Research Article

k-Step Sum and *m*-Step Gap Fibonacci Sequence

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For two given integers *k, m*, we introduce the *k*-step sum and *m*-step gap Fibonacci sequence by presenting a recurrence formula that generates the *n*th term as the sum of *k* successive previous terms starting the sum at the *m*th previous term. Known sequences, like Fibonacci, tribonacci, tetranacci, and Padovan sequences, are derived for specific values of *k, m*. Two limiting properties concerning the terms of the sequence are presented. The limits are related to the spectral radius of the associated {0, 1}-matrix.

1. Introduction

It is well-known that the Fibonacci sequence, the Lucas sequence, the Padovan sequence, the Perrin sequence, the tribonacci sequence, and the tetranacci sequence are very prominent examples of recursive sequences, which are defined as follows.

The Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, \ldots$ are derived by the recurrence relation $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$, with $f_1 = f_2 = 1$, [1], [2, A000045].

The Lucas numbers $2, 1, 3, 4, 7, 11, 18, 29, \ldots$ are derived by the recurrence relation $\ell_n = \ell_{n-1} + \ell_{n-2}$, $n \geq 3$, with $\ell_1 = 2$, and $\ell_2 = 1$, [1], [2, A000032].

The Padovan numbers $1, 0, 0, 1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, \ldots$ are derived by the recurrence relation $a_n = a_{n-2} + a_{n-3}$, $n \geq 4$, with $a_1 = 1$, $a_2 = a_3 = 0$, [2, A000931].

The Perrin numbers $3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, \ldots$ are derived by the recurrence relation $p_n = p_{n-2} + p_{n-3}$, $n \geq 4$, with $p_1 = 3$, $p_2 = 0$, and $p_3 = 2$, [2, A001608].

Both Fibonacci and Lucas numbers as well as both Padovan and Perrin numbers satisfy the same recurrence relation with different initial conditions.

Extending the above definitions, the *k*-step Fibonacci sequences are derived [3]. For $k = 3$, the tribonacci numbers $1, 1, 2, 4, 7, 13, 24, 44, \ldots$ are derived by the recurrence relation $f_n = f_{n-1} + f_{n-2} + f_{n-3}$, $n \geq 4$, with $f_1 = f_2 = 1$, and $f_3 = 2$, [3–5], [2, A000073].

For $k = 4$, the tetranacci numbers $1, 1, 2, 4, 8, 15, 29, 56, \ldots$ are derived by the recurrence relation $f_n = f_{n-1} + f_{n-2} + f_{n-3} + f_{n-4}$, $n \geq 5$, with $f_1 = f_2 = 1$, and $f_3 = 2$, $f_4 = 4$, [3], [2, A000078].

In this paper, we introduce *k*-step sum and *m*-step gap Fibonacci sequence, where the *n*th term of the sequence is the sum of the *k* successive previous terms starting at the *m*th previous term, using 1's as initial conditions. Further the closed formula of the *n*th term of the sequence is given and the ratio of two successive terms tends to the spectral radius of the associated {0, 1}-matrix.

2. Definition of *k*-Step Sum and *m*-Step Gap Fibonacci Sequence

For the integers $k = 1, 2, \ldots, m = 0, 1, \ldots$, we define the *k-step sum and m-step gap Fibonacci sequence* $(f_n^{(k,m)})_{n=1,2,\ldots}$, whose *n*th term is given by the following recurrence relation:

$$
\begin{aligned}
f_n &= f_{n-m-1} + f_{n-m-2} + \cdots + f_{n-m-(k-1)} + f_{n-m-k} \\
&= \sum_{i=m+1}^{k+m} f_{n-i}, \quad \text{for every } n \geq k + m + 1,
\end{aligned}
\tag{1}
$$

with

$$f_1 = \cdots = f_{k+m} = 1. \tag{2}$$

Combining (1) and (2) notice that all the terms f_n of the sequence $(f_n^{(k,m)})_{n=1,2,\ldots}$ are positive integers and f_n is the sum of k terms starting the sum at the mth previous term from f_n; thus, (1) can be written equivalently as

$$f_n = f_{n-m-1} + f_{n-m-2} + \cdots + f_{n-m-(k-1)} + f_{n-m-k}$$

$$= \sum_{j=1}^{k} f_{n-m-j}, \quad \text{for every } n \geq k+m+1. \tag{3}$$

Remark 1. (i) From (2)-(3) it is evident that for $k = 1$ and $m = 0, 1, \ldots$ all the terms of the sequence $(f_n^{(k,m)})_{n=1,2,\ldots}$ are equal to one. Hereafter consider $k \geq 2$, since the case $k = 1$ is trivial.

