International Scholarly Research Network
ISRN Discrete Mathematics
Volume 2012, Article ID 308595, 8 pages
doi:10.5402/2012/308595

Research Article

Bipancyclic Properties of Faulty Hypercubes

Chun-Nan Hung and Min-Kun Hsiao

Department of Computer Science and Information Engineering, Da-Yeh University, Changhua 51591, Taiwan

Correspondence should be addressed to Chun-Nan Hung, chunnan.hung@gmail.com

Received 24 July 2012; Accepted 5 September 2012

Academic Editors: L. Ji, U. Vaccaro, W. Wallis, and X. Yong

A bipartite graph $G = (V, E)$ is bipancyclic if it contains cycles of every even length from 4 to $|V|$ and edge bipancyclic if every edge lies on a cycle of every even length from 4 to $|V|$. Let Q_n denote the n-dimensional hypercube. Let F be a subset of $V(Q_n) \cup E(Q_n)$ such that F can be decomposed into two parts F_{av} and F_e, where F_{av} is a union of f_{av} disjoint adjacent pairs of $V(Q_n)$, and F_e consists of f_e edges. We prove that $Q_n - F$ is bipancyclic if $f_{av} + f_e \leq n-2$. Moreover, $Q_n - F$ is edge bipancyclic if $f_{av} + f_e \leq n-2$ with $f_{av} < n-2$.

1. Introduction

Interconnection networks play an important role in parallel computing/communication systems. The graph embedding problem, which is a central issue in evaluating a network, asks if the guest graph is a subgraph of a host graph. A benefit of graph embedding is that we can apply an existing algorithm for guest graphs to host graphs. This problem has attracted a burst of studies in recent years. Note that cycle networks are useful for designing simple algorithms with low communication costs. Thus, there are many studies on the cycle embedding problem. The cycle embedding problem deals with identifying all possible lengths of the cycles in a given graph. For the graph definition and notation, we follow [1].

Let $\mathbf{u} = u_n u_{n-1} \cdots u_2 u_1$ be n-bit binary strings. The *Hamming weight* of $\mathbf{u}$, denoted by $w(\mathbf{u})$, is the number of i such that $u_i = 1$. Let $\mathbf{u} = u_n u_{n-1} \cdots u_2 u_1$ and $\mathbf{v} = v_n v_{n-1} \cdots v_2 v_1$ be two n-bit binary strings. The *Hamming distance* $h(\mathbf{u}, \mathbf{v})$ between two vertices $\mathbf{u}$ and $\mathbf{v}$ is the number of different bits in the corresponding strings of both vertices. The *n-dimensional hypercube*, denoted by Q_n, has all n-bit binary strings as its vertices; two vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent if and only if $h(\mathbf{u}, \mathbf{v}) = 1$. Obviously, Q_n is a bipartite graph with bipartition $A = \{\mathbf{u} \mid w(\mathbf{u}) \text{ is even}\}$ and $B = \{\mathbf{u} \mid w(\mathbf{u}) \text{ is odd}\}$. A vertex $\mathbf{u}$ of Q_n is *white* if $w(\mathbf{u})$ is odd, otherwise

u is *black*. It is known that the distance between **u** and **v** is $d_{Q_n}(\mathbf{u},\mathbf{v}) = h(\mathbf{u},\mathbf{v})$. For $i = 0, 1$, let Q_n^i denote the subgraph of Q_n induced by $\{\mathbf{u} = u_n u_{n-1} \cdots u_2 u_1 \mid u_j = i \text{ for some } 1 \le j \le n\}$. Obviously, Q_n^i is isomorphic to Q_{n-1}. For any vertex $\mathbf{u} = u_n u_{n-1} \ldots u_2 u_1$, we use $(\mathbf{u})_j$ to denote the bit u_j. Moreover, we use $(\mathbf{u})^k$ to denote the vertex $\mathbf{v} = v_n v_{n-1} \cdots v_2 v_1$ with $v_i = u_i$ for $1 \le i \ne k \le n$ and $v_k = 1 - u_k$. An edge $(\mathbf{u},\mathbf{v}) \in E(Q_n)$ is of *dimension* k if $\mathbf{v} = (\mathbf{u})^k$.

The hypercube Q_n is one of the most popular interconnection networks for parallel computers/communication systems [2]. This is partly due to its attractive properties, such as regularity, recursive structure, vertex and edge symmetry, maximum connectivity as well as effective routing and broadcasting algorithms.

Note that the hypercube Q_n is a bipartite graph for every integer n. The corresponding cycle embedding problem on bipartite graphs is called the bipancyclic property. A bipartite graph G is *bipancyclic* if it contains cycles of every even length from 4 to $|V(G)|$, inclusive.

There are some variations of the bipancyclic property. A bipartite graph G is *edge bipancyclic* if every edge lies on a cycle of every even length from 4 to $|V(G)|$, inclusive. A bipartite graph is *k-edge fault tolerant bipancyclic* if $G - F$ is bipancyclic for any $F \subset E(G)$ with $|F| \le k$. Moreover, a bipartite graph is *k-edge fault tolerant edge bipancyclic* if $G - F$ is edge bipancyclic for any $F \subset E(G)$ with $|F| \le k$. The following theorem is proved.

Theorem 1.1 (see [3]). *Q_n is $(n-2)$-edge fault tolerant edge bipancyclic if $n \ge 2$.*

In this paper, we improve Theorem 1.1 by considering both edge faults and vertex faults. However, we restrict the faults on the vertex set to those occurring only on disjoint adjacent pairs. Let F be a subset of $V(Q_n) \cup E(Q_n)$ such that F can be decomposed into two parts F_{av} and F_e where F_{av} is a union of f_{av} disjoint adjacent pairs of Q_n, and F_e consists of f_e edges. More precisely, $F_{av} = \cup_{i=1}^{f_{av}} \{\mathbf{b}_i, \mathbf{w}_i\}$, where $(\mathbf{b}_i, \mathbf{w}_i) \in E(Q_n)$ and $\{\mathbf{b}_i, \mathbf{w}_i\} \cap \{\mathbf{b}_j, \mathbf{w}_j\} = \emptyset$ for $i \ne j$. Without loss of generality, we assume that $\{\mathbf{b}_i \mid 1 \le i \le f_{av}\}$ is a set of f_{av} black vertices, and $\{\mathbf{w}_i \mid 1 \le i \le f_{av}\}$ is a set of f_{av} white vertices. We will prove that $Q_n - F$ is bipancyclic if $f_{av} + f_e \le n - 2$. Moreover, $Q_n - F$ is edge bipancyclic if $f_{av} + f_e \le n - 2$ with $f_{av} < n - 2$.

