Research Article
On Almost φ-Lagrange Spaces

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We initiate a study on the geometry of an almost φ-Lagrange space (APL-space in short). We obtain the expressions for the symmetric metric tensor, its inverse, semispray coefficients, solution curves of Euler-Lagrange equations, nonlinear connection, differential equation of autoparallel curves, coefficients of canonical metrical d-connection, and h- and v-deflection tensors in an APL-space. Corresponding expressions in a φ-Lagrange space and an almost Finsler Lagrange space (AFL-space in short) have also been deduced.

1. Introduction

In the last three decades, various meaningful generalizations of Finsler spaces have been considered. These generalizations have been found much applicable to mechanics, theoretical physics, variational calculus, optimal control, complex analysis, biology, ecology, and so forth. The geometry of Lagrange spaces is one such generalization of the geometry of Finsler spaces which was introduced and studied by Miron [1, 2]. He [1, 2] introduced the most natural generalization of Lagrange spaces named as generalized Lagrange space. Since the introduction of Lagrange spaces and generalized Lagrange spaces, many geometers and physicists have been engaged in the exploration, development, and application of these concepts [3–13]. Antonelli and Hrimiuc [14, 15] introduced a special type of regular Lagrangian called φ-Lagrangian. Applications of such Lagrangian have been discussed by Antonelli et al. in the monograph [16]. In the present paper, we generalize the notion of φ-Lagrangian and introduce the concept of almost φ-Lagrange spaces. We hope that the results obtained in the paper will be interesting for the researchers working on the application of Lagrange spaces in various fields of science.

Let $F^n = (M, F(x, y))$ be an n-dimensional Finsler space, and let $\varphi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function. The composition $L := \varphi(F^2)$ defines a differentiable Lagrangian. This was

regarded by Antonelli and Hrimiuc [14, 15] as φ-Lagrangian associated to the Finsler space F^n. They [14] proved that if the function φ has the following properties:

(a) $\varphi'(t) \neq 0$,

(b) $\varphi'(t) + \varphi''(t) \neq 0$,   for every $t \in \text{Im}\left(F^2\right)$, $\qquad\qquad$ (1.1)

then L is a regular Lagrangian and thus $L^n = (M, L(x, y))$ is a Lagrange space, called a φ-Lagrange space.

In this paper, we consider a more general Lagrangian as follows:

$$L(x, y) = \varphi\left(F^2\right) + A_i(x)y^i + U(x), \qquad\qquad (1.2)$$

where φ is the same as discussed earlier, $A_i(x)$ is a covector, and $U(x)$ is a smooth function.

In Section 2, we show that if the function φ has the properties (1.1), then $L(x, y)$ is a regular Lagrangian and thus the pair $L^n = (M, L(x, y))$ is a Lagrange space. We call this space as an almost φ-Lagrange space (shortly APL-space).

An APL-space reduces to a φ-Lagrange space if and only if $A_i(x) = 0$ and $U(x) = 0$.
If $\varphi(t) = t$, for all $t \in \text{Im}(F^2)$, then the Lagrangian in (1.2) takes the form

$$L(x, y) = F^2 + A_i(x)y^i + U(x). \qquad\qquad (1.3)$$

This defines a regular Lagrangian, and the pair $L^n = (M, L(x, y))$ is called an almost Finsler Lagrange space (shortly AFL-space). Such Lagrange space was introduced by Miron and Anastasiei (vide Chapter IX of [17]).

We take

$$g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2, \quad a_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L, \quad \dot{\partial}_i \equiv \frac{\partial}{\partial y^i}. \qquad\qquad (1.4)$$

Henceforth, we will indicate all the geometrical objects related to F^n by a small circle "$\circ$" put over them.

In a Finsler space, the geodesics, parameterized by arc length (the extremals of the length integral), coincide with the extremals of action integral or with the autoparallel curves of the Cartan nonlinear connection [16]:

$$\overset{\circ}{N}{}_j^i = \overset{\circ}{\gamma}{}_{j0}^i - \overset{\circ}{C}{}_{jk}^i \overset{\circ}{\gamma}{}_{00}^k, \qquad\qquad (1.5)$$

where

$$\overset{\circ}{\gamma}{}_{jk}^i = \frac{1}{2}g^{ih}(\partial_j g_{hk} + \partial_k g_{jh} - \partial_h g_{jk}); \quad \partial_j \equiv \frac{\partial}{\partial x^j},$$

$$\overset{\circ}{C}{}_{jk}^i = \frac{1}{2}g^{ih}\dot{\partial}_h g_{jk}, \qquad \overset{\circ}{\gamma}{}_{j0}^i = \overset{\circ}{\gamma}{}_{jk}^i y^k, \qquad \overset{\circ}{\gamma}{}_{00}^i = \overset{\circ}{\gamma}{}_{jk}^i y^j y^k. \qquad (1.6)$$

These geodesics are the integral curves of the spray [16] (i.e., (2) p-homogeneous):

$$\overset{\circ}{G}{}^i = \frac{1}{4}\, g^{ij}\left(y^k\dot{\partial}_j\partial_k F^2 - \partial_j F^2\right), \tag{1.7}$$

that is, solutions of the differential equations

$$\frac{d^2 x^i}{ds^2} + 2\overset{\circ}{G}{}^i\left(x(s), \frac{dx}{ds}\right) = 0. \tag{1.8}$$