(ii) For $m = 0$, (3) and (2) give the nth term f_n of the sequence $(f_n^{(k,0)})_{n=1,2,\ldots}$, which is formulated as

$$f_n = f_{n-1} + f_{n-2} + \cdots + f_{n-(k-1)} + f_{n-k}$$

$$= \sum_{i=1}^{k} f_{n-i}, \quad \text{for every } n \geq k+1, \tag{4}$$

with initial values

$$f_1 = \cdots = f_k = 1. \tag{5}$$

Remark 2. The sequence $(f_n^{(k,m)})_{n=1,2,\ldots}$ gives known sequences for various values of the steps k, m:

(i) for $k = 2$, $m = 0$, (4)-(5) give the well-known Fibonacci sequence, $1, 1, 2, 3, 5, 8, 13, \ldots$;

(ii) for $k = 3$, $m = 0$, (4)-(5) give the tribonacci sequence, $1, 1, 1, 3, 5, 9, 17, 31, \ldots$, [2, A000213];

(iii) for $k = 4$, $m = 0$, (4)-(5) give the tetranacci sequence, $1, 1, 1, 1, 4, 7, 13, 25, \ldots$, [2, A000288];

(iv) for $k = 2$, $m = 1$, (2)-(3) give the Padovan sequence, $1, 1, 1, 2, 2, 3, 4, 5, 7, 9, \ldots$, [2, A 000931].

In the following, the Dirac delta function (or δ function) is denoted by $\delta_{n-j} = \left\{ \begin{smallmatrix} 0, & n \neq j \\ 1, & n = j \end{smallmatrix} \right.$ and the Heaviside step function (or the unit step function) $u_{n-j} = \left\{ \begin{smallmatrix} 0, & n < j \\ 1, & n \geq j \end{smallmatrix} \right.$.

Moreover, the nth number of the sequence $(f_n^{(k,m)})_{n=1,2,\ldots}$ follows immediately from (2) and (3) using the above definition of the δ function and considering that the first $k + m$ negative indexed terms are equal to zero:

$$f_{-(k+m-1)} = \cdots = f_{-1} = f_0 = 0, \tag{6}$$

which is formulated in the following proposition.

Proposition 3. *For the given integers $k \geq 2$, $m \geq 0$, for all $n \geq 1$, the nth number, f_n, of the sequence $(f_n^{(k,m)})_{n=1,2,\ldots}$ is given by the following recurrence relation:*

$$f_n = \sum_{i=1}^{k} f_{n-m-i} + \sum_{j=1}^{k+m} \delta_{n-j} - \sum_{j=1}^{k-1}\sum_{i=1}^{k-j} \delta_{n-m-j-i}, \tag{7}$$

with initial values as in (6).

In the following, we are going to demonstrate a close link between matrices and Fibonacci numbers in (3) with initial values in (2).

To this end, consider $k \geq 2$, $m \geq 1$. One can write the following linear system, where (3) constitutes its first equation:

$$f_n = f_{n-m-1} + f_{n-m-2} + \cdots + f_{n-m-k+1} + f_{n-m-k}$$

$$f_{n-1} = f_{n-1}$$

$$\vdots$$

$$f_{n-m} = f_{n-m} \tag{8}$$

$$f_{n-m-1} = f_{n-m-1}$$

$$\vdots$$

$$f_{n-m-(k-1)} = f_{n-m-(k-1)}.$$

Hence, using a $(k + m) \times 1$ vector, the linear system in (8) can be formed as

$$
\begin{bmatrix} f_n \\ f_{n-1} \\ \vdots \\ f_{n-m} \\ f_{n-m-1} \\ \vdots \\ f_{n-m-(k-2)} \\ f_{n-m-(k-1)} \end{bmatrix}
$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & & & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & & \vdots \\ \vdots & & & & \ddots & & & \vdots \\ \vdots & & & & & \ddots & & 0 \\ 0 & 0 & \cdots & \cdots & & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_{n-2} \\ \vdots \\ f_{n-m} \\ f_{n-m-1} \\ \vdots \\ f_{n-m-(k-1)} \\ f_{n-m-k} \end{bmatrix}, \tag{9}$$

whereby it is obvious that the sequence $(f_n^{(k,m)})_{n=1,2,\ldots}$ can be represented by a $(k + m) \times (k + m)$ matrix, $F_{k,m}$, which is a block matrix such that

$$F_{k,m} = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & & 0 \\ \vdots & & 1 & 0 & & \vdots \\ & & & & \ddots & \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix}, \tag{10}$$

where the first row consists of the vector-matrices F_1, F_2; the m entries of the $1 \times m$ vector F_1 are equal to zero and the rest k entries of the $1 \times k$ vector F_2 are equal to one; the $(k + m - 1) \times (k + m - 1)$ matrix F_3 is the identity matrix and the $k + m - 1$ entries of the $(k + m - 1) \times 1$ vector F_4 are equal to zero.