2. Preliminary

We need the following lemmas.

Lemma 2.1 (see [4]). *Let e be any edge of Q_n for $n \ge 2$. There are $n - 1$ cycles of length four that contain e in common.*

Lemma 2.2 (see [5]). *Assume that n is any positive integer with $n \ge 2$, and F is a subset of $E(Q_n)$ with $|F| \le n-2$. Then there exists a Hamiltonian path of $Q_n - F$ joining any two vertices from different bipartite sets. Moreover, there exists a Hamiltonian path joining $\mathbf{y}$ to $\mathbf{z}$ of $Q_n - F - \{\mathbf{x}\}$ for $\mathbf{x}$ in some partite set and $\mathbf{y}$,$\mathbf{z}$ in the other partite set for $|F| \le n - 3$.*

Lemma 2.3 (see [6]). *Assume that n is any positive integer with $n \ge 2$. Let $\mathbf{u}$ and $\mathbf{x}$ be two distinct white vertices of Q_n and $\mathbf{v}$ and $\mathbf{y}$ be two distinct black vertices of Q_n. There are two disjoint paths P_1 and P_2 such that (1) P_1 joins $\mathbf{u}$ to $\mathbf{v}$, (2) P_2 joins $\mathbf{x}$ to $\mathbf{y}$, and (3) $P_1 \cup P_2$ spans Q_n.*

We extend the above lemma by considering the occurrence of edge faults.

Lemma 2.4. *Assume that n is any positive integer with $n \geq 3$, and F is a subset of $E(Q_n)$ with $|F| \leq n-3$. Let $\mathbf{u}$ and $\mathbf{x}$ be two distinct white vertices of Q_n and $\mathbf{v}$ and $\mathbf{y}$ be two distinct black vertices of Q_n. There are two disjoint paths P_1 and P_2 of $Q_n - F$ such that (1) P_1 joins $\mathbf{u}$ to $\mathbf{v}$, (2) P_2 joins $\mathbf{x}$ to $\mathbf{y}$, and (3) $P_1 \cup P_2$ spans $Q_n - F$.*

Proof. We prove this lemma by induction on n. By Lemma 2.3, this lemma is true for $n = 3$. Thus, we assume $n \geq 4$ and $|F| \geq 1$. For $1 \leq i \leq n$, let F_i denote the set of i-dimensional edges in F. Thus, $\sum_{i=1}^{n} |F_i| = |F|$. Without loss of generality, we assume that $|F_n| \geq 1$. For $i = 0, 1$, we use F^i to denote the set $E(Q_n^i) \cap F$. Obviously, $|F^i| \leq n-4$. Let $S^i = \{\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}\} \cap V(Q_n^i)$ for $i = 0$, 1. Without loss of generality, we can assume that $|S^0| \geq |S^1|$. We have the following cases.

Case 1 ($|S^0| = 4$). By induction, there exist two spanning disjoint paths R_1 and R_2 of $Q_n^0 - F^0$ such that R_1 joins $\mathbf{u}$ to $\mathbf{v}$, and R_2 joins $\mathbf{x}$ to $\mathbf{y}$. Since $2^{n-1} - 2 > 2(n-3)$, there exists an edge $(\mathbf{s}, \mathbf{t})$ of R_1 or R_2 such that $\{(\mathbf{s}, (\mathbf{s})^n), (\mathbf{t}, (\mathbf{t})^n)\} \cap F = \emptyset$. Without loss of generality, we can assume that $(\mathbf{s}, \mathbf{t})$ is in R_1. Thus, R_1 can be written as $\langle \mathbf{u}, H_1, \mathbf{s}, \mathbf{t}, H_2, \mathbf{v} \rangle$. By Lemma 2.2, there exists a Hamiltonian path Q joining $(\mathbf{s})^n$ to $(\mathbf{t})^n$ of $Q_n^1 - F^1$. We set P_1 as $\langle \mathbf{u}, H_1, \mathbf{s}, (\mathbf{s})^n, Q, (\mathbf{t})^n, \mathbf{t}, H_2, \mathbf{v} \rangle$ and set P_2 as R_2.

Case 2 ($|S^0| = 3$). Without loss of generality, we can assume that $\mathbf{y} \in Q_n^1$. Since there are 2^{n-2} black vertices in Q_n^0 and $2^{n-2} - 1 > n-3$ for $n \geq 4$, there exists a black vertex $\mathbf{z}$ of Q_n^0 such that $\mathbf{z} \neq \mathbf{v}$ and $(\mathbf{z}, (\mathbf{z})^n) \notin F$. By induction, there exist two spanning disjoint paths P_1 and R_1 of $Q_n^0 - F^0$ such that P_1 joins $\mathbf{u}$ to $\mathbf{v}$, and R_1 joins $\mathbf{x}$ to $\mathbf{z}$. By Lemma 2.2, there exists a Hamiltonian path R_2 joining $(\mathbf{z})^n$ to $\mathbf{y}$ of $Q_n^1 - F^1$. We set P_2 as $\langle \mathbf{x}, R_1, \mathbf{z}, (\mathbf{z})^n, R_2, \mathbf{y} \rangle$. Thus, P_1 and P_2 form the required paths.

Case 3 ($|S^0| = 2$ and $S^i = \{\mathbf{u}, \mathbf{v}\}$ for some $i = 0, 1$). Without loss of generality, we can assume that $S^0 = \{\mathbf{u}, \mathbf{v}\}$. By Lemma 2.2, there exists a Hamiltonian path P_1 of $Q_n^0 - F^0$ joining $\mathbf{u}$ to $\mathbf{v}$, and there exists a Hamiltonian path P_2 of $Q_n^1 - F^1$ joining $\mathbf{x}$ to $\mathbf{y}$. Obviously, P_1 and P_2 form the required paths.

