

## Research Article

# On the Geometry of Almost $\mathcal{S}$ -Manifolds

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An  $f$ -structure on a manifold  $M$  is an endomorphism field  $\varphi$  satisfying  $\varphi^3 + \varphi = 0$ . We call an  $f$ -structure *regular* if the distribution  $T = \ker \varphi$  is involutive and regular, in the sense of Palais. We show that when a regular  $f$ -structure on a compact manifold  $M$  is an almost  $\mathcal{S}$ -structure, it determines a torus fibration of  $M$  over a symplectic manifold. When rank  $T = 1$ , this result reduces to the Boothby-Wang theorem. Unlike similar results for manifolds with  $\mathcal{S}$ -structure or  $\mathcal{K}$ -structure, we do not assume that the  $f$ -structure is normal. We also show that given an almost  $\mathcal{S}$ -structure, we obtain an associated Jacobi structure, as well as a notion of symplectization.

## 1. Introduction

Let  $(M, \eta)$  be a cooriented contact manifold. The Boothby-Wang theorem [1] tells us that if the Reeb field  $\xi$  corresponding to the contact form  $\eta$  is regular (in the sense of Palais [2]), then  $M$  is a prequantum circle bundle  $\pi : M \rightarrow N$  over a symplectic manifold  $(N, \omega)$ , where  $\pi^*\omega = -d\eta$  and  $\eta$  may be identified with the connection 1-form. Conversely, let  $M$  be a prequantum circle bundle over a symplectic manifold  $(N, \omega)$ , and let  $\eta$  be a connection 1-form. Given a choice of compatible almost complex structure  $J$  for  $\omega$ , let  $G(X, Y) = \omega(JX, Y)$  be the associated Riemannian metric on  $N$ , and let  $\tilde{\pi}$  denote the horizontal lift of vector fields defined by  $\eta$ . We can then define an endomorphism field  $\varphi \in \Gamma(M, \text{End}(TM))$  by

$$\varphi X = \tilde{\pi} J \pi_* X, \quad (1.1)$$

and a Riemannian metric  $g$  by  $g = \pi^*G + \eta \otimes \eta$ . If we let  $\xi$  be the vertical vector field satisfying  $\eta(\xi) = 1$ , then  $(\varphi, \xi, \eta, g)$  defines a contact metric structure on  $M$  [3]. In particular, we note that  $\varphi$  is an  $f$ -structure on  $M$ . By construction, we have  $\varphi^2 = -\text{Id}_{TM} + \eta \otimes \xi$ , from which it follows that  $\varphi^3 + \varphi = 0$ .

In [4, 5], Blair et al. consider compact Riemannian manifolds equipped with a regular normal  $f$ -structure  $\varphi$  and show that such manifolds are the total space of a principal torus bundle over a complex manifold  $N$ , and that in addition,  $N$  is a Kähler manifold if the fundamental 2-form of the  $f$ -structure is closed (i.e., if  $M$  is a  $\mathcal{K}$ -manifold). Saenz argued in [6] that if this  $\mathcal{K}$ -structure is an  $\mathcal{S}$ -structure, then the symplectic form of the Kähler manifold  $N$  is integral.

While the results in [5, 6] provide us with a generalization of the Boothby-Wang theorem, the proofs in [5] (and by extension, the argument in [6]) rely in several places on the assumption that the  $f$ -structure  $\varphi$  is normal. Since this assumption is not required in the original Boothby-Wang theorem, it is natural to ask what can be said if this assumption is dropped for  $f$ -structures of higher corank. In this note, we use a theorem of Tanno [7] to show that if  $M$  is a compact almost  $\mathcal{S}$ -manifold, in the sense of [8], then  $M$  is a principal torus bundle over a symplectic manifold whose symplectic form is integral. (More precisely, the symplectic form will be a real multiple of an integral symplectic form.) Not surprisingly, this tells us that requiring  $\varphi$  to be normal is the same as demanding that the base of our torus bundle be Kähler.

This “generalized Boothby-Wang theorem” is one of a number of similarities between manifolds with almost  $\mathcal{S}$ -structure and contact manifolds. In the final section of this paper we demonstrate two more. First, there is a natural notion of symplectization: given an almost  $\mathcal{S}$ -manifold  $M$ , there is an open, conic, symplectic submanifold of  $T^*M$  whose base is  $M$ . Second, a choice of one-form (expressed in terms of the almost  $\mathcal{S}$ -structure) allows us to define a Jacobi bracket on the algebra of smooth functions on  $M$ , giving us in particular a notion of Hamiltonian vector field on manifolds with almost  $\mathcal{S}$ -structure.

