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Research Article

A q-Analogue of Rucinski-Voigt Numbers

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A q-analogue of Rucinski-Voigt numbers is defined by means of a recurrence relation, and some properties including the orthogonality and inverse relations with the q-analogue of the limit of the differences of the generalized factorial are obtained.

1. Introduction

Rucinski and Voigt [1] defined the numbers $S_k^n(\mathbf{a})$ satisfying the relation

$$x^n = \sum_{k=0}^{n} S_k^n(\mathbf{a}) p_k^{\mathbf{a}}(x), \tag{1.1}$$

where $\mathbf{a}$ is the sequence $(a, a+r, a+2r, \ldots)$ and $p_k^{\mathbf{a}}(x) = \prod_{i=0}^{k-1}(x-(a+ir))$ and proved that these numbers are asymptotically normal. We call these numbers *Rucinski-Voigt numbers*. Note that the classical Stirling numbers of the second kind $S(n,k)$ in [2–4] and the r-Stirling numbers of the second kind $\widehat{\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]}_r$ of Broder [5] can be expressed in terms of $S_k^n(\mathbf{a})$ as follows:

$$\begin{gathered} S(n,k) = S_k^n(\mathbf{d}), \\ \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r = S_k^n(\mathbf{e}), \end{gathered} \tag{1.2}$$

where $\mathbf{d}$ and $\mathbf{e}$ are the sequences $(0, 1, 2, \ldots)$ and $(r, r+1, r+2, \ldots)$, respectively. With these observations, $S_k^n(\mathbf{a})$ may be considered as certain generalization of the second kind Stirling-type numbers.

Several properties of Rucinski-Voigt numbers can easily be established parallel to those in the classical Stirling numbers of the second kind. To mention a few, we have the triangular recurrence relation

$$S_k^{n+1}(\mathbf{a}) = S_{k-1}^n(\mathbf{a}) + (kr + a)S_k^n(\mathbf{a}), \tag{$P1$}$$

the exponential and rational generating function

$$\sum_{n\geq 0} S_k^n(\mathbf{a})\frac{x^n}{n!} = \frac{1}{r^k k!}e^{ax}\left(e^{rx}-1\right)^k, \tag{$P2$}$$

$$\sum_{n\geq 0} S_k^n(\mathbf{a})x^n = \frac{x^k}{\prod_{j=0}^k (1-(rj+a)x)}, \tag{$P3$}$$

and explicit formulas

$$S_k^n(\mathbf{a}) = \frac{1}{r^k k!}\sum_{j=0}^k (-1)^{k-j}\binom{k}{j}(rj+a)^n, \tag{$P4$}$$

$$S_k^n(\mathbf{a}) = \sum_{c_0+c_1+\cdots+c_k=n-k}\prod_{j=0}^k (rj+a)^{c_j}. \tag{$P5$}$$

The explicit formula in ($P4$) can be used to interpret $r^k k! S_k^n(\mathbf{a})$ as the number of ways to distribute n distinct balls into the $k+1$ cells (one ball at a time), the first k of which has r distinct compartments and the last cell with a distinct compartments, such that

(i) the capacity of each compartment is unlimited;

(ii) the first k cells are nonempty.

The other explicit formula ($P5$) can also be used to interpret $S_k^n(\mathbf{a})$ as the number of ways of assigning n people to $k+1$ groups of tables where all groups are occupied such that the first group contains a distinct tables and the rest of the group each contains r distinct tables.

The Rucinski-Voigt numbers are nothing else but the r-Whitney numbers of the second kind, denoted by $W_{m,r}(n,k)$, in Mező [6]. That is, $S_k^n(\mathbf{a}) = W_{r,a}(n,k)$. It is worth-mentioning that the r-Whitney numbers of the second kind are generalization of Whitney numbers of the second kind in Benoumhani's papers [7–9].

On the other hand, the limit of the differences of the generalized factorial [10]

$$F_{\alpha,\gamma}(n,k) = \lim_{\beta\to 0}\frac{\left[\Delta_t^k(\beta t+\gamma \mid \alpha)_n\right]_{t=0}}{k!\beta^k}, \qquad (\beta t+\gamma \mid \alpha)_n = \prod_{j=0}^{n-1}(\beta t+\gamma - j\alpha) \tag{1.3}$$

was also known as a generalization of the Stirling numbers of the first kind. That is, all the first kind Stirling-type numbers may also be expressed in terms of $F_{\alpha,\gamma}(n,k)$ by a special choice of the values of α and γ. It was shown in [10] that

$$\sum_{k=0}^{n} F_{\alpha,\gamma}(n,k)t^k = p_n^{\mathbf{b}}(t), \tag{1.4}$$

where $\mathbf{b}$ is the sequence $(-\gamma, -\gamma+\alpha, -\gamma+2\alpha, \ldots)$. Recently, q-analogue and (p,q)-analogue of $F_{\alpha,\gamma}(n,k)$, denoted by $\phi_{\alpha,\gamma}[n,k]_q$ and $\phi_{\alpha,\gamma}[n,k]_{pq}$, respectively, were established by Corcino and Hererra in [10] and obtained several properties including the horizontal generating function for $\phi_{\alpha,\gamma}[n,k]_q$

$$\sum_{k=0}^{n} \phi_{\alpha,\gamma}[n,k]_q t^k = \left\langle t + [\gamma]_q \mid [\alpha]_q \right\rangle_n^q, \tag{1.5}$$

where

$$\left\langle t + [\gamma]_q \mid [\alpha]_q \right\rangle_n^q = \prod_{j=0}^{n-1} \left(t + [\gamma]_q - [j\alpha]_q \right), \qquad \left\langle t + [\gamma]_q \mid [\alpha]_q \right\rangle_0^q = 1. \tag{1.6}$$

The numbers $F_{\alpha,-\gamma}(n,k)$ are equivalent to the r-Whitney numbers of the first kind, denoted by $w_{m,r}(n,k)$, in [6]. More precisely, $F_{\alpha,-\gamma}(n,k) = w_{\alpha,\gamma}(n,k)$. These numbers are generalization of Whitney numbers of the first kind in Benoumhani's papers [7–9].