Case 4 ($|S^0| = 2$ and $S^i \neq \{\mathbf{u}, \mathbf{v}\}$ for each $i = 0, 1$). Without loss of generality, we can assume that $S^0 = \{\mathbf{u}, \mathbf{x}\}$. Since there are 2^{n-2} black vertices in Q_n^0 and $2^{n-2} > n-2$ for $n \geq 4$, there exist two black vertices $\mathbf{s}$ and $\mathbf{t}$ of Q_n^0 such that $\{(\mathbf{s}, (\mathbf{s})^n), (\mathbf{t}, (\mathbf{t})^n)\} \cap F = \emptyset$. By induction, there exist two spanning disjoint paths Q_1 and Q_2 of $Q_n^0 - F^0$ such that Q_1 joins $\mathbf{u}$ to $\mathbf{s}$ and Q_2 joins $\mathbf{x}$ to $\mathbf{t}$. Similarly, there exist two spanning disjoint paths R_1 and R_2 of $Q_n^1 - F^1$ such that R_1 joins $(\mathbf{s})^n$ to $\mathbf{v}$, and R_2 joins $(\mathbf{t})^n$ to $\mathbf{y}$. We set P_1 as $\langle \mathbf{u}, Q_1, \mathbf{s}, (\mathbf{s})^n, R_1, \mathbf{v} \rangle$ and set P_2 as $\langle \mathbf{x}, Q_2, \mathbf{t}, (\mathbf{t})^n, R_2, \mathbf{y} \rangle$. □

Lemma 2.5 (see [7]). *Assume that $n \geq 3$. Let F be a subset of $V(Q_n) \cup E(Q_n)$ such that F can be decomposed into two parts F_{av} and F_e where F_{av} is a union of f_{av} disjoint adjacent pairs of $V(Q_n)$ and F_e consists of f_e edges. Then there exists a Hamiltonian cycle of $Q_n - F$ if $f_{av} + f_e \leq n-2$.*

Lemma 2.6 (see [7]). *Assume that $n \geq 3$. Let F be a subset of $V(Q_n) \cup E(Q_n)$ such that F can be decomposed into two parts F_{av} and F_e, where F_{av} is a union of f_{av} disjoint adjacent pairs of $V(Q_n)$, and F_e consists of f_e edges. Then there exists a Hamiltonian path of $Q_n - F$ between any two vertices from different partite sets of Q_n if $f_{av} + f_e \leq n-3$.*

We improve Lemma 2.6 into the following lemma.

Lemma 2.7. *Assume that $n \geq 3$. Let F be a subset of $V(Q_n) \cup E(Q_n)$ such that F can be decomposed into two parts F_{av} and F_e, where F_{av} is a union of f_{av} disjoint adjacent pairs of $V(Q_n)$, and F_e consists of f_e edges. Then there exists a Hamiltonian path of $Q_n - F$ between any two vertices from different partite sets of Q_n if $f_{av} + f_e \leq n-2$ with $f_{av} \leq n-3$.*

Proof. Let $\mathbf{w}$ be any fault-free white vertex and $\mathbf{b}$ be any fault-free black vertex. We need to construct a Hamiltonian path of $Q_n - F$ joining $\mathbf{w}$ to $\mathbf{b}$ by induction on n. Let $H = F_e \cup \{(\mathbf{b}_i, \mathbf{w}_i) \mid \{\mathbf{b}_i, \mathbf{w}_i\} \subset F_{av}\}$. For $1 \le i \le n$, let H_i denote the set of i-dimensional edges in H. Thus, $\sum_{i=1}^{n} |H_i| = f_e + f_{av}$. Without loss of generality, we assume that $|H_n| = 0$.

By brute force, we can check that the required paths exist for $n = 3, 4$. By Lemmas 2.2 and 2.6, the required paths exist when $f_{av} = 0$ or $f_e + f_{av} \le n - 3$. Therefore, we only need to consider the case $f_{av} \ge 1$ and $f_{av} + f_e = n - 2$ for $n \ge 5$. Thus, $F_e \ne \emptyset$. Let $F_e^i = F_e \cap E(Q_n^i)$, $F_{av}^i = F_{av} \cap V(Q_n^i)$, and $F^i = F_{av}^i \cup F_e^i$ for $i = 0,\ 1$. Let $\mathbf{b}_1$ and $\mathbf{w}_1$ be a pair of F_{av}^0 where $\mathbf{b}_1$ is a black vertex.

Case 1 ($F \not\subset Q_n^i$ and $|F_{av}^i| \le n - 4,$ for $i = 0, 1$). We first consider the case that $\mathbf{w}$ and $\mathbf{b}$ are in the same subcube. Without loss of generality, we can assume that both $\mathbf{w}$ and $\mathbf{b}$ are in Q_n^0. By induction, there exists a Hamiltonian path P_1 of $Q_n^0 - F^0$ joining $\mathbf{w}$ to $\mathbf{b}$. Note that $l(P_1) - 4f_{av}^1 = 2^{n-1} - 2f_{av}^0 - 1 - 4f_{av}^1 \ge 2^{n-1} - 4n + 15 > 0$. We can write P_1 as $\langle \mathbf{w}, R_1, \mathbf{u}, \mathbf{v}, R_2, \mathbf{b} \rangle$ for some $\mathbf{u}$ and $\mathbf{v}$ such that $\{(\mathbf{u})^n, (\mathbf{v})^n\} \cap F_{av}^1 = \emptyset$. By induction, there exists a Hamiltonian path R_3 of $Q_n^1 - F^1$ joining $(\mathbf{u})^n$ to $(\mathbf{v})^n$. Thus, $\langle \mathbf{w}, R_1, \mathbf{u}, (\mathbf{u})^n, R_3, (\mathbf{v})^n, \mathbf{v}, R_2, \mathbf{b} \rangle$ is a desired path.

Thus we consider the case that $\mathbf{w}$ and $\mathbf{b}$ are in different subcubes. Without loss of generality, we can assume that $\mathbf{w} \in Q_n^0$ and $\mathbf{b} \in Q_n^1$. Since there are 2^{n-2} black vertices in Q_n^0 and $2^{n-2} \ge n - 3$ for $n \ge 5$, there exists a black vertex $\mathbf{x}$ in Q_n^0 such that $\{\mathbf{x}, (\mathbf{x})^n\} \cap F_{av} = \emptyset$. By induction, there exists a Hamiltonian path P_1 of $Q_n^0 - F^0$ joining $\mathbf{w}$ to $\mathbf{x}$ and there exists a Hamiltonian path P_2 of $Q_n^1 - F^1$ joining $(\mathbf{x})^n$ to $\mathbf{b}$. Thus, $\langle \mathbf{w}, P_1, \mathbf{x}, (\mathbf{x})^n, P_2, \mathbf{b} \rangle$ is a desired path.