We have the following equalities:

$$(a)\ \overset{\circ}{G}{}^i = \frac{1}{2}\overset{\circ}{\gamma}{}^i_{00}, \tag{1.9}$$
$$(b)\ \overset{\circ}{N}{}^i_j = \dot{\partial}_j\overset{\circ}{G}{}^i.$$

In a general Lagrange space $L^n = (M, L(x, y))$, the geodesics are the extremals of the action integral and coincide with the integral curves of the semispray [17, 18] (i.e., may not be a spray):

$$G^i = \frac{1}{4}\, a^{ij}\left(y^k\dot{\partial}_j\partial_k L - \partial_j L\right). \tag{1.10}$$

As in a Finsler space, a remarkable nonlinear connection can be considered in a Lagrange space:

$$N^i_j = \dot{\partial}_j G^i. \tag{1.11}$$

Such nonlinear connection is a canonical nonlinear connection [17, 18] as it depends only on the fundamental function $L(x, \ y)$ of the Lagrange space.

In general, the autoparallel curves of (N^i_j) are different from the geodesics of $L^n = (M, L(x, y))$ (cf. [17]).

Given a nonlinear connection (N^i_j) on a Lagrange space $L^n = (M, L(x, y))$, there is a unique h- and v-metrical d-connection (cf. [17, 19]) $C\Gamma(N) = (N^i_j, L^i_{jk}, C^i_{jk})$ with torsions $T^i_{jk} = 0$ and $S^i_{jk} = 0$, called the canonical metrical d-connection. This connection is linear and its coefficients are given by

$$L^i_{jk} = \frac{1}{2}a^{ih}\left(\delta_j a_{hk} + \delta_k a_{jh} - \delta_h a_{jk}\right), \tag{1.12}$$

$$C^i_{jk} = \frac{1}{2}a^{ih}\left(\dot{\partial}_j a_{hk} + \dot{\partial}_k a_{jh} - \dot{\partial}_h a_{jk}\right), \tag{1.13}$$

where $\delta_i = \partial_i - N^r_i\dot{\partial}_r$ is the Lagrange differentiation operator.

If $\overset{\circ}{C\Gamma}(\overset{\circ}{N}) = (\overset{\circ}{N}{}_j^i, \overset{\circ}{L}{}_{jk}^i, \overset{\circ}{C}{}_{jk}^i)$ is the Cartan connection of the Finsler space $F^n = (M, F(x, y))$, then its coefficients are given by

$$\overset{\circ}{L}{}_{jk}^i = \frac{1}{2}g^{ih}\left(\overset{\circ}{\delta}_j g_{hk} + \overset{\circ}{\delta}_k g_{jh} - \overset{\circ}{\delta}_h g_{jk}\right), \tag{1.14}$$

$$\overset{\circ}{C}{}_{jk}^i = \frac{1}{2}g^{ih}(\dot{\partial}_j g_{hk} + \dot{\partial}_k g_{jh} - \dot{\partial}_h g_{jk}), \tag{1.15}$$

where $\overset{\circ}{\delta}_i = \partial_i - \overset{\circ}{N}{}_i^r \dot{\partial}_r$.

The h- and v-deflection tensor fields D_j^i and d_j^i, respectively, of a Lagrange space L^n are defined by (cf. [19])

$$D_j^i := y_{|j}^i = y^s L_{sj}^i - N_j^i, \tag{1.16}$$

$$d_j^i := y^i|_j = \delta_j^i + y^s C_{sj}^i, \tag{1.17}$$

where $_|$ and $|$, respectively, denote the h- and v-covariant derivatives with respect to $C\Gamma$.

If $\overset{\circ}{D}{}_j^i$ is the h-deflection tensor field and $\overset{\circ}{d}{}_j^i$ is the v-deflection tensor field of the Finsler space F^n, then

$$\overset{\circ}{D}{}_j^i := y_{\underset{|j}{\circ}}^i = y^s \overset{\circ}{L}{}_{sj}^i - \overset{\circ}{N}{}_j^i = 0, \tag{1.18}$$

$$\overset{\circ}{d}{}_j^i := y^i \overset{\circ}{|}_j = \delta_j^i, \tag{1.19}$$

where $\underset{|}{\circ}$ and $\overset{\circ}{|}$, respectively, denote the h- and v-covariant derivatives with respect to $\overset{\circ}{C\Gamma}$.

For basic terminology and notations related to a Finsler space and a Lagrange space, we refer to the books [17, 20].

2. Almost φ-Lagrange Spaces

As discussed earlier, we consider the Lagrangian given by (1.2) in which the function φ satisfies (1.1). We prove that it is a regular Lagrangian and the pair $L^n = (M, L(x, y))$ is a Lagrange space which we term as an almost φ-Lagrange space (APL-space in short).