In this paper, we establish a q-analogue of $S_k^n(\mathbf{a})$ and obtain some properties including recurrence relations, explicit formulas, generating functions, and the orthogonality and inverse relations.

2. Definition and Some Recurrence Relations

It is known that a given polynomial $a_k(q)$ is a q-analogue of an integer a_k if

$$\lim_{q\to 1} a_k(q) = a_k. \tag{2.1}$$

For example, the polynomials

$$[n]_q = \frac{q^n - 1}{q - 1}, \qquad [n]_q! = \prod_{i=1}^{n} [i]_q, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n-i+1} - 1}{q^i - 1} \tag{2.2}$$

are the q-analogues of the integers n, $n!$, and $\binom{n}{k}$, respectively, since

$$\lim_{q\to 1} [n]_q = n, \qquad \lim_{q\to 1} [n]_q! = n!, \qquad \lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}. \tag{2.3}$$

The last two polynomials in (2.2) are called the *q-factorial* and *q-binomial coefficients*, respectively. With these in mind, it is interesting also that, for a given property of an integer a_k, we can find an analogous property for the polynomial $a_k(q)$. For example, the binomial coefficients $\binom{n}{k}$ satisfy the known inversion formula

$$f_n = \sum_{k=0}^{n}\binom{n}{k}g_k \Longleftrightarrow g_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f_k \tag{2.4}$$

and Vandermondes identity

$$\binom{m+n}{k} = \sum_{r=0}^{k}\binom{m}{r}\binom{n}{k-r}, \tag{2.5}$$

while the q-binomial coefficients $\left[{n \atop k}\right]_q$ satisfy the q-binomial inversion formula [3]

$$\begin{aligned} f_n &= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q g_k \Longleftrightarrow g_n = \sum_{k=0}^{n}(-1)^{n-k}q^{\binom{n-k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q f_k, \\ f_n &= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q^\beta} g_k \Longleftrightarrow g_n = \sum_{k=0}^{n}(-1)^{n-k}q^{\beta\binom{n-k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_{q^\beta} f_k, \end{aligned} \tag{2.6}$$

and q-Vandermondes identity [11]

$$\begin{bmatrix} m+n \\ k \end{bmatrix}_q = \sum_{r=0}^{k}q^{r(m-k+r)}\begin{bmatrix} m \\ r \end{bmatrix}_q \begin{bmatrix} n \\ k-r \end{bmatrix}_q. \tag{2.7}$$

Carlitz [12] defined a q-Stirling number of the second kind in terms of a recurrence relation

$$S_q[n,k] = S_q[n-1,k-1] + [k]_q S_q[n-1,k] \tag{2.8}$$

in connection with a problem in abelian groups, such that when $q \to 1$, this gives the triangular recurrence relation for the classical Stirling numbers of the second kind $S(n,k)$

$$S(n,k) = S(n-1,k-1) + kS(n-1,k). \tag{2.9}$$

This motivates the authors to define a q-analogue of the $S_k^n(\mathbf{a})$ as follows.

Definition 2.1. For nonnegative integers n and k and complex numbers β and r, a q-analogue $\sigma[n,k]_q^{\beta,r}$ of $S_k^n(\mathbf{c})$ is defined by

$$\sigma[n,k]_q^{\beta,r} = \sigma[n-1,k-1]_q^{\beta,r} + \left([k\beta]_q + [r]_q\right)\sigma[n-1,k]_q^{\beta,r}, \tag{2.10}$$

where $\mathbf{c}$ is the sequence $(r, r+\beta, r+2\beta, \ldots)$, $\sigma[0,0]_q^{\beta,r} = 1$, and $\sigma[n,k]_q^{\beta,r} = 0$ for $n < k$ or $n, k < 0$.

The numbers $\sigma[n,k]_q^{\beta,r}$ may be considered as a q-analogue of $S_k^n(\mathbf{c})$ since, when $q \to 1$,

$$[k\beta]_q + [r]_q \longrightarrow k\beta + r \tag{2.11}$$

and, hence, the recurrence relation in (2.10) will give the recurrence relation in $(P1)$ for $S_k^n(\mathbf{c})$ where $\mathbf{c}$ is the sequence $(r, r+\beta, r+2\beta, \ldots)$. This fact will also be verified in Section 3 (Remark 3.4).

The above triangular recurrence relation for the q-Stirling numbers of the second kind can easily be deduced from (2.10) by taking $\beta = 1$ and $r = 0$.

Clearly, using the initial conditions of $\sigma[n,k]_q^{\beta,r}$, we can have

$$\begin{aligned} \sigma[n,0]_q^{\beta,r} &= [r]_q^n, \quad \forall n \geq 0, \\ \sigma[n,n]_q^{\beta,r} &= 1, \quad \forall n \geq 0. \end{aligned} \tag{2.12}$$

By repeated application of (2.10), we obtain the following theorem.

Theorem 2.2. *For nonnegative integers n and k and complex numbers β and r, the q-analogue $\sigma[n,k]_q^{\beta,r}$ satisfies the following vertical recurrence relation:*

$$\sigma[n+1,k+1]_q^{\beta,r} = \sum_{j=k}^{n} \left([(k+1)\beta]_q + [r]_q \right)^{n-j} \sigma[j,k]_q^{\beta,r} \tag{2.13}$$

with initial conditions $\sigma[0,0]_q^{\beta,r} = 1$ and $\sigma[n,n]_q^{\beta,r} = 1$, $\sigma[n,0]_q^{\beta,r} = [r]_q^n$ for all $n \geq 0$.