Case 2 ($F \subset Q_n^i$ or $|F_{av}^i| = n - 3$ for some $i = 0, 1$). Without loss of generality, we can assume that $F \subset Q_n^0$ or $|F_{av}^0| = n - 3$. Thus, $|F_{av}^1| = 0$.

Assume that both $\mathbf{w}$ and $\mathbf{b}$ are in Q_n^0. By induction, there exists a Hamiltonian path P_1 of $Q_n^0 - (F - \{\mathbf{b}_1, \mathbf{w}_1\})$ joining $\mathbf{w}$ to $\mathbf{b}$. Suppose that the edge $(\mathbf{b}_1, \mathbf{w}_1)$ is in P_1. Without loss of generality, we can write P_1 as $\langle \mathbf{w}, Z_1, \mathbf{x}, \mathbf{b}_1, \mathbf{w}_1, \mathbf{y}, Z_2, \mathbf{b} \rangle$. By Lemma 2.2, there exists a Hamiltonian path Z_3 of $Q_n^1 - F_e^1$ joining $(\mathbf{x})^n$ to $(\mathbf{y})^n$. Obviously, $\langle \mathbf{w}, Z_1, \mathbf{x}, (\mathbf{x})^n, Z_3, (\mathbf{y})^n, \mathbf{y}, Z_2, \mathbf{b} \rangle$ is a desired Hamiltonian path. Thus, we consider the case that $(\mathbf{b}_1, \mathbf{w}_1)$ is not in P_1. Without loss of generality, P_1 can be written as $\langle \mathbf{w}, R_1, \mathbf{u}, \mathbf{b}_1, \mathbf{v}, R_2, \mathbf{x}, \mathbf{w}_1, \mathbf{y}, R_3, \mathbf{b} \rangle$. By Lemma 2.4, there are two disjoint spanning paths R_4 and R_5 of $Q_n^1 - F_e^1$ such that R_4 joins $(\mathbf{u})^n$ to $(\mathbf{x})^n$ and R_5 joins $(\mathbf{v})^n$ to $(\mathbf{y})^n$ of $Q_n^1 - F_e^1$. Obviously, $\langle \mathbf{w}, R_1, \mathbf{u}, (\mathbf{u})^n, R_4, (\mathbf{x})^n, \mathbf{x}, R_2^{-1}, \mathbf{v}, (\mathbf{v})^n, R_5, (\mathbf{y})^n, \mathbf{y}, R_3, \mathbf{b} \rangle$ is a desired path.

Now, we consider the case that $\mathbf{w}$ and $\mathbf{b}$ are in different subcubes. Without loss of generality, we can assume that $\mathbf{w} \in Q_n^0$ and $\mathbf{b} \in Q_n^1$. By induction, there exists a Hamiltonian path P_1 of $Q_n^0 - (F - \{\mathbf{b}_1, \mathbf{w}_1\})$ joining $\mathbf{w}$ and $\mathbf{b}_1$. Suppose that the edge $(\mathbf{b}_1, \mathbf{w}_1)$ is in P_1. We can write P_1 as $\langle \mathbf{w}, Z_1, \mathbf{y}, \mathbf{w}_1, \mathbf{b}_1 \rangle$. By Lemma 2.2, there exists a Hamiltonian path Z_2 of $Q_n^1 - F_e^1$ joining $(\mathbf{y})^n$ to $\mathbf{b}$. Obviously, $\langle \mathbf{w}, Z_1, \mathbf{y}, (\mathbf{y})^n, Z_2, \mathbf{b} \rangle$ is a desired path. Thus, we consider the case that $(\mathbf{b}_1, \mathbf{w}_1)$ is not in P_1. We can write P_1 as $\langle \mathbf{w}, R_1, \mathbf{u}, \mathbf{w}_1, \mathbf{v}, R_2, \mathbf{x}, \mathbf{b}_1 \rangle$. Assume that $(\mathbf{x})^n \ne \mathbf{b}$. By Lemma 2.4, there are two disjoint spanning paths R_3 and R_4 of $Q_n^1 - F_e^1$ such that R_3 joins $(\mathbf{u})^n$ to $(\mathbf{x})^n$ and R_4 joins $(\mathbf{v})^n$ to $\mathbf{b}$. Obviously, $\langle \mathbf{w}, R_1, \mathbf{u}, (\mathbf{u})^n, R_3, (\mathbf{x})^n, \mathbf{x}, R_2^{-1}, \mathbf{v}, (\mathbf{v})^n, R_4, \mathbf{b} \rangle$ is a desired path. Assume that $(\mathbf{x})^n = \mathbf{b}$. By Lemma 2.2, there exists a Hamiltonian path R_5 of $Q_n^1 - F_e^1 - \{\mathbf{b}\}$ joining $(\mathbf{u})^n$ to $(\mathbf{v})^n$. Obviously, $\langle \mathbf{w}, R_1, \mathbf{u}, (\mathbf{u})^n, R_5, (\mathbf{v})^n, \mathbf{v}, R_2, \mathbf{x}, (\mathbf{x})^n = \mathbf{b} \rangle$ is a desired path.

Finally, we consider the case that $\mathbf{w}$ and $\mathbf{b}$ are in Q_n^1. Let $e = (\mathbf{x}, \mathbf{y})$ be any faulty edge of F, where $\mathbf{x}$ is a white vertex. By Lemma 2.5, there exists a Hamiltonian cycle C_1 of $Q_n^0 - (F - \{e\})$.

Suppose that e is not an edge of C_1. Since the length of C is at least $2^{n-1} - 2(n-3)$, we can write C_1 as $\langle \mathbf{u}, Z_1, \mathbf{v}, \mathbf{u} \rangle$ such that $\{(\mathbf{u})^n, (\mathbf{v})^n\} \cap \{\mathbf{w}, \mathbf{b}\} = \emptyset$ and $\mathbf{u}$ is a white vertex. By

Lemma 2.4, there exist two disjoint spanning paths Z_2 and Z_3 of $Q_n^1 - F_e^1$ such that Z_2 joins $\mathbf{w}$ to $(\mathbf{u})^n$ and Z_3 joins $(\mathbf{v})^n$ to $\mathbf{b}$. Obviously, $\langle \mathbf{w}, Z_2, (\mathbf{u})^n, \mathbf{u}, Z_1, \mathbf{v}, (\mathbf{v})^n, Z_3, \mathbf{b}\rangle$ is a desired path.