Working as in the above, for $k \geq 2$, $m = 0$, and using (4) with initial values in (5), we can write the following linear system:

$$f_n = f_{n-1} + f_{n-2} + \cdots + f_{n-k+1} + f_{n-k}$$

$$f_{n-1} = f_{n-1}$$

$$\vdots$$

$$f_{n-(k-2)} = f_{n-(k-2)}$$

$$f_{n-(k-1)} = f_{n-(k-1)} \tag{11}$$

$$\Longleftrightarrow$$

$$\begin{bmatrix} f_n \\ f_{n-1} \\ \vdots \\ f_{n-(k-2)} \\ f_{n-(k-1)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & & 1 \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \\ 0 & \cdots & & 1 & 0 \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_{n-2} \\ \vdots \\ f_{n-(k-1)} \\ f_{n-k} \end{bmatrix}.$$

The $k \times k$ matrix, $F_{k,0}$, of the coefficients of the above system, is defined as

$$F_{k,0} = \begin{bmatrix} \widetilde{F}_1 & 1 \\ I_{k-1} & \widetilde{F}_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & & \ddots & & \\ 0 & \cdots & & 1 & 0 \end{bmatrix}, \tag{12}$$

where the $k-1$ entries of the $1 \times (k-1)$ vector $\widetilde{F}_1$ are equal to one, I_{k-1} is the $(k-1) \times (k-1)$ identity matrix, and the $k-1$ entries of the $(k-1) \times 1$ vector $\widetilde{F}_4$ are equal to zero.

Remark 4. (i) The well-known sequences, which are presented in Remark 2, correspond to $F_{k,0}$ in (12) for suitable integer value of $k \geq 2$ and $= 0$;

(a) for $k = 2$, the Fibonacci sequence corresponds to $F_{2,0} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$;

(b) for $k = 3$, the tribonacci sequence corresponds to $F_{3,0} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$;

(c) for $k = 4$, the tetranacci sequence corresponds to $F_{4,0} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

(ii) The Padovan sequence corresponds to the matrix $F_{2,1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ by (10) with $k = 2$, $m = 1$.

(iii) The matrix $F_{k,0}$ in (12) has been defined and the determinant of $F_{k,0}$ has been investigated in [6] and some results on matrices related with Fibonacci numbers and Lucas numbers have been investigated in [7] and the transpose matrix of the general Q-matrix in [8].

Proposition 5. *The $(k+m)$th degree characteristic polynomial $x(\lambda)$ of $F_{k,m}$ in (10) is given by*

$$x(\lambda) = \lambda^{k+m} - \sum_{i=0}^{k-1} \lambda^i = \lambda^{k+m} - \sum_{i=1}^{k} \lambda^{i-1}$$

$$= \lambda^{k+m} - \sum_{i=1}^{k} \lambda^{k-i}. \tag{13}$$

Proof. The proof of (13) is based on the induction method. For $k = 2$, $m = 1$, the characteristic polynomial of $F_{2,1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is $x(\lambda) = \lambda^3 - \lambda - 1$, which satisfies (13). Let m be a fixed integer and assume that the formula in (13) is true for k; that is,

$$x(\lambda) = \det(\lambda I_{k+m} - F_{k,m}) = \lambda^{k+m} - \sum_{i=1}^{k} \lambda^{k-i}. \tag{14}$$