Using the following notation

$$\left\{ [r]_q \mid [\beta]_q \right\}_k = \prod_{j=0}^{n-1} \left([r]_q + [j\beta]_q \right), \qquad \left\{ [r]_q \mid [\beta]_q \right\}_0 = 1, \tag{2.14}$$

we can now state the horizontal recurrence relation for $\sigma[n,k]_q^{\beta,r}$.

Theorem 2.3. *For nonnegative integers n and k and complex numbers β and r, the q-analogue $\sigma[n,k]_q^{\beta,r}$ satisfies the following horizontal recurrence relation:*

$$\sigma[n,k]_q^{\beta,r} = \sum_{j=0}^{n-k} (-1)^j \frac{\left\{ [r]_q \mid [\beta]_q \right\}_{k+j+1}}{\left\{ [r]_q \mid [\beta]_q \right\}_{k+1}} \sigma[n+1,k+j+1]_q^{\beta,r}, \tag{2.15}$$

with initial condition $\sigma[0,0]_q^{\beta,r} = 1$ and $\sigma[n,n]_q^{\beta,r} = 1$, $\sigma[n,0]_q^{\beta,r} = [r]_q^n$ for all $n \geq 0$.

Proof. To prove (2.15), we simply evaluate its right-hand side using (2.10) and obtain $\sigma[n,k]_q^{\beta,r}$. □

It will be shown in Section 3 that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = (q-1)^{n-k}\sigma[n,k]_q^{1,\log_q 2}. \tag{2.16}$$

By taking $\beta = 1$ and $r = \log_q 2$, (2.13) and (2.15) yield

$$\begin{gathered}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q = \sum_{j=k}^{n}\left(q^{k+1}\right)^{n-j}\begin{bmatrix} j \\ k \end{bmatrix}_q, \\ \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{j=0}^{n-k}(-1)^j q^{jk+\binom{j+1}{2}}\begin{bmatrix} n+1 \\ k+j+1 \end{bmatrix}_q,\end{gathered} \tag{2.17}$$

which are exactly the recurrence relations obtained in [13]. When $q \to 1$, these further give the Hockey Stick identities.

3. Explicit Formulas and Generating Functions

The next theorem is analogous to that relation in (1.1). This is necessary in obtaining one of the explicit formulas for $\sigma[n,k]_q^{\beta,r}$ and the orthogonality and inverse relations of $\phi_{\alpha,\gamma}[n,k]_q$ and $\sigma[n,k]_q^{\beta,r}$.

Theorem 3.1. *For nonnegative integers n and k and complex numbers β and r, the q-analogue $\sigma[n,k]_q^{\beta,r}$ satisfies the following relation:*

$$\sum_{k=0}^{n}\sigma[n,k]_q^{\beta,r}\left\langle t \mid [\beta]_q\right\rangle_k^q = \left(t+[r]_q\right)^n. \tag{3.1}$$

Proof. We proceed by induction on n. Clearly, (3.1) is true for $n=0$. Assume that it is true for $n>0$. Then using Definition 2.1,

$$\begin{aligned}&\sum_{k=0}^{n+1}\sigma[n+1,k]_q^{\beta,r}\left\langle t \mid [\beta]_q\right\rangle_k^q \\ &\quad= \sum_{k=0}^{n}\sigma[n,k]_q^{\beta,r}\langle t \mid \left\langle [\beta]_q\right\rangle_{k+1}^q + \sum_{k=0}^{n}\left([k\beta]_q+[r]_q\right)\sigma[n,k]_q^{\beta,r}\left\langle t \mid [\beta]_q\right\rangle_k^q \\ &\quad= \left(t+[r]_q\right)\sum_{k=0}^{n}\sigma[n,k]_q^{\beta,r}\left\langle t \mid [\beta]_q\right\rangle_k^q \\ &\quad= \left(t+[r]_q\right)\left(t+[r]_q\right)^n = \left(t+[r]_q\right)^{n+1}.\end{aligned} \tag{3.2}$$

□

The new q-analogue of Newton's Interpolation Formula in [14] states that, for

$$f_q(x) = a_0 + a_1[x - x_0]_q + \cdots + a_m[x - x_0]_q[x - x_1]_q[x - x_{m-1}]_q, \tag{3.3}$$

we have

$$\begin{aligned} f_q(x) = f_q(x_0) &+ \frac{\Delta_{q^h,h} f_q(x_0)[x - x_0]_q}{[1]_{q^h}![h]_q} + \frac{\Delta^2_{q^h,h} f_q(x_0)[x - x_0]_q[x - x_1]_q}{[2]_{q^h}![h]_q^2} \\ &+ \cdots + \frac{\Delta^m_{q^h,h} f_q(x_0)[x - x_0]_q[x - x_1]_q \ldots [x - x_{m-1}]_q}{[m]_{q^h}![h]_q^m}, \end{aligned} \tag{3.4}$$

where $x_k = x_0 + kh$, $k = 1, 2, \ldots$ such that when $x_0 = 0$, this can be simplified as

$$\begin{aligned} f_q(x) = f_q(0) &+ \frac{\Delta_{q^h,h} f_q(0)[x]_q}{[1]_{q^h}![h]_q} + \frac{\Delta^2_{q^h,h} f_q(0)[x]_q[x - h]_q}{[2]_{q^h}![h]_q^2} \\ &+ \cdots + \frac{\Delta^m_{q^h,h} f_q(0)[x]_q[x - h]_q \ldots [x - (m-1)h]_q}{[m]_{q^h}![h]_q^m}. \end{aligned} \tag{3.5}$$