Thus, we consider the case that e is in C_1. We can write C_1 as $\langle \mathbf{x}, P_1, \mathbf{y}, \mathbf{x}\rangle$. Suppose that $\{(\mathbf{x})^n, (\mathbf{y})^n\} \cap \{\mathbf{w}, \mathbf{b}\} = \emptyset$. By Lemma 2.4, there exist two disjoint spanning paths P_2 and P_3 of $Q_n^1 - F_e^1$ such that P_2 joins $\mathbf{w}$ to $(\mathbf{x})^n$ and P_3 joins $(\mathbf{y})^n$ to $\mathbf{b}$. Obviously, $\langle \mathbf{w}, P_2, (\mathbf{x})^n, \mathbf{x}, P_1, \mathbf{y}, (\mathbf{y})^n, P_3, \mathbf{b}\rangle$ is a desired path. Suppose that $|\{(\mathbf{x})^n, (\mathbf{y})^n\} \cap \{\mathbf{w}, \mathbf{b}\}| = 1$. Without loss of generality, we assume that $(\mathbf{x})^n = \mathbf{b}$. By Lemma 2.2, there exists a Hamiltonian path P_4 of $Q_n^1 - F_e^1 - \{\mathbf{b}\}$ joining $\mathbf{w}$ to $(\mathbf{y})^n$. Thus, $\langle \mathbf{w}, P_4, (\mathbf{y})^n, \mathbf{y}, P_1^{-1}, \mathbf{x}, (\mathbf{x})^n = \mathbf{b}\rangle$ is a Hamiltonian path of $Q_n - F$. Suppose that $\{(\mathbf{x})^n, (\mathbf{y})^n\} = \{\mathbf{w}, \mathbf{b}\}$. Obviously, C_1 can be written as $\langle \mathbf{y}, P_5, \mathbf{u}, \mathbf{v}, P_6, \mathbf{x}, \mathbf{y}\rangle$ for some black vertex $\mathbf{u}$. By induction, there exists a Hamiltonian path P_7 of $Q_n^1 - F_e^1 - \{\mathbf{w}, \mathbf{b}\}$ joining $(\mathbf{u})^n$ to $(\mathbf{v})^n$. Obviously, $\langle \mathbf{w} = (\mathbf{y})^n, \mathbf{y}, P_5, \mathbf{u}, (\mathbf{u})^n, P_7, (\mathbf{v})^n, \mathbf{v}, P_6, \mathbf{x}, (\mathbf{x})^n = \mathbf{b}\rangle$ is a desired path. □

3. Bipancyclic Properties

Theorem 3.1. *Assume that $n \geq 3$. Let F be a subset of $V(Q_n) \cup E(Q_n)$ such that F can be decomposed into two parts F_{av} and F_e, where F_{av} is a union of f_{av} disjoint adjacent pairs of $V(Q_n)$, and F_e consists of f_e edges. Then $Q_n - F$ is bipancyclic if $f_{av} + f_e \leq n - 2$.*

Proof. To prove this theorem, we will construct a cycle in $Q_n - F$ of length t for every even integer with $4 \leq t \leq 2^n - 2f_{av}$ by induction. Let $H = F_e \cup \{(\mathbf{b}_i, \mathbf{w}_i) \mid \{\mathbf{b}_i, \mathbf{w}_i\} \subset F_{av}\}$. For $1 \leq i \leq n$, let H_i denote the set of i-dimensional edges in H. Thus, $\sum_{i=1}^n |H_i| = f_e + f_{av}$. Without loss of generality, we assume that $|H_n| = 0$. Let $F_{av}^i = F_{av} \cap V(Q_n^i)$, $F_e^i = F_e \cap E(Q_n^i)$, $F^i = F_{av}^i \cup F_e^i$ and $f_{av}^i = |F_{av}^i|$ for $i = 0, 1$.

By brute force, we can check that the required cycles exist for $n = 3,\ 4$. By Theorem 1.1, the required cycles exist if $f_{av} = 0$. Therefore, we consider the case $f_{av} \geq 1$ and $n \geq 5$. Assume t is an even integer from 4 to $2^n - 2f_{av}$. Let $H^i = H \cap E(Q_n^i)$ for $i = 0,\ 1$. Without loss of generality, we can assume $|H^0| \geq |H^1|$.

Suppose that $|H^0| \leq n-3$. We first consider the case that $t \leq 2^{n-1} - 2f_{av}^1$. By induction, the desired cycle exists in $Q_n^1 - F^1$. Suppose that $t > 2^{n-1} - 2f_{av}^1$. Let C be a cycle of length $2^{n-1} - 2f_{av}^0$ in $Q_n^0 - F^0$. Let $t' = t - 2^{n-1} + 2f_{av}^0 - 1$. Note that there are $2^{n-1} - 2f_{av}^0 + 1$ sections in C of length t' depending on choice of the beginning and terminating vertices. Since $2^{n-1} > 4(n-2) \geq 2f_{av}^0 + 4f_{av}^1$, one such section, say P, joins vertex $\mathbf{p}$ to vertex $\mathbf{q}$ such that $\{(\mathbf{p})^n, (\mathbf{q})^n\} \cap F_{av}^1 = \emptyset$. By Lemma 2.6, there exists a Hamiltonian path R of $Q_n^1 - F^1$ joining $(\mathbf{q})^n$ to $(\mathbf{p})^n$. Obviously, $\langle \mathbf{p}, P, \mathbf{q}, (\mathbf{q})^n, R, (\mathbf{p})^n, \mathbf{p}\rangle$ is a desired cycle.

Thus, we consider $|H^0| = n - 2$. Therefore, $|H^1| = 0$. Assume that $4 \leq t \leq 2^{n-1}$. By Theorem 1.1, the desired cycle exists in Q_n^1. Thus, we consider $t > 2^{n-1}$. Let $\{\mathbf{b}_1, \mathbf{w}_1\}$ be a pair of adjacent vertices of F_{av} with $\mathbf{b}_1$ being a black vertex. By Lemma 2.5, there exists a Hamiltonian cycle C in $Q_n^0 - (H - \{\mathbf{b}_1, \mathbf{w}_1\})$. Obviously, C can be written as $\langle \mathbf{b}_1, P_1, \mathbf{w}_1, P_2, \mathbf{b}_1\rangle$. Without loss of generality, we assume that $l(P_1) \leq l(P_2)$. Thus, $2^{n-2} - f_{av}^0 \leq l(P_2) \leq 2^{n-1} - 2f_{av}^0 - 1$. We can write P_2 as $\langle \mathbf{w}_1, \mathbf{x}, P_2', \mathbf{y}, \mathbf{b}_1\rangle$.