Then, $\det(\lambda I_{k+m+1} - F_{k+1,m})$ of the $(k+m+1) \times (k+m+1)$ matrix $\lambda I_{k+m+1} - F_{k+1,m}$ can be computed by using the Laplace expansion along the $(k+m+1)$th column and the assumption of induction. Thus, we have

$$x(\lambda) = \det(\lambda I_{k+m+1} - F_{k+1,m})$$

$$= (-1)^{k+m+2}(-1) \det \begin{bmatrix} -1 & \lambda & 0 & \cdots & 0 \\ 0 & -1 & \lambda & & 0 \\ 0 & 0 & -1 & \ddots & \vdots \\ \vdots & & & \ddots & \lambda \\ 0 & 0 & \cdots & 0 & -1 \end{bmatrix}$$

$$+ (-1)^{2(k+m+1)} \lambda \det(\lambda I_{k+m} - F_{k,m})$$

$$= (-1)^{k+m+3}(-1)^{k+m} + \lambda \left(\lambda^{k+m} - \sum_{i=1}^{k} \lambda^{k-i} \right)$$

$$= \lambda^{k+m+1} - \sum_{i=1}^{k} \lambda^{k+1-i} - 1 = \lambda^{k+m+1} - \sum_{i=1}^{k+1} \lambda^{k+1-i}, \tag{15}$$

hence, (13) holds for $k + 1$, too. Thus the result follows by the induction method. □

The set of all eigenvalues of $F_{k,m}$ is denoted by $\sigma(F_{k,m})$ and called the *spectrum* of $F_{k,m}$; the nonnegative real number $\rho(F_{k,m}) = \max\{|\lambda| : \lambda \in \sigma(F_{k,m})\}$ is called *spectral radius* of $F_{k,m}$. Here, $\rho(F_{k,m})$ is an eigenvalue of $F_{k,m}$, since the entries of $F_{k,m}$ are 0 or 1, [11, Theorem 8.3.1]; further

$$1 < \rho(F_{k,m}) < 2, \tag{16}$$

since

$$\min_{1 \leq i \leq k+m} \sum_{j=1}^{k+m} \varphi_{ij} < \rho(F_{k,m}) < \max_{1 \leq i \leq k+m} \sum_{j=1}^{k+m} \varphi_{ij},$$

$$\min_{1 \leq j \leq k+m} \sum_{i=1}^{k+m} \varphi_{ij} < \rho(F_{k,m}) < \max_{1 \leq j \leq k+m} \sum_{i=1}^{k+m} \varphi_{ij}, \tag{17}$$

where φ_{ij} denotes the ijth entry of $F_{k,m}$, [11, Theorem 8.1.22], [12, Theorem 7], and [13].

Notice that if $\lambda_j \in \sigma(F_{k,m})$ is an eigenvalue of $F_{k,m}$, then $\overline{\lambda}_j \in \sigma(F_{k,m})$, because $x(\lambda)$ has real coefficients. Further, since $x(\lambda)$ in (13) has the constant term equal to -1, it is evident that

$$\det F_{k,m} = (-1)^{k+m}(-1) = (-1)^{k+m+1}. \qquad (18)$$

Hence, $F_{k,m}$ is a nonsingular and all the eigenvalues are nonzero.

Remark 6. Notice that, for $m = 0$,

(i) the kth degree characteristic polynomial $x(\lambda)$ of the matrix $F_{k,0}$ in (12) is formulated by (13), which has presented in [9, 10];

(ii) the authors in [10] have shown bounds for $\rho(F_{k,0})$; the lower bound is more accurate than the associated bound in (16); in particular,

$$\sqrt{\frac{2k-1}{k}} < \rho(F_{k,0}) < 2, \qquad (19)$$

(iii) the determinant of $F_{k,0}$ is computed by (18) and derived the same result as in [6].

Example 7. Consider $k = 2$, $m = 0$, and the well-known Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$, as in **Remark 2**. According to **Remark 4(i)**, the 2×2 matrix

$$F_{2,0} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad (20)$$

is derived by (12). It is evident that the characteristic polynomial is given by $x(\lambda) = \lambda^2 - \lambda - 1$ and its roots are $\lambda_1 = (1 - \sqrt{5})/2$ and $\rho(F_{2,0}) \equiv \lambda_2 = (1 + \sqrt{5})/2$, the well-known number as the golden ratio.

Example 8. Consider $k = 2$, $m = 1$. By (2)-(3) the associated sequence $(f_n^{(2,1)})_{n=1,2,\ldots}$ is formed as $f_1 = f_2 = f_3 = 1$ and $f_n = f_{n-3} + f_{n-2}$, for all $n \geq 4$, which is well-known as the Padovan sequence $1, 1, 1, 2, 2, 3, 4, 5, \ldots$ (see, **Remark 2**). According to **Remark 4(ii)**, the associated 3×3 matrix is given by

$$F_{2,1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \qquad (21)$$

The characteristic polynomial is given by (13) as $x(\lambda) = \lambda^3 - \lambda - 1$ and its spectrum $\sigma(F_{2,1}) = \{\rho(F_{2,1}) = 1.32472, \lambda_1 = -0.66235 + 0.56227i, \lambda_2 = \overline{\lambda}_1 = -0.66235 - 0.56227i\}$.