Theorem 2.1. *If the function φ satisfies the conditions (1.1), then $L(x, y)$, given by (1.2), is a regular Lagrangian and $L^n = (M, L(x, y))$ is a Lagrange space.*

Proof. Differentiating (1.2) partially with respect to y^i, we get

$$\dot{\partial}_i L = \varphi'\left(F^2\right)\dot{\partial}_i F^2 + A_i(x). \tag{2.1}$$

Again differentiating (2.1) partially with respect to y^j, we obtain

$$\dot{\partial}_j \dot{\partial}_i L = \varphi''\left(F^2\right)\dot{\partial}_i F^2 \dot{\partial}_j F^2 + \varphi'\left(F^2\right)\dot{\partial}_i \dot{\partial}_j F^2, \tag{2.2}$$

which, in view of (1.4), provides

$$a_{ij} = 2F^2 \varphi''\left(F^2\right)\dot{\partial}_i F \dot{\partial}_j F + \varphi'\left(F^2\right)g_{ij}. \tag{2.3}$$

Now

$$F\dot{\partial}_i F = \frac{1}{2}\dot{\partial}_i F^2 = \frac{1}{2}\dot{\partial}_i\left(g_{jk}y^j y^k\right) = g_{ik}y^k := \overset{\circ}{y}_i. \tag{2.4}$$

In view of (2.4), (2.3) takes the form

$$a_{ij} = \varphi' \cdot \left(g_{ij} + \frac{2\varphi''}{\varphi'}\overset{\circ}{y}_i\overset{\circ}{y}_j\right). \tag{2.5}$$

Under the hypothesis, the matrix (a_{ij}) is invertible and its inverse is (see Lemma 6.2.2.1, page 891 in [20])

$$a^{ij} = \frac{1}{\varphi'}\left(g^{ij} - \frac{2\varphi''}{\varphi' + 2F^2\varphi''}y^i y^j\right). \tag{2.6}$$

This proves the theorem. $\square$

Remarks 1. (i) If $A_i(x) = 0$ and $U(x) = 0$ in (1.2), then expression (2.5) remains unchanged. Hence, the symmetric metric tensor of a φ-Lagrange space is the same as that of an APL-space.

(ii) If $\varphi(F^2) = F^2$, then $\varphi' = 1$ and $\varphi'' = 0$. Hence, the symmetric metric tensor of an AFL-space coincides with that of the associated Finsler space.

3. Semispray, Integral Curves of Euler-Lagrange Equations

In this section, we obtain the coefficients of the canonical semispray of the APL-space $L^n = (M, L(x, y))$ and deduce corresponding expressions for a φ-Lagrange space and an AFL-space. Next, we obtain the differential equations whose solution curves are the integral curves of Euler-Lagrange equations in an APL-space. We deduce corresponding differential equations for a φ-Lagrange space and an AFL-space.

If we differentiate (1.2) partially with respect to x^k, we have

$$\partial_k L = \varphi'\left(F^2\right)\partial_k F^2 + y^i \partial_k A_i(x) + \partial_k U(x). \tag{3.1}$$

Differentiating (3.1) partially with respect to y^j, we obtain

$$\dot{\partial}_j \partial_k L = 2\varphi''\left(F^2\right) F \dot{\partial}_j F \partial_k F^2 + \varphi'\left(F^2\right) \dot{\partial}_j \partial_k F^2 + \partial_k A_j(x), \tag{3.2}$$

which, in view of (2.4), takes the form

$$\dot{\partial}_j \partial_k L = 2\varphi''\left(F^2\right) \mathring{y}_j \partial_k F^2 + \varphi'\left(F^2\right) \dot{\partial}_j \partial_k F^2 + \partial_k A_j(x). \tag{3.3}$$

Using (3.1) and (3.3) in (1.10), we have

$$G^i = \frac{1}{4} a^{ij} \left\{ 2\varphi''\left(F^2\right) \mathring{y}_j y^k \partial_k F^2 + \varphi'\left(F^2\right) \left(y^k \dot{\partial}_j \partial_k F^2 - \partial_j F^2\right) - 2y^k F_{jk} - \partial_j U \right\}, \tag{3.4}$$

where

$$F_{jk}(x) = \frac{1}{2} \left(\partial_j A_k - \partial_k A_j\right) \tag{3.5}$$

is electromagnetic tensor field of the potentials $A_i(x)$.

Applying (2.6) in (3.4) and using $\mathring{y}_i y^i = F^2$, $g^{ij}\mathring{y}_j = y^i$, and $y^j \dot{\partial}_j \partial_k F^2 = 2\partial_k F^2$ (by Euler's theorem on homogeneous functions), we obtain

$$G^i = \frac{1}{2} \frac{\varphi''}{\varphi'} \left(1 - \frac{2\varphi'' F^2}{\varphi' + 2F^2 \varphi''}\right) y^i y^k \partial_k F^2 + \frac{1}{4} \left\{ g^{ij}\left(y^k \dot{\partial}_j \partial_k F^2 - \partial_j F^2\right) - \frac{2\varphi'' F^2}{\varphi' + 2F^2 \varphi''} y^i y^k \partial_k F^2 \right\}$$

$$- \frac{1}{4} a^{ij} \left(2F_{jk} y^k + \partial_j U\right). \tag{3.6}$$

Using (1.7) in (3.6) and simplifying, we get

$$G^i = \mathring{G}^i - \frac{1}{4} a^{ij} \left(2F_{jk} y^k + \partial_j U\right). \tag{3.7}$$

Thus, we have the following.