Suppose that $2^{n-1} < t \leq 2^{n-1} + l(P_2) - 1$. Let $t' = t - 2^{n-1} - 1$. Let R be the section of P_2' joining vertex $\mathbf{x}$ to $\mathbf{z}$ of length t'. By Lemma 2.2, there exists a Hamiltonian path S of Q_n^1 joining $(\mathbf{z})^n$ to $(\mathbf{x})^n$. Obviously, $\langle \mathbf{x}, R, \mathbf{z}, (\mathbf{z})^n, S, (\mathbf{x})^n, \mathbf{x}\rangle$ is a desired cycle.

Suppose that $2^{n-1} + l(P_2) < t \leq 2^n - 2f_{av}$. Since $l(P_1) + l(P_2) = 2^{n-1} - 2f_{av}$, $l(P_1) \geq 3$. We can write P_1 as $\langle \mathbf{w}_1, \mathbf{u}, P_1', \mathbf{v}, \mathbf{b}_1\rangle$. Let $t' = t - 2^{n-1} - l(P_2)$. Let T be the section of P'_1 joining vertex $\mathbf{u}$ to $\mathbf{c}$ of length t'. By Lemma 2.4, there exist two spanning disjoint paths R_1 and R_2 of Q_n^1 such that R_1 joins $(\mathbf{y})^n$ to $(\mathbf{u})^n$ and R_2 joins $(\mathbf{c})^n$ to $(\mathbf{x})^n$. Thus, $\langle \mathbf{x}, P'_2, \mathbf{y}, (\mathbf{y})^n, R_1, (\mathbf{u})^n, \mathbf{u}, T, \mathbf{c}, (\mathbf{c})^n, R_2, (\mathbf{x})^n, \mathbf{x}\rangle$ forms a desired cycle.

The theorem is proved. □

Theorem 3.2. *Assume that $n \geq 3$. Let F be a subset of $V(Q_n) \cup E(Q_n)$ such that F can be decomposed into two parts F_{av} and F_e where F_{av} is a union of f_{av} disjoint adjacent pairs of $V(Q_n)$ and F_e consists of f_e edges. Then $Q_n - F$ is edge bipancyclic if $f_{av} + f_e \leq n-2$ and $f_{av} \leq n-3$.*

Proof. Let $e = (\mathbf{u}, \mathbf{v})$ be any fault-free edge of Q_n where $\mathbf{u}$ is a black vertex. For $4 \leq t \leq 2^n - 2f_{av}$, we will construct a cycle containing $(\mathbf{u}, \mathbf{v})$ of length t in $Q_n - F$ by induction to prove this theorem. Suppose $t = 4$, the desired cycle exists by Lemma 2.1. Thus, we consider $t \geq 6$.

Let $H = F_e \cup \{(\mathbf{b}_i, \mathbf{w}_i) \mid \{\mathbf{b}_i, \mathbf{w}_i\} \subset F_{av}\} \cup \{(\mathbf{u}, \mathbf{v})\}$. For $1 \leq i \leq n$, let H_i denote the set of i-dimensional edges in H. Since $f_e + f_{av} \leq n-2$, we can assume without loss of generality that $|H_n| = 0$. Thus, $\mathbf{u}, \mathbf{v}$ are vertices in Q_n^i for some $i \in \{0, 1\}$. Let $F_{av}^i = V(Q_n^i) \cap F_{av}$ and $f_{av}^i = |F_{av}^i|$ for $i = 0, 1$.

By Theorem 1.1, the desired cycles exist if $f_{av} = 0$. Thus, the desired cycles exist for $n = 3$. By brute force, we can check that the desired cycles exist for $n = 4$. Thus, we only consider the case $f_{av} \geq 1$ with $n \geq 5$. Without loss of generality, we can assume that $f_{av}^0 + f_e^0 \geq f_{av}^1 + f_e^1$.

Suppose that $f_{av}^0 + f_e^0 \leq n-3$ and $f_{av}^0 \leq n-4$. Without loss of generality, we can assume that both $\mathbf{u}$ and $\mathbf{v}$ are in Q_n^0. Suppose that $6 \leq t \leq 2^{n-1} - 2f_{av}^0$. By induction, the desired cycle exists in $Q_n^0 - F^0$. Thus, we consider that $2^{n-1} - 2f_{av}^0 + 2 \leq t \leq 2^n - 2f_{av}$. Let $t' = t - 2^{n-1} - 2f_{av}^0 - 1$. By induction, there exists a Hamiltonian cycle C of $Q_n^0 - F^0$ containing the edge $(\mathbf{u}, \mathbf{v})$. Since $2^{n-1} > 4(n-3)+1 \geq 2f_{av}^0 + 4f_{av}^1 + 1$, there exists an edge $(\mathbf{x}, \mathbf{y})$ in C such that $\{(\mathbf{x})^n, (\mathbf{y})^n\}\} \cap F_{av}^1 = \emptyset$. Thus, C can be written as $\langle \mathbf{u}, \mathbf{v}, P_1, \mathbf{x}, \mathbf{y}, P_2, \mathbf{u}\rangle$. By induction, there exist cycles $\langle (\mathbf{x})^n, Q, (\mathbf{y})^n, (\mathbf{x})^n\rangle$ of length t' of $Q_n^1 - F^1$. Obviously, $\langle \mathbf{u}, \mathbf{v}, P_1, \mathbf{x}, (\mathbf{x})^n, Q, (\mathbf{y})^n, \mathbf{y}, P_2, \mathbf{u}\rangle$ forms the desired cycle.

Thus, we consider $f_{av}^0 + f_e^0 = n-2$ or $f_{av}^0 = n-3$. We have the following two cases.

Case 1 ($(\mathbf{u}, \mathbf{v}) \in Q_n^1$). Suppose that $6 \leq t \leq 2^{n-1}$. By Theorem 1.1, the desired cycle exists. Thus, we consider $t \geq 2^{n-1}+2$. Let $\mathbf{b}_1$ and $\mathbf{w}_1$ be a pair of adjacent vertices in F_{av}^0. By induction, there exists a Hamiltonian cycle C in $Q_n^0 - F_e^0 - (F_{av}^0 - \{\mathbf{b}_1, \mathbf{w}_1\})$ containing the edge $(\mathbf{b}_1, \mathbf{w}_1)$. Thus, we can write C as $\langle \mathbf{b}_1, \mathbf{w}_1, \mathbf{x}, P, \mathbf{y}, \mathbf{b}_1\rangle$.

