ)$ be an idempotent (not necessarily commutative) quasi-group, where $Q = \{1, 2, \ldots, n\}$. Let $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$. We denote B which contains the following two types of triples.

Type 1. The four triples $\{\infty, (x,1), (x,2)\}$, $\{\infty, (x,2), (x,3)\}$, $\{\infty, (x,1), (x,3)\}$, $\{(x,1), (x,2), (x,3)\} \in B$ for all $x \in Q$

Type 2. If $x \neq y$, the six triples $\{(x,1), (y,1), (x \circ y,2)\}$, $\{(y,1), (x,1), (y \circ x,2)\}$, $\{(x,2), (y,2), (x \circ y,3)\}$, $\{(y,2), (x,2), (y \circ x,3)\}$, $\{(x,3), (y,3), (x \circ y,1)\}$, $\{(y,3), (x,3), (y \circ x,1)\} \in B$.

Then (S, B) is a 2-fold triple system of order $3n+1$.

2. Main Results

$6n+4$ Construction

Let $(Q, \circ)$ be an idempotent commutative quasi-group of order $2n+1$, where $Q = \{1, 2, \ldots, 2n+1\}$. Now we construct (S, B) where $S = \{\infty\} \cup \{Q \times \{1, 2, 3\}\}$ and B contains the following blocks.

Type 1. $\{\infty, (i,1), (i,2), (i,3)\} \in B$, for $1 \le i \le 2n+1$.

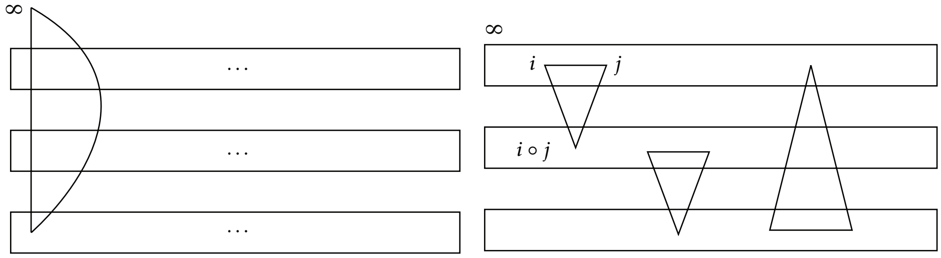

Figure 1: Graphical representation of $(6n+4)$ construction.

Type 2. $\{(i,1),(j,1),(i\circ j,2)\},\{(i,2),(j,2),(i\circ j,3)\},\{(i,3),(j,3),(i\circ j,1)\}\in B$ for $1\le i<j\le 2n+1$.

Theorem 2.1. *The ordered pair* (S,B) *is a pairwise balanced design* $2-(6n+4,\{4,3\},1)$.

Proof. The number of points v is $6n+4$ (see Figure 1). The blocks of Type 1 are of length 4, whereas the blocks of Type 2 are of length 3. It is clear from the construction that any pair of points of S occurs in exactly one block. Hence (S,B) is a PBD $2-(6n+4,\{4,3\},1)$.

The number of blocks of the PBD $2-(6n+4,\{4,3\},1)$ are $(2n+1)+(3.^{2n+1}C_2)=(2n+1)(3n+1)$. ∎

For example, $n=2$, that is, $v=6.2+4=16$, the quasi-group of order 5 is

$$\begin{array}{c|ccccc} o & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 5 & 2 & 3 & 4 \\ 2 & 5 & 2 & 4 & 1 & 3 \\ 3 & 2 & 4 & 3 & 5 & 1 \\ 4 & 3 & 1 & 5 & 4 & 2 \\ 5 & 4 & 3 & 1 & 2 & 5 \end{array} \tag{2.1}$$

Let $S=\{\infty,1,2,\ldots,15\}$ and B contains the following blocks.

Type 1. $\{\infty,1,6,11\},\{\infty,2,7,12\},\{\infty,3,8,13\},\{\infty,4,9,14\},\{\infty,5,10,15\}\in B$.

Type 2. $\{1,2,10\},\{1,3,7\},\{1,4,8\},\{1,5,9\},\{2,3,9\},\{2,4,6\},\{2,5,8\},\{3,4,10\},\{3,5,6\},\{4,5,7\},\{6,7,15\},\{6,8,12\},\{6,9,13\},\{6,10,14\},\{7,8,14\},\{7,9,11\},\{7,10,13\},\{8,9,15\},\{8,10,11\},\{9,10,12\},\{11,12,5\},\{11,13,2\},\{11,14,4\},\{11,15,4\},\{12,13,4\},\{12,14,1\},\{12,15,3\},\{13,14,5\},\{13,15,1\},\{14,15,2\}\in B$.