Theorem 3.1. *The canonical semispray of an APL-space has the local coefficients given by*

$$G^i = \mathring{G}^i - \frac{1}{4} a^{ij} \left(2F_{jk} y^k + \partial_j U\right), \tag{3.8}$$

where $\mathring{G}^i$ are the local coefficients of the spray of F^n.

For a φ-Lagrange space, $A_i(x) = 0$ and $U(x) = 0$. Hence, from (3.5), we have $F_{jk} = 0$. Therefore, (3.7) reduces to

$$G^i = \overset{\circ}{G}{}^i.$$ (3.9)

Thus, we may state the following.

Corollary 3.2 (see [14]). *The canonical semispray of a φ-Lagrange space becomes a spray and coincides with that of the associated Finsler space.*

For an AFL-space, $a^{ij} = g^{ij}$ (see Remark (ii)). Hence, (3.7) takes the form

$$G^i = \overset{\circ}{G}{}^i - \frac{1}{4}g^{ij}\left(2F_{jk}y^k + \partial_j U\right).$$ (3.10)

Thus, we have the following.

Corollary 3.3 (see [17, 20]). *The canonical semispray of an AFL-space has the local coefficients given by* (3.10).

In a Lagrange space, the integral curves of the Euler-Lagrange equations:

$$E_i(L) := \partial_i L - \frac{d}{dt}\left(\dot{\partial}_i L\right) = 0$$ (3.11)

are the solution curves of the equations [20]

$$\frac{d^2 x^i}{dt^2} + 2G^i(x, y) = 0.$$ (3.12)

Using (3.7) in (3.12), we obtain

$$\frac{d^2 x^i}{dt^2} + 2\overset{\circ}{G}{}^i = \frac{1}{2}\left(2F_k^i y^k + a^{ij}\partial_j U\right),$$ (3.13)

where $F_k^i = a^{ij}F_{jk}$.
Using (1.9) (a) in (3.13), we have

$$\frac{d^2 x^i}{dt^2} + \overset{\circ}{\gamma}{}^i_{00} = \frac{1}{2}\left(2F_k^i y^k + a^{ij}\partial_j U\right).$$ (3.14)

Thus, we have the following.

Theorem 3.4. *In an APL-space $L^n = (M, L(x, y))$, the integral curves of the Euler-Lagrange equations $E_i(L) = 0$ are the solution curves of* (3.14).

For a φ-Lagrange space, equations (3.14) take the following simple form:

$$\frac{d^2 x^i}{dt^2} + \overset{\circ}{\gamma}{}^i_{00} = 0.$$

(3.15)

This enables us to state the following.

Corollary 3.5 (see [14]). *In a φ-Lagrange space, the integral curves of the Euler-Lagrange equations are the solution curves of* (3.15).

For an AFL-space, $a^{ij} = g^{ij}$. Therefore, equations (3.14) become

$$\frac{d^2 x^i}{dt^2} + \overset{\circ}{\gamma}{}^i_{00} = F^i_k y^k + \frac{1}{2} g^{ij} \partial_j U,$$

(3.16)

where $F^i_k = g^{ij} F_{jk}$.

Thus, we have the following.

Corollary 3.6 (see [17, 20]). *In an AFL-space, the integral curves of the Euler-Lagrange equations $E_i(L) = 0$ are the solution curves of* (3.16).

4. Nonlinear Connection, Autoparallel Curves

In this section, we find the coefficients of the nonlinear connection of an APL-space and obtain the differential equations of the autoparallel curves of the nonlinear connection. Corresponding results have been deduced for a φ-Lagrange space and an AFL-space.

Partial differentiation of (2.5) with respect to y^k yields

$$\dot{\partial}_k a_{ij} =: 2C_{ijk} = 2\varphi' \overset{\circ}{C}_{ijk} + 2\varphi'' \left(g_{ij} \overset{\circ}{y}_k + g_{jk} \overset{\circ}{y}_i + g_{ki} \overset{\circ}{y}_j \right) + 4\varphi''' \overset{\circ}{y}_i \overset{\circ}{y}_j \overset{\circ}{y}_k.$$

(4.1)

Using (3.7) in (1.11) and taking (1.9) (b), (2.6), (4.1), $\overset{\circ}{C}_{pqj} y^j = 0$, $y^i \overset{\circ}{y}_i = F^2$, and $g^{ij} \overset{\circ}{y}_j = y^i$ into account, we obtain

$$N^i_j = \overset{\circ}{N}{}^i_j - \frac{1}{2} F^i_j + \left[\frac{1}{2\varphi'} \overset{\circ}{C}{}^i_{qj} g^{qr} + \frac{1}{2} \frac{\varphi''}{\varphi'^2} g^{ir} \overset{\circ}{y}_j + \frac{\varphi''}{2\varphi'(\varphi' + 2F^2\varphi'')} \left(\delta^r_j y^i + \delta^i_j y^r \right) \right.$$
$$\left. + \frac{\varphi'^2 \varphi''' - 2\varphi''^3 F^2 - 4\varphi'\varphi''^2}{2\varphi'^2(\varphi' + 2F^2\varphi'')^2} y^i \overset{\circ}{y}_j y^r \right] \left(2F_{rk} y^k + \partial_r U \right).$$

(4.2)