## 2. Preliminaries

### 2.1. Regular Involutive Distributions

Let  $F \subset TM$  be an involutive distribution of rank  $k$ . We briefly recall the notion of a regular distribution in the sense of Palais and refer the reader to [2] for the details. Roughly speaking, the involutive distribution  $F$  is *regular* if each point  $p \in M$  has a coordinate neighbourhood  $(U, x^1, \dots, x^n)$  such that

$$\left\{ \left( \frac{\partial}{\partial x^1} \right)_p, \dots, \left( \frac{\partial}{\partial x^k} \right)_p \right\} \quad (2.1)$$

forms a basis for  $F_p \subset T_pM$ , and such that the integral submanifold of  $F$  through  $p$  intersects  $U$  in only one  $k$ -dimensional slice. When  $F$  is regular, the leaf space  $\mathcal{F} = M/F$  is a smooth Hausdorff manifold, and the quotient mapping  $\pi_F : M \rightarrow \mathcal{F}$  is smooth and closed. When  $M$  is compact and connected, the leaves of  $F$  are compact and isomorphic and are the fibres of the smooth fibration  $\pi_F : M \rightarrow \mathcal{F}$ .

In particular, a vector field  $X$  on  $M$  is regular if each  $p \in M$  has a neighbourhood  $U$  through which the integral curve of  $X$  through  $p$  passes only once. If  $M$  is compact, the integral curves of a regular vector field are thus diffeomorphic to circles. Applying this fact to the Reeb vector field of a contact manifold gives part of the proof of the Boothby-Wang theorem.

## 2.2. $f$ -Structures

An  $f$ -structure on  $M$  is an endomorphism field  $\varphi \in \Gamma(M, \text{End } TM)$  such that

$$\varphi^3 + \varphi = 0. \quad (2.2)$$

Such structures were introduced by Yano in [9]; many of the facts regarding  $f$ -structures are collected in the book [10]. By a result of Stong [11], every  $f$ -structure is of constant rank. If  $\text{rank } \varphi = \dim M$ , then  $\varphi$  is an almost complex structure on  $M$ , while if  $\text{rank } \varphi = \dim M - 1$ , then  $\varphi$  determines an almost contact structure on  $M$ .

It is easy to check that the operators  $l = -\varphi^2$  and  $m = \varphi^2 + \text{Id}_{TM}$  are complementary projection operators; letting  $E = l(TM) = \text{im } \varphi$  and  $T = m(TM) = \text{ker } \varphi$ , we obtain the splitting

$$TM = E \oplus T = \text{im } \varphi \oplus \text{ker } \varphi \quad (2.3)$$

of the tangent bundle. Since  $(\varphi|_E)^2 = -\text{Id}_E$ ,  $\varphi$  is necessarily of even rank. When the corank of  $\varphi$  is equal to one, the distribution  $T$  is automatically trivial and involutive. However, if  $\text{rank } T > 1$ , this need not be the case, and one often makes additional simplifying assumptions about  $T$ . An  $f$ -structure such that  $T$  is trivial is called an  $f$ -structure with parallelizable kernel (or  $f$ -pk-structure for short) in [8]. We will assume that an  $f$ -pk-structure includes a choice of a trivializing frame  $\{\xi_i\}$  and corresponding coframe  $\{\eta^i\}$  for  $T^*$ , with

$$\eta^i(\xi_j) = \delta_j^i, \quad \varphi(\xi_i) = \eta^j \circ \varphi = 0, \quad \varphi^2 = -\text{Id} + \sum \eta^i \otimes \xi_i. \quad (2.4)$$

(This is known as an  $f$ -structure with complemented frames in [4]; such a choice of frame and coframe always exists.) Given an  $f$ -pk-structure, it is always possible [10] to find a Riemannian metric  $g$  that is compatible with  $(\varphi, \xi_i, \eta^i)$  in the sense that, for all  $X, Y \in \Gamma(M, TM)$ , we have

$$g(X, Y) = g(\varphi X, \varphi Y) + \sum_{i=1}^k \eta^i(X) \eta^i(Y). \quad (2.5)$$