So (S,B) is a PBD $2-(16,\{4,3\},1)$ with a total of 35 blocks in which $2.2+1(=5)$ blocks of size 4 and $3.^5C_2(=30)$ blocks of size 3.

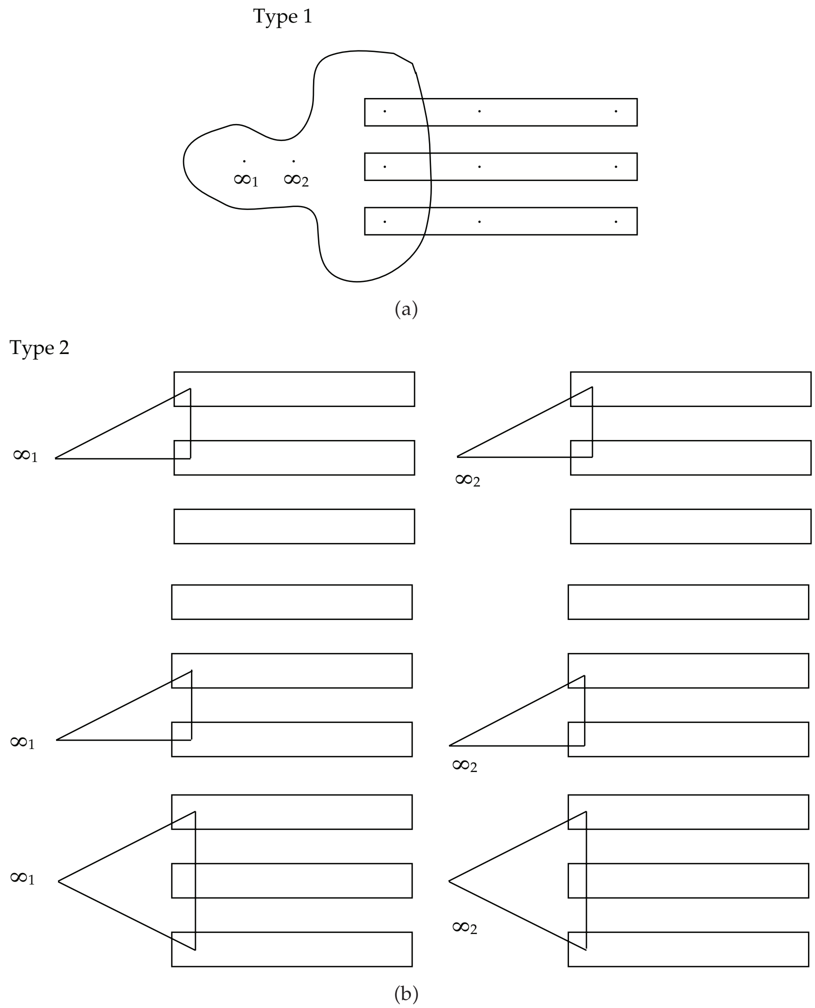

Figure 2: Graphical representation of $(3n+2)$ construction (Type 1 and Type 2).

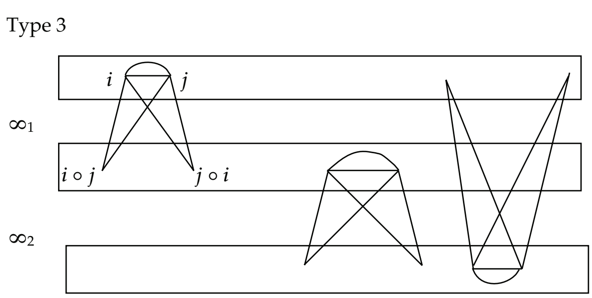

Figure 3: Graphical representation of $(3n+2)$ construction (Type 3).

$3n + 2$ Construction

Let $(Q, \circ)$ be an idempotent (not necessarily commutative) quasi-group, where $Q = \{1, 2, \dots, n\}$. Let $S = \{\infty_1, \infty_2\} \cup (Q \times \{1, 2, 3\})$. We define B as follows.