Using (3.1) with $t = [x]_q$, we get

$$\sum_{k=0}^{n} \sigma[n,k]_q^{\beta,r} \left\langle [x]_q \mid [\beta]_q \right\rangle_k^q = \left([x]_q + [r]_q\right)^n, \tag{3.6}$$

which can be expressed further as

$$\sum_{k=0}^{n} \sigma[n,k]_q^{\beta,r} q^{\beta\binom{k}{2}} [x]_q [x - \beta]_q \cdots [x - (k-1)\beta]_q = \left([x]_q + [r]_q\right)^n. \tag{3.7}$$

Applying the above Newton's Interpolation Formula and the identity in [14]

$$\Delta^n_{q,h} f(x) = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q f(x + (n-k)h), \tag{3.8}$$

we get

$$\begin{aligned} \sigma[n,k]_q^{\beta,r} q^{\beta\binom{k}{2}} &= \frac{\Delta^k_{q^\beta,\beta} f_q(0)}{[k]_{q^\beta}![\beta]_q^k} \\ &= \frac{1}{[k]_{q^\beta}![\beta]_q^k} \sum_{j=0}^{k} (-1)^{k-j} q^{\beta\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^\beta} \left([j\beta]_q + [r]_q\right)^n. \end{aligned} \tag{3.9}$$

With $\langle [k\beta]_q \mid [\beta]_q \rangle_k^q = q^{\beta\binom{k}{2}} [k\beta]_q [(k-1)\beta]_q \cdots [\beta]_q = q^{\beta\binom{k}{2}} [k]_{q^\beta}! [\beta]_q^k$, we obtain the following explicit formula.

Theorem 3.2. *For nonnegative integers n and k and complex numbers β and r, the q-analogue $\sigma[n,k]_q^{\beta,r}$ is equal to*

$$\sigma[n,k]_q^{\beta,r} = \frac{1}{\left\langle [k\beta]_q \mid [\beta]_q \right\rangle_k^q} \sum_{j=0}^{k} (-1)^{k-j} q^{\beta\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^\beta} \left([j\beta]_q + [r]_q \right)^n. \tag{3.10}$$

Remark 3.3. We can also prove Theorem 3.2 using the q-binomial inversion formula in (2.6). That is, by taking $t = [k\beta]_q$, (3.1) gives

$$\begin{aligned} \left([k\beta]_q + [r]_q \right)^n &= \sum_{j=0}^{n} \sigma[n,j]_q^{\beta,r} \left\langle [k\beta]_q \mid [\beta]_q \right\rangle_j^q \\ &= \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_{q^\beta} \left\{ \frac{\sigma[n,j]_q^{\beta,r} \left\langle [k\beta]_q \mid [\beta]_q \right\rangle_j^q}{\begin{bmatrix} k \\ j \end{bmatrix}_{q^\beta}} \right\}. \end{aligned} \tag{3.11}$$

Applying (2.6), we obtain

$$\frac{\sigma[n,j]_q^{\beta,r} \left\langle [k\beta]_q \mid [\beta]_q \right\rangle_k^q}{\begin{bmatrix} k \\ k \end{bmatrix}_{q^\beta}} = \sum_{j=0}^{k} (-1)^{k-j} q^{\beta\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^\beta} \left([j\beta]_q + [r]_q \right)^n. \tag{3.12}$$

This is precisely the explicit formula in Theorem 3.2.

Remark 3.4. Note that $\langle [k\beta]_q \mid [\beta]_q \rangle_k^q \to k!\beta^k$, $\begin{bmatrix} k \\ j \end{bmatrix}_{q^\beta} \to \binom{k}{j}$, and $([j\beta]_q + [r]_q)^n \to (j\beta + r)^n$ as $q \to 1$. Thus, using property $(P4)$, $\sigma[n,k]_q^{\beta,r} \to S_k^n(\mathbf{c})$ as $q \to 1$. This implies that $\sigma[n,k]_q^{\beta,r}$ is a proper q-analogue of $S_k^n(\mathbf{c})$.

Now, using the explicit formula in Theorem 3.2, we obtain

$$\begin{aligned}\sum_{n\geq 0}\sigma[n,j]_q^{\beta,r}\frac{t^n}{n!} &= \frac{1}{\left\langle [k\beta]_q \mid [\beta]_q\right\rangle_k^q}\sum_{n\geq 0}\left\{\sum_{j=0}^{k}(-1)^{k-j}q^{\beta\binom{k-j}{2}}\begin{bmatrix}k\\ j\end{bmatrix}_{q^\beta}\left([j\beta]_q+[r]_q\right)^n\right\}\frac{t^n}{n!}\\ &= \frac{\sum_{j=0}^{k}(-1)^{k-j}q^{\beta\binom{k-j}{2}}\begin{bmatrix}k\\ j\end{bmatrix}_{q^\beta}\left\{\sum_{n\geq 0}\left([j\beta]_q+[r]_q\right)^n t^n/n!\right\}}{\left\langle [k\beta]_q \mid [\beta]_q\right\rangle_k^q}\\ &= \frac{\sum_{j=0}^{k}(-1)^{k-j}q^{\beta\binom{k-j}{2}}\begin{bmatrix}k\\ j\end{bmatrix}_{q^\beta}\left\{\sum_{n\geq 0}\left(\sum_{i=0}^{n}\left(\left([r]_q t\right)^i/i!\right)\left(\left([j\beta]_q t\right)^{n-i}/(n-i)!\right)\right)\right\}}{\left\langle [k\beta]_q \mid [\beta]_q\right\rangle_k^q}.\end{aligned} \tag{3.13}$$

Applying Cauchy's formula for the product of two power series [3], we get

$$\sum_{n\geq 0}\sigma[n,k]_q^{\beta,r}\frac{t^n}{n!} = \frac{1}{\left\langle [k\beta]_q \mid [\beta]_q\right\rangle_k^q}\sum_{j=0}^{k}(-1)^{k-j}q^{\beta\binom{k-j}{2}}\begin{bmatrix}k\\ j\end{bmatrix}_{q^\beta}\left\{\sum_{\lambda\geq 0}\frac{\left([r]_q t\right)^\lambda}{\lambda!}\sum_{\mu\geq 0}\frac{\left([j\beta]_q t\right)^\mu}{\mu!}\right\}. \tag{3.14}$$