If we take

$$S^{ir}_j = \frac{1}{2\varphi'} \overset{\circ}{C}{}^i_{qj} g^{qr} + \frac{1}{2} \frac{\varphi''}{\varphi'^2} g^{ir} \overset{\circ}{y}_j + \frac{\varphi'' \left(\delta^r_j y^i + \delta^i_j y^r \right)}{2\varphi'(\varphi' + 2F^2\varphi'')} + \frac{\varphi'^2 \varphi''' - 2\varphi''^3 F^2 - 4\varphi'\varphi''^2}{2\varphi'^2(\varphi' + 2F^2\varphi'')^2} y^i \overset{\circ}{y}_j y^r,$$

(4.3)

the last expression becomes

$$N_j^i = \overset{\circ}{N}{}_j^i - \frac{1}{2}F_j^i + S_j^{ir}\left(2F_{rk}y^k + \partial_r U\right), \tag{4.4}$$

that is,

$$N_j^i = \overset{\circ}{N}{}_j^i - V_j^i, \tag{4.5}$$

where

$$V_j^i = \frac{1}{2}F_j^i - S_j^{ir}\left(2F_{rk}y^k + \partial_r U\right). \tag{4.6}$$

Thus, we have the following.

Theorem 4.1. *The canonical nonlinear connection of an APL-space L^n has the local coefficients given by* (4.5).

For a φ-Lagrange space, we have $F_{rk} = 0, F_j^i = 0$ and $U = 0$ and hence $V_j^i = 0$. Therefore, (4.5) reduces to

$$N_j^i = \overset{\circ}{N}{}_j^i. \tag{4.7}$$

Thus, we have the following.

Corollary 4.2 (see [14]). *The canonical nonlinear connection of a φ-Lagrange space coincides with the nonlinear connection of the associated Finsler space.*

For an AFL-space, (4.3) reduces to

$$S_j^{ir} = \frac{1}{2}\overset{\circ}{C}{}_{qj}^i g^{qr} \tag{4.8}$$

and hence (4.6) gives

$$V_j^i = \frac{1}{2}F_j^i - \overset{\circ}{C}{}_{qj}^i F_k^q y^k - \frac{1}{2}\overset{\circ}{C}{}_{qj}^i g^{qr}\partial_r U := B_j^i. \tag{4.9}$$

Therefore, (4.5) takes the form

$$N_j^i = \overset{\circ}{N}{}_j^i - B_j^i. \tag{4.10}$$

Thus, we have the following.

Corollary 4.3 (see [17, 20]). *The canonical nonlinear connection of an AFL-space L^n has the local coefficients given by* (4.10).

Transvecting (4.5) by y^i and using $\overset{\circ}{N}{}^i_j y^j = \overset{\circ}{\gamma}{}^i_{00}$, we obtain

$$N^i_j y^j = \overset{\circ}{\gamma}{}^i_{00} - V^i_0, \tag{4.11}$$

where $V^i_0 = V^i_j y^j$.

The autoparallel curves of the canonical nonlinear connection $N = (N^i_j)$ of a Lagrange space are given by the following system of differential equations (vide [20]):

$$\frac{d^2 x^i}{dt^2} + N^i_j(x, y) y^j = 0. \tag{4.12}$$

Equations (4.12), in view of (4.11), take the form

$$\frac{d^2 x^i}{dt^2} + \overset{\circ}{\gamma}{}^i_{00} = V^i_0. \tag{4.13}$$

Thus, we have the following.

Theorem 4.4. *The autoparallel curves of the canonical nonlinear connection $N = (N^i_j)$ of an APL-space $L^n = (M, L(x, y))$ are given by the system of differential equations* (4.13).

For a φ-Lagrange space, $V^i_j = 0$ and hence $V^i_0 = 0$. Therefore, (4.13) reduces to

$$\frac{d^2 x^i}{dt^2} + \overset{\circ}{\gamma}{}^i_{00} = 0. \tag{4.14}$$

Thus, we have the following.

Corollary 4.5 (see [14]). *The autoparallel curves of the canonical nonlinear connection of a φ-Lagrange space $L^n = (M, L(x, y))$ are given by the system of differential equations* (4.14).

For an AFL-space,

$$V^i_j = B^i_j =: \frac{1}{2} F^i_j - \overset{\circ}{C}{}^i_{qj} F^q_k y^k - \frac{1}{2} \overset{\circ}{C}{}^i_{qj} g^{qr} \partial_r U \tag{4.15}$$

and hence, by virtue of $\overset{\circ}{C}{}^i_{qj} y^j = 0$, we have $V^i_0 = (1/2) F^i_j y^j$. Therefore, equations (4.12) take the form

$$\frac{d^2 x^i}{dt^2} + \overset{\circ}{\gamma}{}^i_{00} = \frac{1}{2} F^i_j y^j. \tag{4.16}$$

Thus, we deduce the following.

Corollary 4.6 (see [17, 20]). *The autoparallel curves of the nonlinear connection $N = (N_j^i)$ of an AFL-space $L^n = (M, L(x, y))$ are given by the system of differential equations* (4.16).

If we compare (3.14), (3.15), and (3.16), respectively, with (4.13), (4.14), and (4.16), we observe that, in an APL-space as well as in an AFL-space, solution curves of Euler-Lagrange equations do not coincide with the autoparallel curves of the canonical nonlinear connection whereas in a φ-Lagrange space they do. Therefore, in a φ-Lagrange space, geodesics are autoparallel curves whereas in an APL-space and in an AFL-space they are not so.