For the integers $k \geq 2$ and $m \geq 0$, it is worth noting that, since the entries of the matrix $(I_{k+m} + F_{k,m})^{k+m-1}$ are positive integers, $F_{k,m}$ is an irreducible matrix [11, Lemma 8.4.1]; it follows that the spectral radius $\rho(F_{k,m})$ is a positive, simple (without multiplicity) eigenvalue of $F_{k,m}$ [11, Theorem 8.4.4]. In addition, the entries of $F^{(k+m)^2-2(k+m)+2}$ are positive

integers; thus $F_{k,m}$ is a primitive matrix [11, Corollary 8.5.9]; that is, $\rho(F_{k,m})$ is the unique eigenvalue with maximum modulus [11, Definition 8.5.0]. Hence, in the following, we denote $\lambda_1, \lambda_2, \ldots, \lambda_{k+m-1}, \rho(F_{k,m})$ all the distinct eigenvalues of $F_{k,m}$, for which the following inequality holds:

$$0 < \left|\lambda_j\right| < \rho(F_{k,m}); \quad j = 1, 2, \ldots, k + m - 1. \qquad (22)$$

Furthermore, rewriting (7) as

$$f_n - \sum_{i=1}^{k} f_{n-m-i}$$
$$= \sum_{j=1}^{k+m} \delta_{n-j} - \sum_{i=1}^{k-1} \delta_{n-m-1-i} - \sum_{i=1}^{k-2} \delta_{n-m-2-i} \qquad (23)$$
$$- \cdots - \sum_{i=1}^{2} \delta_{n-m-(k-2)-i} - \delta_{n-k-m},$$

the z-transform on both sides of (23) yields

$$F(z)$$
$$= \left(z^{-1} + \cdots + z^{-(k+m-1)} + z^{-(k+m)} - \sum_{i=1}^{k-1} z^{-(m+1+i)} \right.$$
$$- \sum_{i=1}^{k-2} z^{-(m+2+i)} - \cdots - \sum_{i=1}^{3} z^{-(k+m+i-3)}$$
$$\left. - \sum_{i=1}^{2} z^{-(k+m+i-2)} - z^{-(k+m)} \right)$$
$$\times \left(1 - z^{-(m+1)} - z^{-(m+2)} - \cdots - z^{-(k+m)} \right)^{-1}$$
$$= \left(z^{-1} + \cdots + z^{-(k+m-1)} - \sum_{i=1}^{k-2} z^{-(m+1+i)} \right.$$
$$- \sum_{i=1}^{k-3} z^{-(m+2+i)} - \cdots - \sum_{i=1}^{2} z^{-(k+m+i-3)}$$
$$\left. - z^{-(k+m-1)} - (k-2) z^{-(k+m)} \right)$$
$$\times \left(1 - z^{-(m+1)} - z^{-(m+2)} - \cdots - z^{-(k+m)} \right)^{-1}$$
$$= \left(z^{-1} + \cdots + z^{-(k+m-1)} - \sum_{i=1}^{k-2} z^{-(m+1+i)} \right.$$
$$- \sum_{i=1}^{k-3} z^{-(m+2+i)} - \cdots - \sum_{i=1}^{2} z^{-(k+m+i-3)}$$
$$\left. - z^{-(k+m-1)} - (k-2) z^{-(k+m)} \right)$$
$$\times \left(\frac{1}{z^{k+m}} \left(z^{k+m} - z^{k-1} - z^{k-2} - \cdots - z - 1 \right) \right)^{-1}$$

$$= \left(z^{-1} + \cdots + z^{-(m+1)} - \sum_{i=1}^{k-3} z^{-(m+2+i)} \right.$$

$$- \sum_{i=1}^{k-4} z^{-(m+3+i)} - \cdots - \sum_{i=1}^{2} z^{-(k+m+i-3)}$$

$$\left. - z^{-(k+m-1)} - (k-2) z^{-(k+m)} \right)$$

$$\times \left(z^{-(k+m)} x(z) \right)^{-1}. \tag{24}$$

From (24) it is worth noting that the poles of $F(z)$ are the eigenvalues of $F_{k,m}$, which are all simple (distinct) and the complex eigenvalues are conjugate; furthermore, the degrees of the polynomials of numerator and denominator of $F(z)$ coincide. Thus, the partial-fraction decomposition of (24) is given by

$$F(z) = k - 2 + \frac{c}{1 - \rho(F_{k,m}) z^{-1}} + \sum_{j=1}^{k+m-1} \frac{c_j}{1 - \lambda_j z^{-1}}, \tag{25}$$

where c, $\rho(F_{k,m})$ are real and the others coefficients c_j are complex or real numbers.