Type 1. $\{\infty_1, \infty_2, (1,1), (1,2), (1,3)\}$ occurs exactly twice in B.

Type 2. For $1 \leq x \leq n$, $\{\infty_1, (x,1), (x,2)\}, \{\infty_1, (x,2), (x,3)\}, \{\infty_1, (x,1), (x,3)\}$, $\{\infty_2, (x,1), (x,2)\}, \{\infty_2, (x,2), (x,3)\}, \{\infty_2, (x,1), (x,3)\} \in B$ for all $x \in Q$.

Type 3. If $x \neq y$, the six triples $\{(x,1), (y,1), (x \circ y, 2)\}, \{(y,1), (x,1), (y \circ x, 2)\}$, $\{(x,2), (y,2), (x \circ y, 3)\}, \{(y,2), (x,2), (y \circ x, 3)\}, \{(x,3), (y,3), (x \circ y, 1)\}, \{(y,3), (x,3), (y \circ x, 1)\} \in B$.

Theorem 2.2. *The ordered pair (S, B) is a 2-fold system of order $3n + 2$.*

Proof. The number of points is $v = 3n + 2$. The Type 1 of new (S, B) construction consists of block length 5. The Type 2 and Type 3 of the above construction consist of length 3. Any pair of points of S occurs in exactly two blocks. Hence the construction (S, B) is a 2-fold system. Also the ordered pair (S, B) holds $2 - (3n + 2, \{3, 5\}, 2)$ design. $\square$

For example, $n = 4$, the quasi-group of order 4 is

$$\begin{array}{c|cccc} o & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 3 & 4 & 2 \\ 2 & 4 & 2 & 1 & 3 \\ 3 & 2 & 4 & 3 & 1 \\ 4 & 3 & 1 & 2 & 4 \end{array} \tag{2.2}$$

Let $S = \{\infty_1, \infty_2, 1, 2, \dots, 12\}$ and B contains the following blocks.

Type 1. $\{\infty_1, \infty_2, 1, 5, 9\}$, $\{\infty_1, \infty_2, 1, 5, 9\} \in B$.

Type 2. $\{\infty_1, 2, 6\}, \{\infty_1, 2, 10\}, \{\infty_1, 6, 10\}, \{\infty_1, 3, 7\}, \{\infty_1, 7, 11\}, \{\infty_1, 3, 11\}, \{\infty_1, 4, 8\}$, $\{\infty_1, 8, 12\}, \{\infty_1, 4, 12\}, \{\infty_2, 2, 6\}, \{\infty_2, 2, 10\}, \{\infty_2, 6, 10\}, \{\infty_2, 3, 7\}, \{\infty_2, 7, 11\}, \{\infty_2, 3, 11\}$, $\{\infty_2, 4, 8\}, \{\infty_2, 8, 12\}, \{\infty_2, 4, 12\} \in B$.

Type 3. $\{1,2,7\}, \{2,1,8\}, \{1,3,8\}, \{3,1,6\}, \{1,4,6\}, \{4,1,7\}, \{2,3,5\}, \{3,2,8\}, \{2,4,7\}, \{4,2,5\}$, $\{3,4,5\}, \{4,3,6\}, \{5,6,11\}, \{6,5,12\}, \{5,7,12\}, \{7,5,10\}, \{5,8,10\}, \{8,5,11\}, \{6,7,9\}, \{7,6,12\}$, $\{6,8,11\}, \{8,6,9\}, \{7,8,9\}, \{8,7,10\}, \{9,10,3\}, \{10,9,4\}, \{9,11,4\}, \{11,9,2\}, \{9,12,2\}$, $\{12,9,3\}, \{10,11,1\}, \{11,10,4\}, \{10,12,3\}, \{12,10,1\}, \{11,12,1\}, \{12,11,2\} \in B$.

So (S, B) is a PBD $2 - (14, \{5, 3\}, 1)$ with 2 blocks of size 5 and the rest of the blocks of size 3.

3. Conclusion

There already exist STS for $v = 6n + 1$ and $v = 6n + 3$ and PBD for $v = 6n + 5$. In this paper, we construct PBD for $v = 6n + 4$. The PBD for $v = 6n$ and, $v = 6n + 2$ is still now open. The 2-fold triple system exists for $v \equiv 0$ or $1 \pmod 3$ and, in this paper, we construct 2-fold system exits for $3n+2$. These complete that the 2-fold system holds for all natural numbers n.

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