Thus,

$$\sum_{n\geq 0}\sigma[n,k]_q^{\beta,r}\frac{t^n}{n!} = \frac{1}{\left\langle [k\beta]_q \mid [\beta]_q\right\rangle_k^q}\sum_{j=0}^{k}(-1)^{k-j}q^{\beta\binom{k-j}{2}}\begin{bmatrix}k\\ j\end{bmatrix}_{q^\beta}e^{([j\beta]_q+[r]_q)t}. \tag{3.15}$$

Applying the above identity for $\Delta_{q,h}^n f$ to the function f defined by

$$f(x) = \frac{e^{([x\beta]_q+[r]_q)t}}{\left\langle [k\beta]_q \mid [\beta]_q\right\rangle_k^q}, \tag{3.16}$$

we can further express the above generating function in terms of a q-difference operator. More precisely,

$$\sum_{n\geq 0}\sigma[n,k]_q^{\beta,r}\frac{t^n}{n!} = \left\{\Delta_q^k\left(\frac{e^{([x\beta]_q+[r]_q)t}}{\left\langle [k\beta]_q \mid [\beta]_q\right\rangle_k^q}\right)\right\}_{x=0}. \tag{3.17}$$

This is a kind of exponential generating function for $\sigma[n,k]_q^{\beta,r}$ which is included in the next theorem. Together with this, a rational generating function for $\sigma[n,k]_q^{\beta,r}$ is also stated in the theorem that will be used to derive another explicit formula for $\sigma[n,k]_q^{\beta,r}$ in homogeneous symmetric function form.

Theorem 3.5. *For nonnegative integers n and k and complex numbers β and r, the q-analogue $\sigma[n,k]_q^{\beta,r}$ satisfies the exponential generating function*

$$\Phi_k(t) = \sum_{n\geq 0}\sigma[n,k]_q^{\beta,r}\frac{t^n}{n!} = \left[\Delta_{q^\beta}^k\left(\frac{e^{([x\beta]_q+[r]_q)t}}{\left\langle [k\beta]_q \mid [\beta]_q\right\rangle_k^q}\right)\right]_{x=0}, \tag{3.18}$$

and the rational generating function

$$\psi_k(t) = \sum_{n\geq k}\sigma[n,k]_q^{\beta,r}t^n = \frac{t^k}{\prod_{j=0}^k\left(1-\left([j\beta]_q+[r]_q\right)t\right)}. \tag{3.19}$$

Proof. We are done with the proof of the first generating function. We are left to prove the second one and we are going to prove this by induction on k. For $k=0$, we have

$$\psi_0(t) = \sum_{n\geq 0}\sigma[n,0]_q^{\beta,r}t^n = \sum_{n\geq 0}[r]_q^n t^n = \frac{1}{\left(1-[r]_q t\right)}. \tag{3.20}$$

With $k>0$ and using Definition 2.1, we obtain

$$\begin{aligned}\psi_k(t) &= \sum_{n\geq k}\sigma[n,k]_q^{\beta,r}t^n\\ &= t\sum_{n\geq k}\sigma[n-1,k-1]_q^{\beta,r}t^{n-1} + \left([k\beta]_q+[r]_q\right)t\sum_{n\geq k}\sigma[n-1,k]_q^{\beta,r}t^{n-1}\\ &= t\psi_{k-1}(t) + \left([k\beta]_q+[r]_q\right)t\psi(t).\end{aligned} \tag{3.21}$$

Hence,

$$\psi_k(t) = \frac{t}{1-\left([k\beta]_q+[r]_q\right)t}\psi_{k-1}(t), \tag{3.22}$$

which gives

$$\psi_k(t) = \frac{t^k}{\prod_{j=0}^k\left(1-\left([j\beta]_q+[r]_q\right)t\right)}. \tag{3.23}$$

□

The rational generating function in Theorem 3.5 can then be expressed as

$$\sigma[n,k]_q^{\beta,r} = \sum_{s_1+s_2+\cdots+s_k=n-k} \prod_{j=0}^{k} \left([j\beta]_q + [r]_q \right)^{s_j}. \tag{3.24}$$

This sum may be written further as follows.

Theorem 3.6. *For nonnegative integers n and k and complex numbers β and r, the explicit formula for $\sigma[n,k]_q^{\beta,r}$ in homogeneous symmetric function form is given by*

$$\sigma[n,k]_q^{\beta,r} = \sum_{0\le j_1\le j_2\le\cdots\le j_{n-k}\le k} \prod_{i=1}^{n-k} \left([j_i\beta]_q + [r]_q \right). \tag{3.25}$$

This explicit formula is necessary in giving combinatorial interpretation of $\sigma[n,k]_q^{\beta,r}$ in the context of 0-1 tableau. Note that when $\beta = 1$ and $r = 0$, Theorem 3.6 yields

$$\sigma[n,k]_q^{1,0} = \sum_{0\le j_1\le j_2\le\cdots\le j_{n-k}\le k} \prod_{i=1}^{n-k} [j_i]_q = S_q[n,k], \tag{3.26}$$

the q-Stirling numbers of the second kind [12]. Moreover, taking $\beta = 1$ and $r = \log_q 2$, Theorem 3.6 reduces to

$$\begin{aligned}\sigma[n,k]_q^{1,\log_q 2} &= \sum_{0\le j_1\le j_2\le\cdots\le j_{n-k}\le k} \prod_{i=1}^{n-k} \left([j_i]_q + \left[\log_q 2\right]_q \right) \\ &= (q-1)^{k-n} \sum_{0\le j_1\le j_2\le\cdots\le j_{n-k}\le k} \prod_{i=1}^{n-k} q^{j_i}.\end{aligned} \tag{3.27}$$

Using the representation given in [15] for the q-binomial coefficients, we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = (q-1)^{n-k} \sigma[n,k]_{1q}^{1,\log_q 2}. \tag{3.28}$$

This is the identity that we used in Section 2.