In the following theorem, we are able to present the closed formula of the terms of the sequence $(f_n^{(k,m)})_{n=1,2,\ldots}$, which depends on all the eigenvalues of $F_{k,m}$.

Theorem 9. *Let $\lambda_1, \lambda_2, \ldots, \lambda_{k+m-1}, \rho(F_{k,m})$ be the eigenvalues of $F_{k,m}$ and the fixed integers k, m, with $k \geq 2$, $m \geq 0$. The nth number of the sequence $(f_n^{(k,m)})_{n=1,2,\ldots}$ is given by*

$$f_n = c \left(\rho(F_{k,m}) \right)^n + \sum_{j=1}^{k+m-1} c_j (\lambda_j)^n, \tag{26}$$

where c, c_j, for all $j = 1, 2, \ldots, k + m - 1$, are the determined coefficients of the partial-fraction decomposition in (25).

Proof. The inverse z-transform on both sides of (25) for all $n = 1, 2, \ldots$ yields

$$f_n = (k-2) \delta_n + c \left(\rho(F_{k,m}) \right)^n u_n$$

$$+ \sum_{j=1}^{k+m-1} c_j (\lambda_j)^n u_n. \tag{27}$$

The closed formula of f_n in (26) follows from the above equation and the definitions of δ and Heaviside step functions. $\square$

3. Limiting Properties of k-Step Sum and m-Step Gap Fibonacci Sequence

The spectral radius of $F_{k,m}$ in (10) is a characteristic quantity, which appears in (26) and for some cases of k, m is computed in Table 1.

TABLE 1: The spectral radius $\rho(F_{k,m})$ of $F_{k,m}$ with respect to k and m.

	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$m = 0$	1.6180	1.8393	1.9276	1.9659
$m = 1$	1.3247	1.4656	1.5342	1.5701
$m = 2$	1.2207	1.3247	1.3808	1.4122
$m = 3$	1.1673	1.2499	1.2965	1.3247

From the values in Table 1 observe that the spectral radius $\rho(F_{k,m})$

(i) increases as k increases and m remains constant;

(ii) decreases as m increases and k remains constant;

(iii) lies in the interval $(1, 2)$ verifying (16).

Note that for $k = 3$, $m = 0$, the spectral radius $\rho(F_{k,m})$ is the tribonacci constant and for $k = 4$, $m = 0$, the spectral radius $\rho(F_{k,m})$ is the tetranacci constant [2].

The significance of $\rho(F_{k,m})$ is presented in the following theorem.

Theorem 10. *For the fixed integers k, m, with $k \geq 2$, $m \geq 0$, the positive numbers f_n of the sequence $(f_n^{(k,m)})_{n=1,2,\ldots}$ in (26) satisfy the following limit properties:*

$$\lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \rho(F_{k,m}), \tag{28}$$

$$\lim_{n \to \infty} \sqrt[n]{f_n} = \rho(F_{k,m}), \tag{29}$$

where $\rho(F_{k,m})$ is the spectral radius of $F_{k,m}$ in (10).

Proof. Consider that the polar form of the determined coefficients c_j in (25) is denoted by $c_j = |c_j| e^{i\theta_j}$, and the eigenvalues (except the spectral radius) $\lambda_j = |\lambda_j| e^{i\omega_j}$, for all $j = 1, 2, \ldots, k+m-1$. The substitution of c_j, λ_j from the polar forms in (26) yields

$$f_n = c \left(\rho(F_{k,m}) \right)^n + \sum_{j=1}^{k+m-1} |c_j| e^{i\theta_j} \left(|\lambda_j| e^{i\omega_j} \right)^n$$

$$= c \left(\rho(F_{k,m}) \right)^n + \sum_{j=1}^{k+m-1} |c_j| |\lambda_j|^n e^{i(\theta_j + n\omega_j)}$$

$$= c \left(\rho(F_{k,m}) \right)^n$$

$$+ \sum_{j=1}^{k+m-1} |c_j| |\lambda_j|^n \left(\cos(\theta_j + n\omega_j) + i \sin(\theta_j + n\omega_j) \right). \tag{30}$$