Following [8], we will call the 4-tuple  $(\varphi, \xi_i, \eta^i, g)$  a *metric  $f$ -pk structure*. Given a metric  $f$ -pk-structure  $(\varphi, \xi_i, \eta^i, g)$ , we can define the *fundamental 2-form*  $\Phi_g \in \mathcal{A}^2(M)$  by

$$\Phi_g(X, Y) = g(\varphi X, Y). \quad (2.6)$$

*Remark 2.1.* Our definition of  $\Phi_g$  is chosen to agree with our preferred sign conventions in symplectic geometry; however, many authors place  $\varphi$  in the second slot, so our convention here uses the opposite sign of that found for example in [5, 8].

We will call an  $f$ -structure  $\varphi$  *regular* if the distribution  $T = \ker \varphi$  is regular in the sense of Palais [2]. An  $f$ -pk-structure is regular if the vector fields  $\xi_i$  are regular and independent. An  $f$ -pk-structure is called *normal* [4] if the tensor  $N$  defined by

$$N = [\varphi, \varphi] + \sum_{i=1}^k d\eta^i \otimes \xi_i \quad (2.7)$$

vanishes identically. Here  $[\varphi, \varphi]$  denotes the Nijenhuis torsion of  $\varphi$ , which is given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \quad (2.8)$$

When  $\varphi$  is normal, the  $+i$ -eigenbundle of  $\varphi$  (extended by  $\mathbb{C}$  linearity to  $T_{\mathbb{C}}M$ ) defines a CR structure  $E_{1,0} \subset T_{\mathbb{C}}M$ . Regular normal  $f$ -structures are studied in [5], where it is proved that a compact manifold with regular normal  $f$ -structure is a principal torus bundle over a complex manifold  $N$ . If the fundamental 2-form  $\Phi_g$  of a normal  $f$ -structure is closed, then the  $f$ -structure is called a  $\mathcal{K}$ -structure, and  $M$  a  $\mathcal{K}$ -manifold. For a compact regular  $\mathcal{K}$ -manifold  $M$ , the base  $N$  of the torus fibration is a Kähler manifold. A special case of a  $\mathcal{K}$ -manifold is an  $\mathcal{S}$ -manifold. On an  $\mathcal{S}$  manifold, there exist constants  $\alpha^1, \dots, \alpha^k$  such that  $d\eta^i = -\alpha^i \Phi_g$  for  $i = 1, \dots, k$ . Two commonly considered cases are the case  $\alpha^i = 0$  for all  $i$ , and the case  $\alpha^i = 1$  for all  $i$ . In the language of CR geometry, the former case is analogous to a ‘‘Levi-flat’’ CR manifold,

while the latter defines an analogue of a strongly pseudoconvex CR manifold (typically, strongly pseudoconvex CR manifolds are assumed to be of ‘‘hypersurface type,’’ meaning that the complementary distribution  $T$  has rank one; see [12]).

A refinement of the notion of  $\mathcal{S}$ -structure was introduced in [8]: a metric  $f$ -pk-structure  $(\varphi, \xi_i, \eta^i, g)$  which is not necessarily normal is called an *almost*  $\mathcal{S}$ -structure if  $d\eta^i = -\Phi_g$  for each  $i = 1, \dots, k$ . An  $f$ -structure  $\varphi$  is called CR-integrable in [8] if the  $+i$ -eigenbundle  $E_{1,0} \subset T_{\mathbb{C}}M$  of  $\varphi$  is involutive (and hence, defines a CR structure). It is shown in [8] that an  $f$ -pk-structure is CR-integrable if and only if the tensor  $N$  given by (2.7) satisfies  $N(X, Y) = 0$  for all  $X, Y \in \Gamma(M, E)$ , where  $E = \text{im } \varphi$ , whereas for a normal  $f$ -pk-structure,  $N$  must vanish for all  $X, Y \in \Gamma(M, TM)$ . In [13] it is proved that a CR-integrable almost  $\mathcal{S}$ -manifold admits a canonical connection analogous to the Tanaka-Webster connection of a strongly pseudoconvex CR manifold. For the relationship between this connection and the  $\bar{\partial}_b$  operator of the corresponding tangential Cauchy-Riemann complex, as well as an application of this relationship to defining an analogue of geometric quantization for almost  $\mathcal{S}$ -manifolds, see [14].