4. Combinatorial Interpretation of $\sigma[n,k]_q^{\beta,r}$

Definition 4.1 (see [15]). A *0-1 tableau* is a pair $\varphi = (\lambda, f)$, where

$$\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k) \tag{4.1}$$

0	0	0	1	1
0	0	1		
1	0	0		
0	1			
0				

Figure 1: The 0-1 tableau $\varphi = (\lambda, f)$.

is a partition of an integer m and $f = (f_{ij})_{1\le j\le\lambda_i}$ is a "filling" of the cells of corresponding Ferrers diagram of the shape λ with 0's and 1's, such that there is exactly one "1" in each column.

Using the partition $\lambda = (5, 3, 3, 2, 1)$, we can construct 60 distinct 0-1 tableaux. Figure 1 below shows one of these tableaux with $f_{14} = f_{15} = f_{23} = f_{31} = f_{42} = 1$, $f_{ij} = 0$ elsewhere such that $1 \le j \le \lambda_i$.

Definition 4.2 (see [15]). An *A-tableau* is a list ϕ of column c of a Ferrer's diagram of a partition λ (by decreasing order of length) such that the lengths $|c|$ are part of the sequence $A = (a_i)_{i\ge 0}$, a strictly increasing sequence of nonnegative integers.

Let ω be a function from the set of nonnegative integers N to a ring K. Suppose Φ is an A-tableau with r columns of lengths $|c| \le h$. Then, we set

$$\omega_A(\Phi) = \prod_{c\in\Phi} \omega(|c|). \tag{4.2}$$

Note that Φ might contain a finite number of columns whose lengths are zero since $0 \in A = \{0, 1, 2, \dots, k\}$ and if $\omega(0) \neq 0$.

From this point onward, whenever an A-tableau is mentioned, it is always associated with the sequence $A = \{0, 1, 2, \dots, k\}$.

We are now ready to mention the following theorem.

Theorem 4.3. *Let $\omega : N \to K$ denote a function from N to a ring K (column weights according to length) which is defined by $\omega(|c|) = [|c|\beta]_q + [r]_q$ where β and γ are complex numbers, and $|c|$ is the length of column c of an A-tableau in $T^A(k, n-k)$. Then*

$$\sigma[n,k]_q^{\beta,r} = \sum_{\phi\in T^A(k,n-k)} \prod_{c\in\phi} \omega(|c|). \tag{4.3}$$

Proof. This can easily be proved using Definition 4.2 and Theorem 3.6. □

Now, we demonstrate simple combinatorics of 0-1 tableaux to obtain certain relation for $\sigma[n,k]_q^{\beta,r}$. To start with, we have, from Theorem 4.3,

$$\sigma[n,k]_q^{\beta,r} = \sum_{\phi\in T^A(k,n-k)} \omega_A(\Phi), \tag{4.4}$$

where

$$\omega_A(\Phi) = \prod_{c\in\phi}\Big(\big[|c|\beta\big]_q + [r]_q\Big), \quad |c| \in \{0,1,2,\ldots,k\}. \tag{4.5}$$

Substituting $j_i = |c|$, we obtain

$$\omega_A(\Phi) = \prod_{i=1}^{n-k}\Big(\big[j_i\beta\big]_q + [r]_q\Big), \quad j_i \in \{0,1,2,\ldots,k\}. \tag{4.6}$$

Let $[r]_q = c_1 + c_2$ where $c_1 = [r]_q - [r_2]_q$ and $c_2 = [r_2]_q$ for some numbers r_1 and r_2. Then, with $\omega^*(j) = [j\beta]_q + c_2$, we have

$$\omega_A(\Phi) = \sum_{l=0}^{n-k} c_1^{n-k-l} \sum_{q_1\le q_2\le\cdots\le q_l, q_i\in\{j_1,j_2,\ldots,j_{n-k}\}} \prod_{i=1}^{l}\omega^*(q_i). \tag{4.7}$$

Now, we are going to count the number of tableaux with $n-k$ columns such that $n-k-r$ columns are of weight c_1 and r columns are of weight $\omega^*(q_i)$, $q_i \in \{0,1,2,\ldots,k\}$. Note that there are $\binom{n-k}{r}$ tableaux with r columns whose lengths are taken from the lengths of the columns of Φ. Since there is a one-to-one correspondence between weights $\omega(j_i)$ and A-tableaux, the number of A-tableaux Φ in $T^A(k,n-k)$ is equal to the number of possible multisets $\{j_1, j_2, \ldots, j_{n-k}\}$ with j_i in $\{0,1,2,\ldots,k\}$. That is,

$$\left|T^A(k,n-k)\right| = \binom{n}{k}. \tag{4.8}$$

Thus, for all $\Phi \in T^A(k,n-k)$, we can generate $\binom{n}{k}\binom{n-k}{r}$ tableaux with r columns whose weights are $\omega^*(j_i)$, $j_i \in \{0,1,2,\ldots,k\}$. However, there are only

$$\left|T^A(k,r)\right| = \binom{r+k}{r} \tag{4.9}$$

distinct tableaux with r columns whose lengths are in $\{0,1,2,\ldots,k\}$. Hence, every distinct tableau with $n-k$ columns, r of which are of weight other than c_1, appears

$$\frac{\binom{n}{k}\binom{n-k}{r}}{\binom{r+k}{r}} = \binom{n}{r+k} \tag{4.10}$$