Using (30) and the property of the spectral radius $\rho(F_{k,m}) > 0$ from (22), we can write

$$
\lim_{n \to \infty} \frac{f_{n+1}}{f_n}
$$

$$
= \lim_{n \to \infty} \Bigg((\rho(F_{k,m}))^{n+1}
$$

$$
\times \Bigg(c + \sum_{j=1}^{k+m-1} |c_j| \left| \frac{\lambda_j}{\rho(F_{k,m})} \right|^{n+1}
$$

$$
\times \Big(\cos(\theta_j + (n+1)\omega_j)
$$

$$
+ i \sin(\theta_j + (n+1)\omega_j) \Big) \Bigg)
$$

$$
\times \Bigg((\rho(F_{k,m}))^{n}
$$

$$
\times \Bigg(c + \sum_{j=1}^{k+m-1} |c_j| \left| \frac{\lambda_j}{\rho(F_{k,m})} \right|^{n}
$$

$$
\times \Big(\cos(\theta_j + n\omega_j) + i \sin(\theta_j + n\omega_j) \Big) \Bigg) \Bigg)^{-1} \Bigg).
$$

(31)

Since $(\cos(\theta_j + n\omega_j))_{n=1,2,\dots}$ and $(\sin(\theta_j + n\omega_j))_{n=1,2,\dots}$ are bounded sequences as well as the inequality (22) implies $|\lambda_j / \rho(F_{k,m})| < 1$ for every $j = 1, 2, \dots, k+m-1$, it is obvious that

$$
\lim_{n \to \infty} \Bigg(\left| \frac{\lambda_j}{\rho(F_{k,m})} \right|^{n+1}
$$

$$
\times \Big(\cos(\theta_j + (n+1)\omega_j) + i \sin(\theta_j + (n+1)\omega_j) \Big) \Bigg)
$$

$$
= \lim_{n \to \infty} \Bigg(\left| \frac{\lambda_j}{\rho(F_{k,m})} \right|^{n} \cos(\theta_j + n\omega_j) + i \sin(\theta_j + n\omega_j) \Bigg)
$$

$$
= 0.
$$

(32)

Thus, the validity of (28) follows from (31) and (32).

Furthermore, it is well known that for a sequence $(a_n)_{n \geq 1}$ of nonzero complex numbers, if $\lim_{n \to \infty}(|a_{n+1}|/|a_n|) = \alpha$, then $\lim_{n \to \infty} \sqrt[n]{|a_n|} = \alpha$ [14, Chapter 1], whereby it is evident that for the sequence $(f_n^{(k,m)})_{n=1,2,\dots}$ of the positive integers ($f_n > 0$), the equality (29) follows immediately from (28). $\quad\square$

Remark 11. Notice that for every $n = 1, 2, \dots$ the formulas of f_n in (26) and (30) are equivalent. Additionally, notice the following.

    (i) If $k + m$ is odd, then the characteristic polynomial in (13) has one real root, $\rho(F_{k,m})$, and the others are

complex conjugate. Thus, the complex eigenvalues λ_j and the coefficients c_j in (25) appear in r complex conjugate pairs, which are denoted by $\lambda_1, \lambda_2 = \bar{\lambda}_1$, $\lambda_3, \lambda_4 = \bar{\lambda}_3, \dots, \lambda_{r-1}, \lambda_r = \bar{\lambda}_{r-1}$ and $c_1, c_2 = \bar{c}_1, c_3, c_4 = \bar{c}_3, \dots, c_{r-1}, c_r = \bar{c}_{r-1}$, respectively. Then, using the complex conjugate properties, (30) follows

$$
f_n = c(\rho(F_{k,m}))^n + 2 \sum_{i=1}^{r} |c_i| |\lambda_i|^n \cos(\theta_i + n\omega_i), \qquad (33)
$$

where $r = (k + m - 1)/2$.

    (ii) If $k + m$ is even, then the characteristic polynomial in (13) has two real roots and the others are complex conjugate. The one real root is the unique real positive root $\rho(F_{k,m})$; it lies in the interval $(1, 2)$ by (16) and has maximum modulus. The other real root is negative and lies in the interval $[-1, 0)$ (see in Acknowledgements). Thus, the complex eigenvalues λ_j and the coefficients c_j in (25) appear in r complex conjugate pairs and λ_j, c_j are denoted as in (i). Then, using the complex conjugate properties, (30) follows

$$
f_n = c(\rho(F_{k,m}))^n + c_{k+m-1}(\lambda_{k+m-1})^n
$$

$$
+ 2 \sum_{i=1}^{r} |c_i| |\lambda_i|^n \cos(\theta_i + n\omega_i),
$$

(34)

where $r = (k + m - 2)/2$.