5. Canonical Metrical d-Connection

Let $C\Gamma(N) = (N_j^i, L_{jk}^i, C_{jk}^i)$ be the canonical metrical d-connection of the APL-space $L^n = (M, L(x, y))$, and let $\overset{\circ}{C\Gamma}(N) = (\overset{\circ}{N}_j, \overset{\circ}{L}_{jk}^i, \overset{\circ}{C}_{jk}^i)$ be the Cartan connection of the associated Finsler space $F^n = (M, F(x, y))$. In this section, we obtain the expressions for the coefficients of $C\Gamma(N)$ and we investigate some properties of $C\Gamma(N)$. We deduce corresponding results for a φ-Lagrange space and an AFL-space.

Using (4.1) in (1.13) and taking (1.15) into account, we find

$$C_{jk}^i = \overset{\circ}{C}_{jk}^i + \frac{\varphi''}{\varphi'}\left(\delta_j^i \overset{\circ}{y}_k + \delta_k^i \overset{\circ}{y}_j\right) + \frac{\varphi''}{\varphi' + 2F^2\varphi''} g_{jk} y^i + \frac{2(\varphi'''\varphi' - 2\varphi''^2)}{\varphi'(\varphi' + 2F^2\varphi'')} y^i \overset{\circ}{y}_j \overset{\circ}{y}_k. \tag{5.1}$$

For any C^∞-class function $\psi : \mathbb{R}^+ \to \mathbb{R}$, taking $f(x, y) = \psi(F^2(x, y))$, we have

$$\overset{\circ}{\delta}_k f = f' F_{\overset{\circ}{|k}}^2 \tag{5.2}$$

which, in view of $F_{\overset{\circ}{|k}}^2 = 0$ (see proposition 9.4, page 1037 of [20]), gives

$$\overset{\circ}{\delta}_k f = 0. \tag{5.3}$$

Since $0 = y_{\overset{\circ}{i|k}} := \overset{\circ}{\delta}_k \overset{\circ}{y}_i - \overset{\circ}{L}_{ik}^r \overset{\circ}{y}_r$ (see proposition 9.4, page 1037 of [20]), we have

$$\overset{\circ}{\delta}_k \overset{\circ}{y}_i = \overset{\circ}{L}_{ik}^r \overset{\circ}{y}_r. \tag{5.4}$$

If we operate $\overset{\circ}{\delta}_k$ on (2.5) and utilize (5.3) and (5.4), it follows that

$$\overset{\circ}{\delta}_k a_{ij} = \varphi' \overset{\circ}{\delta}_k g_{ij} + 2\varphi'' \overset{\circ}{y}_r \left(\overset{\circ}{L}_{ik}^r \overset{\circ}{y}_j + \overset{\circ}{L}_{jk}^r \overset{\circ}{y}_i\right). \tag{5.5}$$

In view of $\delta_i = \partial_i - N_i^r \dot{\partial}_r$, (4.5), and $\overset{\circ}{\delta}_i = \partial_i - \overset{\circ}{N}_i{}^r \dot{\partial}_r$, we get

$$\delta_k a_{ij} = \overset{\circ}{\delta}_k a_{ij} + V_k^r \dot{\partial}_r a_{ij}, \tag{5.6}$$

which, on account of (4.1) and (5.5), becomes

$$\delta_k a_{ij} = \varphi' \overset{\circ}{\delta}_k g_{ij} + 2\varphi'' \overset{\circ}{y}_r \left(\overset{\circ}{L}_{ik}{}^r \overset{\circ}{y}_j + \overset{\circ}{L}_{jk}{}^r \overset{\circ}{y}_i \right) + 2V_k^r C_{ijr}. \tag{5.7}$$

Using (5.7) in (1.12) and taking (1.14) and $a^{il} C_{jkl} = C_{jk}^i$ into account, we obtain

$$L_{jk}^i = \overset{\circ}{L}_{jk}{}^i + V_k^r C_{jr}^i + V_j^r C_{kr}^i + V_p^r a^{ip} C_{rkj}. \tag{5.8}$$

Equations (5.1) and (5.8) enable us to state the following.

Theorem 5.1. *The coefficients of the canonical metrical d-connection $C\Gamma(N)$ of an APL-space L^n are given by (5.1) and (5.8).*

For a φ-Lagrange space, $V_j^i = 0$. Hence, (5.1) remains unchanged whereas (5.8) reduces to

$$L_{jk}^i = \overset{\circ}{L}_{jk}{}^i. \tag{5.9}$$

Thus, we have the following.

Corollary 5.2 (see [14]). *The coefficients of the canonical metrical d-connection $C\Gamma(N)$ of a φ-Lagrange space L^n are given by (5.1) and (5.9).*

For an AFL-space, $\varphi(F^2) = F^2$, $\varphi'(F^2) = 1$, $\varphi''(F^2) = 0$, and $a^{ij} = g^{ij}$. Therefore, we have $C_{ijk} = \overset{\circ}{C}_{ijk}$ and $V_j^r = B_j^r$.
In view of these facts, (5.1) reduces to

$$C_{jk}^i = \overset{\circ}{C}_{jk}{}^i, \tag{5.10}$$

whereas (5.8) gives the following:

$$L_{jk}^i = \overset{\circ}{L}_{jk}{}^i + B_k^r \overset{\circ}{C}_{jr}{}^i + B_j^r \overset{\circ}{C}_{kr}{}^i + B_p^r g^{ip} \overset{\circ}{C}_{rkj}, \tag{5.11}$$

where B_k^r is given by (4.9). Thus, we have the following.