Suppose that $2^{n-1} + 2 \leq t \leq 2^n - 2f_{av}$. Let $t' = t - 2^{n-1} - 1$. Obviously, there exists a section P' of P joining $\mathbf{s}$ and $\mathbf{t}$ of length t' such that $\mathbf{s}$ is a black vertex. Suppose that $\{(\mathbf{s})^n, (\mathbf{t})^n\} \cap \{\mathbf{u}, \mathbf{v}\} = \emptyset$. By Lemma 2.4, there exist two spanning disjoint paths Q_1 and Q_2 of $Q_n^1 - F_e^1$ such that Q_1 joins $\mathbf{u}$ to $(\mathbf{s})^n$ and Q_2 joins $(\mathbf{t})^n$ to $\mathbf{v}$. Obviously, $\langle \mathbf{u}, Q_1, (\mathbf{s})^n, \mathbf{s}, P', \mathbf{t}, (\mathbf{t})^n, Q_2, \mathbf{v}, \mathbf{u}\rangle$ forms the desired cycle. Suppose that $|\{(\mathbf{s})^n, (\mathbf{t})^n\} \cap \{\mathbf{u}, \mathbf{v}\}| = 1$. Without loss of generality, we assume that $\mathbf{u} = (\mathbf{t})^n$. By Lemma 2.2, there exists a Hamiltonian path Q of $Q_n^1 - F_e^1 - \{\mathbf{u}\}$ joining $(\mathbf{s})^n$ to $\mathbf{v}$. Obviously, $\langle \mathbf{u} = (\mathbf{t})^n, \mathbf{t}, P'^{-1}, \mathbf{s}, (\mathbf{s})^n, Q, \mathbf{v}, \mathbf{u}\rangle$ forms the desired cycle. Suppose that $\{(\mathbf{s})^n, (\mathbf{t})^n\} = \{\mathbf{u}, \mathbf{v}\}$. Thus, $\mathbf{u} = (\mathbf{t})^n$ and $\mathbf{v} = (\mathbf{s})^n$. We can write P'^{-1} as $\langle \mathbf{t}, R_1, \mathbf{c}, \mathbf{d}, R_2, \mathbf{s}\rangle$ for some $\mathbf{c}$ and $\mathbf{d}$. By induction, there exists a Hamiltonian cycle C' of $Q_n^1 - F_e^1 - \{\mathbf{u}, \mathbf{v}\}$ containing the edge $((\mathbf{c})^n, (\mathbf{d})^n)$. Thus, C' can be written as $\langle (\mathbf{c})^n, Q', (\mathbf{d})^n, (\mathbf{c})^n\rangle$. Obviously, $\langle \mathbf{u} = (\mathbf{t})^n, t, R_1, \mathbf{c}, (\mathbf{c})^n, Q', (\mathbf{d})^n, \mathbf{d}, R_2, \mathbf{s}, (\mathbf{s})^n = \mathbf{v}, \mathbf{u}\rangle$ forms the desired cycle.

Case 2 ($(\mathbf{u}, \mathbf{v})$ in Q_n^0). Suppose that $6 \leq t \leq 2^{n-1} + 2$. Let $t' = t - 2$. By Theorem 1.1, there exists a cycle C of length t' in $Q_n^1 - F_e^1$ containing the edge $((\mathbf{u})^n, (\mathbf{v})^n)$. We can write C as $\langle (\mathbf{u})^n, (\mathbf{v})^n, R, (\mathbf{u})^n\rangle$. Obviously, $\langle \mathbf{u}, \mathbf{v}, (\mathbf{v})^n, R, (\mathbf{u})^n, \mathbf{u}\rangle$ forms the desired cycle.

Now, we consider $2^{n-1}+4 \le t \le 2^n - 2f_{av}$. Let $\mathbf{b}_1$ and $\mathbf{w}_1$ be a pair of adjacent vertices in F_{av}^0. By induction, there exists a Hamiltonian cycle C in $Q_n^0 - F_e - (F_{av} - \{\mathbf{b}_1, \mathbf{w}_1\})$ containing $(\mathbf{u}, \mathbf{v})$.

Suppose that $(\mathbf{b}_1, \mathbf{w}_1) \in C$. Obviously, C can be written as $\langle \mathbf{w}_1, \mathbf{b}_1, \mathbf{y}, P, \mathbf{x}\rangle$. Let $t' = t - 2^{n-1} - 1$. Obviously, there exists a section Q of P joining $\mathbf{s}$ to $\mathbf{t}$ of length t' containing $(\mathbf{u}, \mathbf{v})$. By Lemma 2.2, there exists a Hamiltonian path R joining $(\mathbf{t})^n$ to $(\mathbf{s})^n$ of $Q_n^1 - F_e^1$. Obviously, $\langle \mathbf{s}, Q, \mathbf{t}, (\mathbf{t})^n, R, (\mathbf{s})^n, \mathbf{s}\rangle$ forms the desired cycle.

Suppose that $(\mathbf{b}_1, \mathbf{w}_1) \notin C$. The cycle C can be written as $\langle \mathbf{w}_1, \mathbf{x}, P_1, \mathbf{y}, b_1, \mathbf{s}, P_2, \mathbf{t}, \mathbf{w}_1\rangle$. Without loss of generality, we can assume that $(\mathbf{u}, \mathbf{v})$ is on the path P_1. Suppose that $2^{n-1}+4 \le t \le 2^{n-1} + 1 + l(P_1)$. Let $t' = t - 2^{n-1} - 1$. There exists a section P_1' of P_1 such that (1) P_1' joins $\mathbf{c}$ to $\mathbf{d}$, (2) P_1' contains $(\mathbf{u}, \mathbf{v})$, and (3) P_1' is of length t'. By Lemma 2.2, there exists a Hamiltonian path R of $Q_n^1 - F_e^1$ joining $(\mathbf{d})^n$ to $(\mathbf{c})^n$. Obviously, $\langle \mathbf{c}, P_1', \mathbf{d}, (\mathbf{d})^n, R, (\mathbf{c})^n, \mathbf{c}\rangle$ forms the desired cycle. Suppose that $2^{n-1}+3+l(P_1) \le t \le 2^n - 2f_{av}$. Let $t' = t - 2^{n-1} - 2 - l(P_1)$. There exists a section P_2' of P_2 joining $\mathbf{r}$ to $\mathbf{t}$ of length t'. By Lemma 2.4, there exist two spanning disjoint paths R_1 and R_2 of $Q_n^1 - F_e^1$ such that R_1 joins $(\mathbf{y})^n$ to $(\mathbf{r})^n$ and R_2 joins $(\mathbf{t})^n$ to $(\mathbf{x})^n$. Obviously, $\langle \mathbf{x}, P_1, \mathbf{y}, (\mathbf{y})^n, R_1, (\mathbf{r})^n, \mathbf{r}, P_2', \mathbf{t}, (\mathbf{t})^n, R_2, (\mathbf{x})^n, \mathbf{x}\rangle$ forms the desired cycle. □