times in the collection. Thus,

$$\sum_{\Phi\in T^A(k,n-k)} \omega_A(\Phi) = \sum_{r=0}^{n-k}\binom{n}{r+k} c_1^{n-k-r} \sum_{\phi\in\overline{B}_r}\prod_{c\in\phi}\omega^*(|c|), \tag{4.11}$$

where $\overline{B}_r$ denotes the set of all tableaux ψ having r columns of weights $\omega^*(j_i) = [j_i\beta]_q + c_2$. Reindexing the double sum, we get

$$\sum_{\Phi\in T^A(k,n-k)} \omega_A(\Phi) = \sum_{j=k}^{n} \binom{n}{j} c_1^{n-j} \sum_{\phi\in\overline{B}_{j-k}} \prod_{\overline{c}\in\phi} \omega^*(|c|), \tag{4.12}$$

where $\overline{B}_{j-k}$ is the set of all tableaux with $j-k$ columns of weights $\omega^*(j_i) = [j_i\beta]_q + c_2$ for each $i = 1, 2, \ldots, j-k$. Clearly, $\overline{B}_{j-k} = T^A(k, j-k)$. Therefore,

$$\sum_{\Phi\in T^A(k,n-k)} \omega_A(\Phi) = \sum_{j=k}^{n} \binom{n}{j} c_1^{n-j} \sum_{\phi\in T^A(k,j-k)} \omega_A(\phi). \tag{4.13}$$

Applying Theorem 4.3 completes the proof of the following theorem.

Theorem 4.4. *For nonnegative integers n and k and complex numbers β and r, the q-analogue $\sigma[n,k]_q^{\beta,r}$ satisfies the following identity:*

$$\sigma[n,k]_q^{\beta,r} = \sum_{j=k}^{n} \binom{n}{j} q^{(n-j)r_2} [r_1]_q^{n-j} \sigma[j,k]_q^{\beta,r_2}, \tag{4.14}$$

where $r = r_1 + r_2$.

Taking $\beta = 1$, $r_2 = 0$, and $r = r_1 = \log_q 2$, Theorem 4.4 gives

$$(q-1)^{n-k}\sigma[n,k]_q^{1,\log_q 2} = \sum_{j=k}^{n} \binom{n}{j} (q-1)^{j-k} \sigma[j,k]_q^{1,0}. \tag{4.15}$$

Using (2.16) and (3.26), we obtain

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{j=k}^{n} \binom{n}{j} (q-1)^{j-k} S_q[j,k] \tag{4.16}$$

the Carlitz identity in [12]. Hence, we can consider the identity in Theorem 4.4 as a generalization of the above Carlitz identity.

5. Orthogonality and Inverse Relations

We notice that (1.4) can be written as

$$\sum_{k=0}^{m} F_{r,-a}(m,k) t^k = p_m^{\mathbf{a}}(t). \tag{5.1}$$

Using (1.1), it can easily be shown that

$$\sum_{k=n}^{m} F_{r,-a}(m,k) S_n^k(\mathbf{a}) = \sum_{k=n}^{m} S_k^m(\mathbf{a}) F_{r,-a}(k,n) = \delta_{mn}, \tag{5.2}$$

where δ_{mn} is the Kronecker delta defined by $\delta_{mn} = 1$ if $m = n$, and $\delta_{mn} = 0$ if $m \neq n$. Moreover, the following inverse relations hold:

$$f_n = \sum_{k=0}^{n} S_k^n(\mathbf{a}) g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} F_{r,-a}(n,k) f_k, \tag{5.3}$$

$$f_k = \sum_{n \geq k} S_k^n(\mathbf{a}) g_n \Longleftrightarrow g_k = \sum_{n \geq k} F_{r,-a}(n,k) f_n. \tag{5.4}$$

Relation (5.2) is exactly the orthogonality relation for r-Whitney numbers that appeared in [6]. Consequently, the generating functions in ($P2$) and ($P3$) can be transformed, respectively, using (5.4) into the following identities:

$$\begin{aligned} &\sum_{n \geq k} F_{r,-a}(n,k) \frac{k!}{x^k} \frac{1}{r^n n!} e^{ax} (e^{rx} - 1)^n = 1, \\ &\sum_{n \geq k} F_{r,-a}(n,k) \frac{x^{n-k}}{\prod_{j=0}^{n} (1 - (rj + a)x)} = 1, \end{aligned} \tag{5.5}$$

which will reduce to the following interesting identities for $F_{\alpha,\gamma}(n,k)$ when $x = 1$:

$$\begin{aligned} &\sum_{n \geq k} \frac{F_{\alpha,\gamma}(n,k) k! (e^{\alpha} - 1)^n}{\alpha^n n! e^{\gamma}} = 1, \\ &\sum_{n \geq k} \frac{F_{\alpha,\gamma}(n,k)}{(1 + \gamma \mid \alpha)_{n+1}} = 1. \end{aligned} \tag{5.6}$$

Note that the number $F_{\alpha,\gamma}(n,k)$ can be expressed in terms of the unified generalization of Stirling numbers by Hsu and Shiue [16] as $F_{\alpha,\gamma}(n,k) = S(n,k;\alpha,0,\gamma)$. Hence, the identity in (5.6) coincides with the identity in [17, Theorem 9] by taking $x = 1 + \gamma$.

Parallel to (5.2), (5.3), and (5.4), we will establish in this section the orthogonality and inverse relations of $\phi_{\alpha,\gamma}[n,k]_q$ and $\sigma[n,k]_q^{\beta,r}$.