Corollary 5.3 (see [17, 20]). *The coefficients of the canonical metrical d-connection $C\Gamma(N)$ of an AFL-space L^n are given by (5.10) and (5.11).*

Now, we investigate some properties of the canonical metrical d-connection $C\Gamma(N)$ of an APL-space and deduce the corresponding properties for a φ-Lagrange space and an AFL-space.

Theorem 5.4. *The canonical metrical d-connection $C\Gamma(N)$ of an APL-space has the following properties:*

$$(1)\ D_k^i := y_{|k}^i = V_k^i + V_k^p C_{pr}^i y^r + V_r^p C_{kp}^i y^r + V_s^p a^{is} C_{pkr} y^r, \tag{5.12}$$

$$y_{i|k} = V_k^s \left(a_{si} + C_{sij} y^j \right) - V_i^s C_{ksp} y^p - V_p^s C_{ski} y^p, \tag{5.13}$$

where $y_i =: a_{ij} y^j$,

$$(2)\ d_k^i := y^i|_k = \frac{\varphi' + \varphi'' F^2}{\varphi'} \delta_k^i + B\mathring{y}_k y^i, \qquad y_i|_k = a_{ik} + C_{ikj} y^j, \tag{5.14}$$

where $B = 2\{\varphi'\varphi'' + F^2(\varphi'''\varphi' - \varphi''^2)\}/\varphi'(\varphi' + 2F^2\varphi'')$,

$$(3)\ L_{|k} = X_k + \frac{2\varphi'}{(\varphi' + 2F^2\varphi'')} V_k^r y_r, \qquad L|_k = \frac{2\varphi'}{(\varphi' + 2F^2\varphi'')} y_k + A_k, \tag{5.15}$$

where $X_k = y^r \partial_k A_r - N_k^p A_p + \partial_k U$.

Proof. (1) Using (5.8) and (4.5) in (1.16), we have

$$D_k^i = y^r \left(\mathring{L}_{rk}^i + V_k^p C_{rp}^i + V_r^p C_{kp}^i + V_s^p a^{is} C_{pkr} \right) - \mathring{N}_k^i + V_k^i, \tag{5.16}$$

which, in view of (1.18), reduces to

$$D_k^i = V_k^i + y^r \left(V_k^p C_{rp}^i + V_r^p C_{kp}^i + V_s^p a^{is} C_{pkr} \right). \tag{5.17}$$

Next, if we use (2.5) in $y_i = a_{ij} y^j$, then it follows that

$$y_i = \left(\varphi' + 2F^2\varphi'' \right) \mathring{y}_i. \tag{5.18}$$

Now, applying successively $\delta_i = \partial_i - N_i^r \mathring{\partial}_r$, (4.5), and $\mathring{\delta}_i = \partial_i - \mathring{N}_i^r \mathring{\partial}_r$ in $y_{i|k} = \delta_k y_i - y_r L_{ik}^r$ and keeping (5.8) and (5.18) in view, we have

$$y_{i|k} = \mathring{\delta}_k \left\{ \left(\varphi' + 2F^2\varphi'' \right) \mathring{y}_i \right\} - \left(\varphi' + 2F^2\varphi'' \right) \mathring{y}_r \mathring{L}_{ik}^r + V_k^r \mathring{\partial}_r y_i - y_r \left(V_k^s C_{is}^r + V_i^s C_{ks}^r + V_p^s a^{rp} C_{ski} \right). \tag{5.19}$$

Differentiating $y_i = a_{ij}y^j$ partially with respect to y^r, we have

$$\dot{\partial}_r y_i = a_{ir} + 2C_{irj}y^j. \tag{5.20}$$

Also,

$$y_l C_{jk}^l = a_{lh} y^h C_{jk}^l = y^h C_{hjk}. \tag{5.21}$$

In view of (5.3), we have

$$\overset{\circ}{\delta}_k \left(\varphi' + 2F^2 \varphi'' \right) = 0. \tag{5.22}$$

Using (5.20), (5.21), and (5.22) in (5.19), we obtain

$$y_{i|k} = \left(\varphi' + 2F^2 \varphi'' \right) \left(\overset{\circ}{\delta}_k \overset{\circ}{y}_i - \overset{\circ}{y}_r \overset{\circ}{L}_{ik}^r \right) + V_k^s \left(a_{si} + C_{sij}y^j \right) - \left(V_i^s C_{ksp} + V_p^s C_{ksi} \right) y^p, \tag{5.23}$$

which, in view of (5.4), gives the desired result.

(2) Using (5.1) in (1.17), we get

$$d_k^i = \frac{\varphi' + \varphi'' F^2}{\varphi'} \delta_k^i + B\overset{\circ}{y}_k y^i, \tag{5.24}$$

where $B = 2\{\varphi'\varphi'' + F^2(\varphi'''\varphi' - \varphi''^2)\}/\varphi'(\varphi' + 2F^2\varphi'')$.