In this paper, we will define an almost  $\mathcal{K}$ -structure to be a metric  $f$ -pk-structure for which  $d\Phi_g = 0$ , and we will define an almost  $\mathcal{S}$ -structure more generally to be an almost  $\mathcal{K}$ -structure such that  $d\eta^i = -\alpha^i \Phi_g$  for constants  $\alpha^i \in \mathbb{R}$ , for  $i = 1, \dots, k$ .

### 3. Properties of Almost $\mathcal{K}$ and Almost $\mathcal{S}$ -Structures

Let  $(\varphi, \xi_i, \eta^i)$  be an  $f$ -pk-structure on a compact, connected manifold  $M$ . Let  $g$  be a Riemannian metric satisfying the compatibility condition (2.5), and let  $\Phi_g$  denote the corresponding fundamental 2-form. Let  $E = \text{im } \varphi$ , and  $T = \ker \varphi$  denote the distribution

spanned by the  $\xi_i$ . It is easy to check that the distributions  $E$  and  $T$  are orthogonal with respect to  $g$ , and that the restriction of  $\Phi_g$  to  $E \otimes E$  is nondegenerate, from which we have the following lemma.

**Lemma 3.1.**  $X \in \Gamma(M, T)$  if and only if  $\iota(X)\Phi_g = 0$ .

**Proposition 3.2.** Let  $(\varphi, \xi_i, \eta^i, g)$  be a metric  $f$ - $pk$ -structure. Then  $T = \ker \varphi$  is involutive whenever  $d\Phi_g = 0$ .

*Proof.* Let  $X, Y \in \Gamma(M, T)$ , and let  $Z \in \Gamma(M, TM)$ . Then, using Lemma 3.1 above, we have

$$\begin{aligned} d\Phi_g(X, Y, Z) &= X \cdot \Phi_g(Y, Z) + Y \cdot \Phi_g(Z, X) + Z \cdot \Phi_g(X, Y) \\ &\quad - \Phi_g([X, Y], Z) - \Phi_g([Y, Z], X) - \Phi_g([Z, X], Y) \\ &= -\Phi_g([X, Y], Z). \end{aligned} \quad (3.1)$$

Therefore, if  $d\Phi_g = 0$ , then  $\iota([X, Y])\Phi_g = 0$ , and thus  $[X, Y] \in \Gamma(M, T)$ , which proves the proposition.  $\square$

Let us now suppose that  $(\varphi, \xi_i, \eta^i, g)$  is an almost  $\mathcal{S}$ -structure, so that the 1-forms  $\eta^i$  satisfy  $d\eta^i = -\alpha^i\Phi_g$  for constants  $\alpha^i$ , some of which may be zero. The following results were proved in [8] in the case that  $\alpha^i = 1$  for all  $i$ ; we easily see that the results remain true in our more general setting.

**Proposition 3.3.** If  $(\varphi, \xi_i, \eta^i, g)$  is an almost  $\mathcal{S}$ -structure, then  $\mathcal{L}(\xi_i)\xi_j = [\xi_i, \xi_j] = 0$  for all  $i, j = 1, \dots, k$ .

*Proof.* Since the fundamental 2-form  $\Phi_g$  of an almost  $\mathcal{S}$ -structure is closed, the distribution  $T$  is involutive. Thus we may write  $[\xi_i, \xi_j] = \sum c_{ij}^a \xi_a$ . But for any  $a, i, j \in \{1, \dots, k\}$ , we have

$$c_{ij}^a = \eta^a([\xi_i, \xi_j]) = \xi_i \cdot \eta^a(\xi_j) - \xi_j \cdot \eta^a(\xi_i) - d\eta^a(\xi_i, \xi_j) = \alpha^a \Phi_g(\xi_i, \xi_j) = 0. \quad (3.2)$$

**Proposition 3.4.** If  $(\varphi, \xi_i, \eta^i, g)$  is an almost  $\mathcal{S}$ -structure, then  $\mathcal{L}(\xi_i)\eta^j = 0$  for all  $i, j = 1, \dots, k$ .

*Proof.* We have  $\mathcal{L}(\xi)\eta^j = d(\eta^j(\xi_i)) + \iota(\xi_i)d\eta^j = -\alpha^j \iota(\xi_i)\Phi_g = 0$ .  $\square$

We remark that several other results from [8] hold in this more general setting, but they are not needed here. To conclude this section, we state a theorem due to Tanno [7].