To derive the orthogonality relation for $\phi_{\alpha,\gamma}[n,k]_q$ and $\sigma[n,k]_q^{\beta,r}$, we need to rewrite first (1.5) and (3.1). By taking $\gamma = \log_q(2 - q^r)$, (1.5) gives

$$\sum_{k=0}^{n} \phi_{\alpha,\log_q(2-q^r)}[n,k]_q t^k = \left\langle t - [r]_q \mid [\alpha] \right\rangle_n^q, \tag{5.7}$$

and, by replacing t with $t-[r]_q$, (3.1) yields

$$\sum_{k=0}^{n}\sigma[n,k]_q^{\beta,r}\left\langle t-[r]_q \mid [\beta]\right\rangle_k^q = t^n. \tag{5.8}$$

Using (5.8), (5.7) can be expressed as

$$\begin{aligned}\left\langle t-[r]_q \mid [\beta]\right\rangle_m^q &= \sum_{k=0}^{m}\phi_{\beta,\log_q(2-q^r)}[m,k]_q\left\{\sum_{n=0}^{k}\sigma[k,n]_q^{\beta,r}\left\langle t-[r]_q \mid [\beta]_q\right\rangle_n^q\right\} \\ &= \sum_{n=0}^{m}\left\{\sum_{k=n}^{m}\phi_{\beta,\log_q(2-q^r)}[m,k]_q\sigma[k,n]_q^{\beta,r}\right\}\left\langle t-[r]_q \mid [\beta]_q\right\rangle_n^q.\end{aligned} \tag{5.9}$$

Thus

$$\sum_{k=n}^{m}\phi_{\beta,\log_q(2-q^r)}[m,k]_q\sigma[k,n]_q^{\beta,r} = \delta_{mn} \quad (m\ge n). \tag{5.10}$$

Theorem 5.1. *For nonnegative integers m, n, and k and complex numbers β and r, the following orthogonality relation holds:*

$$\sum_{k=n}^{m}\phi_{\beta,\bar{r}}[m,k]_q\sigma[k,n]_q^{\beta,r} = \sum_{k=n}^{m}\sigma[m,k]_q^{\beta,r}\phi_{\beta,\bar{r}}[k,n]_q = \delta_{mn} \quad (m\ge n), \tag{5.11}$$

where $\bar{r}=\log_q(2-q^r)$.

Remark 5.2. It can easily be shown that $\bar{r}=\log_q(2-q^r)\to -r$ as $q\to 1$. This implies that $\phi_{\beta,\bar{r}}[m,k]_q \to F_{\beta,-r}(m,k)$ as $q\to 1$. Since $\sigma[k,n]_q^{\beta,r}\to S_n^k(\mathbf{c})$ as $q\to 1$, (5.11) yields (5.2) easily.

Remark 5.3. Let M_1 and M_2 be two $n\times n$ matrices whose entries are $\phi_{\beta,\bar{r}}[i,j]_q$ and $\sigma[i,j]_q^{\beta,r}$, respectively. That is, $M_1=(\phi_{\beta,\bar{r}}[i,j]_q)_{0\le i,j\le n}$ and $M_2=(\sigma[i,j]_q^{\beta,r})_{0\le i,j\le n}$. Then using Theorem 5.1, $M_1M_2=M_2M_1=I_n$, the identity matrix of order n. This implies that M_1 and M_2 are orthogonal matrices.

Using the orthogonality relation in Theorem 5.1, we can easily prove the following inverse relation.

Theorem 5.4. *For nonnegative integers m, n, and k, and complex numbers β and r, the following inverse relation holds:*

$$f_n=\sum_{k=0}^{n}\sigma[n,k]_q^{\beta,r}g_k \Longleftrightarrow g_n=\sum_{k=0}^{n}\phi_{\beta,\bar{r}}[n,k]_q f_k, \tag{5.12}$$

where $\bar{r}=\log_q(2-q^r)$.

Proof. Given $f_n = \sum_{k=0}^{n} \sigma[n,k]_q^{\beta,r} g_k$, we have

$$\begin{aligned}
\sum_{k=0}^{n} \phi_{\beta,\overline{r}}[n,k]_q f_k &= \sum_{k=0}^{n} \phi_{\beta,\overline{r}}[n,k]_q \left\{ \sum_{j=0}^{k} \sigma[k,j]_q^{\beta,r} g_j \right\} \\
&= \sum_{j=0}^{k} \left\{ \sum_{k=j}^{n} \phi_{\beta,\overline{r}}[n,k]_q \sigma[k,j]_q^{\beta,r} \right\} g_j \\
&= \sum_{j=0}^{k} \delta_{nj} g_j = g_n.
\end{aligned} \tag{5.13}$$

□

The converse can be shown similarly.

One can easily prove the following inverse relation.

Theorem 5.5. *For nonnegative integers m, n, and k and complex numbers β and r, the following inverse relation holds:*

$$f_k = \sum_{n=0}^{\infty} \sigma[n,k]_q^{\beta,r} g_n \Longleftrightarrow g_k = \sum_{n=0}^{\infty} \phi_{\beta,\overline{r}}[n,k]_q f_n, \tag{5.14}$$

where $\overline{r} = \log_q(2 - q^r)$.

Remark 5.6. The exponential and rational generating functions in Theorem 3.5 can be transformed into the following identities for the q-analogue of $F_{\alpha,\gamma}(n,k)$:

$$\begin{gathered}
\sum_{n\geq 0} \phi_{\beta,\overline{r}}[n,k]_q \frac{k!}{t^k} \Delta_q^n \left(\frac{e^{([x\beta]_q + [r]_q)t}}{\left\langle [n\beta]_q \mid [\beta] \right\rangle_n^q} \right)_{x=0} = 1, \\
\sum_{n\geq 0} \phi_{\beta,\overline{r}}[n,k]_q \frac{t^{n-k}}{\prod_{j=0}^{n} \left(1 - \left([j\beta]_q + [r]_q\right)t\right)} = 1,
\end{gathered} \tag{5.15}$$

when $q \to 1$, (5.15) will exactly give (5.5), respectively.

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