4. Conclusion

In this paper, we study the bipancyclic property of faulty hypercubes. We improve previous results by considering both edge faults and vertex faults. Let F be a subset of $V(Q_n) \cup E(Q_n)$ such that F can be decomposed into two parts F_{av} and F_e where F_{av} is a union of f_{av} disjoint adjacent pairs of $V(Q_n)$ and F_e consists of f_e edges. We prove that $Q_n - F$ is bipancyclic if $f_{av} + f_e \le n-2$. This result is optimal. Let $\mathbf{x}$ be any vertex of Q_n. Assume that $F = \{(\mathbf{x}, (\mathbf{x})^i) \mid 1 \le i \le n-1\}$ forms a set of $(n-1)$ faulty edges. Obviously, $\deg_{Q_n - F}(\mathbf{x}) = 1$. $Q_n - F$ is not bipancyclic.

We also prove that $Q_n - F$ is edge bipancyclic if $f_{av} + f_e \le n-2$ with $f_{av} < n-2$. Again, this result is optimal. Assume that $F = \cup_{i=2}^{n-1}\{(\mathbf{x})^i, ((\mathbf{x})^1)^i\}$ forms a set of $(n-2)$ adjacent faulty vertices. Obviously, $\deg_{Q_n-F}(\mathbf{x}) = \deg_{Q_n-F}((\mathbf{x})^1) = 2$. Thus, any Hamiltonian cycle of $Q_n - F$ contains the path $\langle (\mathbf{x})^n, \mathbf{x}, (\mathbf{x})^1, ((\mathbf{x})^1)^n\rangle$. Therefore, there is no Hamiltonian cycle containing the edge $((\mathbf{x})^n, ((\mathbf{x})^1)^n)$ for $n \ge 3$. Thus, $Q_n - F$ is not edge bipancyclic.

Two interesting observations, Lemmas 2.4 and 2.7, are used in this paper. In the following, we claim that these two lemmas are also optimal.

Let $\mathbf{z}$, $\mathbf{u}$, and $\mathbf{x}$ be three distinct white vertices of Q_n. Assume that $F = \{(\mathbf{z}, (\mathbf{z})^i) \mid \text{for } 2 \le i \le n-1\}$ forms a set of $(n-2)$ faulty edges. It is observed that $\{(\mathbf{z})^1, (\mathbf{z})^n\}$ are the neighbors of $\mathbf{z}$ in $Q_n - F$. Moreover, $(\mathbf{z})^1$ and $(\mathbf{z})^n$ are black vertices. Obviously, $\mathbf{z}$ cannot be any vertex in $P_1 \cup P_2$ for any two disjoint paths P_1 and P_2 of $Q_n - F$ such that P_1 joins $\mathbf{u}$ to $(\mathbf{z})^1$ and P_2 joins $\mathbf{x}$ to $(\mathbf{z})^n$. Therefore, there do not exist two disjoint paths P_1 and P_2 of $Q_n - F$ such that P_1 joins $\mathbf{u}$ to $(\mathbf{z})^1$ and P_2 joins $\mathbf{x}$ to $(\mathbf{z})^n$. Thus, the number of faulty edges of Lemma 2.4 is optimal.

Assume that $n \ge 3$. Let $\mathbf{x}$ be any vertex of Q_n and $F = \cup_{i=2}^{n-1}\{(\mathbf{x})^i, ((\mathbf{x})^1)^i\}$ be a set of $(n-2)$ adjacent faulty vertices. Obviously, $(\mathbf{x})^n$ and $((\mathbf{x})^1)^n$ are vertices in different partite sets. Assume that there is a Hamiltonian path P joining $(\mathbf{x})^n$ to $((\mathbf{x})^1)^n$. Since $\deg_{Q_n-F}(\mathbf{x}) = \deg_{Q_n-F}((\mathbf{x})^1) = 2$, P must include the section $\langle (\mathbf{x})^1, \mathbf{x}, (\mathbf{x})^n, ((\mathbf{x})^n)^1\rangle$. Obviously, P is not a Hamiltonian path and we get a contradiction. Thus, the number of adjacent faulty vertices of Lemma 2.7 is optimal.

Acknowledgment

This work was supported in part by the National Science Council of Republic of China under NSC 100-2221-E-212-014.

References

[1] L.-H. Hsu and C.-K. Lin, *Graph Theory and Interconnection Networks*, CRC Press, Boca Raton, Fla, USA, 2008.

[2] F. T. Leighton, *Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes*, Morgan Kaufmann, San Mateo, Calif, USA, 1992.

[3] T.-K. Li, C.-H. Tsai, J. J. M. Tan, and L.-H. Hsu, "Bipanconnectivity and edge-fault-tolerant bipancyclicity of hypercubes," *Information Processing Letters*, vol. 87, no. 2, pp. 107–110, 2003.

[4] C.-H. Tsai and Y.-C. Lai, "Conditional edge-fault-tolerant edge-bipancyclicity of hypercubes," *Information Sciences*, vol. 177, no. 24, pp. 5590–5597, 2007.

[5] C.-H. Tsai, J. J. M. Tan, T. Liang, and L.-H. Hsu, "Fault-tolerant Hamiltonian laceability of hypercubes," *Information Processing Letters*, vol. 83, no. 6, pp. 301–306, 2002.

[6] C.-D. Park and K.-Y. Chwa, "Hamiltonian properties on the class of hypercube-like networks," *Information Processing Letters*, vol. 91, no. 1, pp. 11–17, 2004.

[7] C.-M. Sun, C.-N. Hung, H.-M. Huang, L.-H. Hsu, and Y.-D. Jou, "Hamiltonian laceability of faulty hypercubes," *Journal of Interconnection Networks*, vol. 8, no. 2, pp. 133–145, 2007.