**Theorem 3.5.** For a regular and proper vector field  $X$  on a manifold  $M$ , the following are equivalent.

- (i) The period function  $\lambda_X$  of  $X$  is constant.
- (ii) There exists a 1-form  $\eta$  such that  $\eta(X) = 1$  and  $\mathcal{L}(X)\eta = 0$ .
- (iii) There exists a Riemannian metric  $g$  such that  $g(X, X) = 1$  and  $\mathcal{L}(X)g = 0$ .

In the above theorem, the period function  $\lambda_X : M \rightarrow \mathbb{R}$  is defined by

$$\lambda_X(p) = \inf\{t > 0 \mid \exp(tX) \cdot p = p\}. \quad (3.3)$$

If  $M$  is noncompact, the value  $\lambda_X(p) = \infty$  is possible. Part (iii) of the above tells us that  $X$  is a unit Killing field for the metric  $g$ . Using this result, Tanno was able to give a simple proof (which is reproduced in [3]) of the Boothby-Wang theorem [1].

#### 4. The Structure of Regular Almost $\mathcal{S}$ -Manifolds

As noted above, from [5], a compact manifold with regular normal  $f$ -structure is a principal torus bundle over a complex manifold  $N$ , and  $N$  is Kähler if  $M$  is a  $\mathcal{K}$ -manifold. If  $M$  is an  $\mathcal{S}$ -manifold with  $\Phi_g = -d\eta^i$  for each  $i$ , then by [6], the symplectic form on  $N$  is integral. We now dispense with the requirement that the  $f$ -structure on  $M$  be normal, and state a similar result for almost  $\mathcal{S}$ -manifolds.

**Theorem 4.1.** *Let  $M$  be a compact manifold of dimension  $2n + k$  equipped with a regular almost  $\mathcal{S}$ -structure  $(\varphi, \tilde{\xi}_i, \tilde{\eta}^i, \tilde{g})$  of rank  $2n$ . Then there exists an almost  $\mathcal{S}$ -structure  $(\varphi, \xi_i, \eta^i, g)$  on  $M$  for which the vector fields  $\xi_1, \dots, \xi_k$  are the infinitesimal generators of a free and effective  $\mathbb{T}^k$ -action on  $M$ . Moreover, the quotient  $N = M/\mathbb{T}^k$  is a smooth symplectic manifold of dimension  $2n$ , and if the  $\alpha^i$  such that  $d\tilde{\eta}^i = -\alpha^i \Phi_{\tilde{g}}$  are not all zero, then the symplectic form on  $N$  is a real multiple of an integral symplectic form.*

*Proof.* By assumption, the vector fields  $\tilde{\xi}_1, \dots, \tilde{\xi}_k$  are regular, independent, and proper, and by Proposition 3.2, the distribution  $T = \text{span}\{\tilde{\xi}_1, \dots, \tilde{\xi}_k\}$  is involutive. Thus, by the results of Palais,  $N = M/T$  is a smooth manifold, and  $\pi : M \rightarrow N$  is a smooth fibration whose fibres are the leaves of the distribution  $T$ . Since  $M$  is compact, the fibres are compact and isomorphic [2]. For each  $i = 1, \dots, k$ , we have  $\tilde{\eta}^i(\tilde{\xi}_i) = 1$  and  $\mathcal{L}(\tilde{\xi}_i)\tilde{\eta}^i = 0$ . Thus, by Theorem 3.5, the period functions  $\lambda_i = \lambda_{\tilde{\xi}_i}$  are constant. We rescale by setting  $\xi_i = \lambda_i \tilde{\xi}_i$  and  $\eta^i = (1/\lambda_i)\tilde{\eta}^i$ . We still have  $\eta^i(\xi_j) = \delta_j^i$ , and note that the associated metric  $g$  for which  $(\varphi, \xi_i, \eta^i, g)$  is an almost  $\mathcal{S}$ -structure differs from  $\tilde{g}$  only along  $T$ , so that  $\Phi_g = \Phi_{\tilde{g}}$ . Each  $\xi_i$  now has period 1, and since the vector fields  $\xi_i$  all commute, they are the generators of a free and effective  $\mathbb{T}^k$ -action on  $M$ . The argument for local triviality is the same as in [5], so we do not repeat it here. Thus, we have that  $M$  is a principal  $\mathbb{T}^k$ -bundle over  $N = M/T$ . The infinitesimal action of  $\mathbb{R}^k$  is given by

$$X = (t^1, \dots, t^k) \mapsto X_M = \sum t^i \xi_i, \quad (4.1)$$

from which we see that  $\boldsymbol{\eta} = (\eta^1, \dots, \eta^k)$  is a connection 1-form on  $M$ : we have  $\iota(X_M)\boldsymbol{\eta} = X$  and  $\mathcal{L}(X_M)\boldsymbol{\eta} = 0$  for all  $X \in \mathbb{R}^k$ .