In view of (5.20) and (5.21), it follows, from $y_i|_k = \dot{\partial}_k y_i - y_r C_{ik}^r$, that

$$y_i|_k = a_{ik} + 2C_{ikj}y^j - C_{ijk}y^j, \tag{5.25}$$

that is, $y_i|_k = a_{ik} + C_{ikj}y^j$ as C_{ijk} is totally symmetric.

(3) Utilizing successively $\delta_i = \partial_i - N_i^r \dot{\partial}_r$, (4.5), and $\overset{\circ}{\delta}_i = \partial_i - \overset{\circ}{N}_i^r \dot{\partial}_r$ in $L_{|k} = \delta_k L$, we get

$$L_{|k} = \overset{\circ}{\delta}_k L + V_k^r \dot{\partial}_r L. \tag{5.26}$$

Using (1.2) and (2.1) in (5.26), we have

$$L_{|k} = \overset{\circ}{\delta}_k \left(\varphi + A_r y^r + U \right) + V_k^r \left(2\varphi' \overset{\circ}{y}_r + A_r \right), \tag{5.27}$$

which, in view of (5.3), gives

$$L_{|k} = \overset{\circ}{\delta}_k \left(A_r y^r + U \right) + V_k^r \left(2\varphi' \overset{\circ}{y}_r + A_r \right). \tag{5.28}$$

Using $\overset{\circ}{\delta}_k = \partial_k - \overset{\circ}{N}{}^r_k \dot{\partial}_r$ and (5.18) in (5.28) and keeping (4.5) in view, we find

$$L_{|k} = y^r \partial_k A_r - N^p_k A_p + \partial_k U + \frac{2\varphi'}{\varphi' + 2F^2 \varphi''} V^r_k y_r. \tag{5.29}$$

If we take $X_k = y^r \partial_k A_r - N^p_k A_p + \partial_k U$, then the last expression takes the form

$$L_{|k} = X_k + \frac{2\varphi'}{\varphi' + 2F^2 \varphi''} V^r_k y_r. \tag{5.30}$$

Next, using (2.1) in $L_{|k} = \dot{\partial}_k L$, we get

$$L|_k = 2\varphi' \overset{\circ}{y}_k + A_k, \tag{5.31}$$

which, in view of (5.18), gives the required result. □

Corollary 5.5 (see [14]). *The canonical metrical d-connection $C\Gamma(N)$ of a φ-Lagrange space has the following properties:*

$$(1)\ D^i_k := y^i_{|k} = 0, \qquad y_{i|k} = 0, \tag{5.32}$$

$$(2)\ d^i_k := y^i|_k = \frac{\varphi' + \varphi'' F^2}{\varphi'} \delta^i_k + B \overset{\circ}{y}_k y^i, \qquad y_i|_k = a_{ik} + C_{ikj} y^j, \tag{5.33}$$

where $B = 2\{\varphi'\varphi'' + F^2(\varphi'''\varphi' - \varphi''^2)\} / \varphi'(\varphi' + 2F^2\varphi'')$,

$$(3)\ L_{|k} = 0, \qquad L|_k = \frac{2\varphi'}{(\varphi' + 2F^2\varphi'')} y_k. \tag{5.34}$$

Proof. Applying $A_i(x) = 0$, $U(x) = 0$, and $V^i_j = 0$ in **Theorem 5.4**, we have the corollary. □

Corollary 5.6. *The canonical metrical d-connection $C\Gamma(N)$ of an AFL-space has the following properties:*

$$(1)\ D^i_k = B^i_k + B^p_r \overset{\circ}{C}{}^i_{kp} y^r, \qquad y_{i|k} = g_{si}\left(B^s_k - B^l_p y^p \overset{\circ}{C}{}^s_{lk}\right), \tag{5.35}$$

where $y_i = g_{ij}y^j$,

$$(2)\ d^i_k = \delta^i_k, \qquad y_i|_k = g_{ik}, \tag{5.36}$$

$$(3)\ L_{|k} = y^r \partial_k A_r - N^p_k A_p + \partial_k U + 2B^r_k y_r, \qquad L|_k = 2y_k + A_k. \tag{5.37}$$
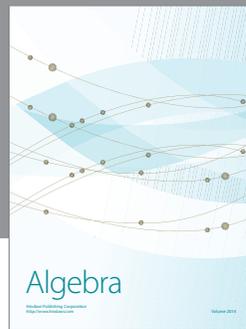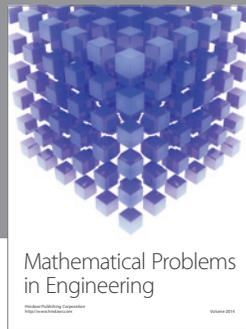
Proof. Using $\varphi(F^2) = F^2$, $\varphi'(F^2) = 1$, $\varphi''(F^2) = 0 = \varphi'''(F^2)$, $a^{ij} = g^{ij}$, $C_{ijk} = \overset{\circ}{C}_{ijk}$, $\overset{\circ}{C}_{ijk}y^j = 0$, $\overset{\circ}{C}{}^i_{jk}y^k = 0$, and $V^r_j = B^r_j$ in Theorem 5.4, we have the corollary. $\qquad\square$

Acknowledgment

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