Example 12. Consider the Padovan sequence of the Example 8. Notice that $k = 2$, $m = 1$ and $k + m$ is odd. The eigenvalues of $F_{2,1}$ are given in Example 8, $\rho(F_{2,1}) = 1.32472$, $\lambda_1 = -0.66235 + 0.56227i = 0.86883(\cos(2.43773) + i \sin(2.43773))$, and $\lambda_2 \equiv \bar{\lambda}_1$. Since $|\lambda_1| = 0.86883$, it is evident that the inequality (22) is verified. The partial-fraction decomposition as in (25) yields $c = 0.54511$, $c_1 = -0.27255 + 0.07397i = 0.28241(\cos(2.87657) + i \sin(2.87657))$, and $c_2 = -0.27255 - 0.07397i \equiv \bar{c}_1$.

Thus, for $n \geq 1$ the nth number of the Padovan sequence is computed by (33) and given by

$$
f_n = c(\rho(F_{2,1}))^n + 2 |c_1| |\lambda_1|^n \cos(\theta_1 + n\omega_1)
$$

$$
= 0.54511(1.32472)^n \qquad (35)
$$

$$
+ 0.56483(0.86883)^n \cos(2.87657 + 2.43773n).
$$

Now, the limited properties of the Padovan sequence are derived by (28) and (29):

$$
\lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \lim_{n \to \infty} \sqrt[n]{f_n} = 1.32472. \qquad (36)
$$

Example 13. Consider the 2-step sum and 2-step gap Fibonacci sequence. Notice that $k + m$ is even. The eigenvalues of $F_{2,2}$ are $\rho(F_{2,2}) = 1.22074$, $\lambda_1 = -0.24812 + 1.0339i = 1.06333(\cos(1.80631) + i \sin(1.80631))$, $\lambda_2 \equiv \bar{\lambda}_1$,

and $\lambda_3 = -0.72449$. The partial-fraction decomposition as in (25) yields $c = 0.59122$, $c_1 = -0.13687 + 0.00951i = 0.13720(\cos(3.07219) + i\sin(3.07219))$, $c_2 = \bar{c}_1$, and $c_3 = -0.31747$. Thus, for $n \geq 1$ the nth number of the sequence is computed by (34) and given by

$$f_n = 0.59122(1.22074)^n - 0.31747(-0.72449)^n$$
$$+ 0.27441(1.06336)^n \cos(3.07219 + 1.80631n). \tag{37}$$

Now, the limited properties of the sequence are derived by (28) and (29):

$$\lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \lim_{n \to \infty} \sqrt[n]{f_n} = 1.22074. \tag{38}$$

4. Conclusions

The k-step sum and m-step gap Fibonacci sequence was introduced. A recurrence formula was presented generating the nth term of the sequence as the sum of k successive previous terms starting the sum at the mth previous term. It was noticed that known sequences, like Fibonacci, tribonacci, tetranacci, and Padovan sequences, are derived for specific values of k, m. A closed formula of the nth term of the sequence was given. The limiting properties concerning the ratio of two successive terms as well as the nth root of the nth term of the sequence were presented. It was shown that these two limits are equal to each other and are related to the spectral radius of the associated $\{0, 1\}$-matrix. These limits can be regarded as the k-step sum and m-step gap Fibonacci sequence constants, like the tribonacci constant and the tetranacci constant.
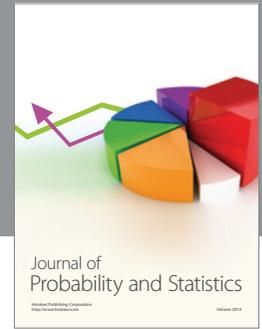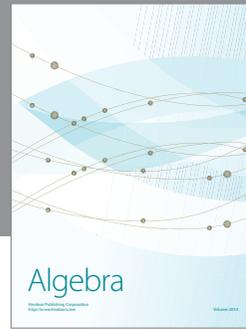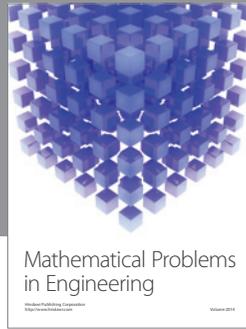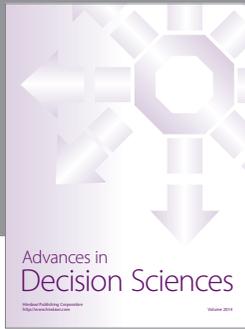
Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

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