Now, we note that the fundamental 2-form  $\Phi_g$  is horizontal and invariant, since  $\iota(X)\Phi_g = \mathcal{L}(X)\Phi_g = 0$  for all  $X \in \Gamma(M, T)$ , and thus there exists a 2-form  $\Omega$  on  $N$  such that  $\pi^*\Omega = \Phi_g$ . Since  $\pi^*d\Omega = d\Phi_g = 0$ ,  $\Omega$  is closed, and since  $\pi^*\Omega^n = \Phi_g^n \neq 0$ ,  $\Omega$  is nondegenerate, and hence symplectic.

Finally, let us suppose that one of the  $\alpha^i$  is nonzero; without loss of generality, let us say  $\alpha^1 \neq 0$ . By the same argument as above, the vector fields  $\xi_2, \dots, \xi_k$  generate a free  $\mathbb{T}^{k-1}$ -action on  $M$ , giving us a fibration  $p : M \rightarrow P$ . Now, since  $\mathcal{L}(\xi_i)\xi_1 = \mathcal{L}(\xi_i)\eta^1 = 0$  for  $i = 2, \dots, k$ , the vector field  $\xi_1$  and 1-form  $\eta^1$  are invariant under the  $\mathbb{T}^{k-1}$ -action. We can thus define a 1-form  $\eta$  on  $P$  by  $\eta(X) = \eta^1(\tilde{p}X)$ , where  $\tilde{p}X$  denotes the horizontal lift of  $X$  with respect to the connection 1-form defined by  $\eta^2, \dots, \eta^k$ , and a vector field  $\xi$  on  $P$  by  $\xi = p_*\xi_1$ . Note that

$d\eta(X, Y) = d\eta^1(\tilde{p}X, \tilde{p}Y)$ . We then have  $\eta(\xi) = 1$ , and  $\mathcal{L}(\xi)\eta = \iota(\xi^1)d\eta^1 = 0$ , so that Theorem 3.5 applies to the pair  $(\eta, \xi)$ . It follows that  $\xi$  generates a free action of  $S^1 = \mathbb{R}/\mathbb{Z}$  on  $P$ , giving us the  $\mathbb{T}^1$ -bundle structure  $q : P \rightarrow N$ . Since  $\pi = q \circ p$ , it follows that

$$d\eta(X, Y) = d\eta^1(\tilde{p}X, \tilde{p}Y) = -\frac{\alpha^1}{\lambda_1}(\pi^*\Omega)(\tilde{p}X, \tilde{p}Y) = -\frac{\alpha^1}{\lambda_1}q^*\Omega(X, Y). \quad (4.2)$$

Thus,  $P$  is a Boothby-Wang fibration over  $(N, (\alpha^1/\lambda_1)\Omega)$ , from which it follows that the symplectic form  $(\alpha^1/\lambda)\Omega$  must be integral (see [15]), and hence  $\Omega$  is a real multiple of an integral symplectic form.  $\square$

*Remark 4.2.* Note that since the last part of the argument is valid for any pair of nonzero constants  $\alpha^i, \alpha^j$ , from which it follows that for each  $i, j$  for which  $\alpha^i$  and  $\alpha^j$  are nonzero, we must have  $\alpha^i/\lambda_i \cdot \lambda_j/\alpha^j \in \mathbb{Q}$ .

Conversely, we have the following theorem.

**Theorem 4.3.** *Suppose that  $M$  is a principal  $\mathbb{T}^k$ -bundle over a symplectic manifold  $(N, \omega)$ , equipped with connection 1-form  $\boldsymbol{\eta} = (\eta^1, \dots, \eta^k)$  such that there exist constants  $\alpha^1, \dots, \alpha^k$  for which  $d\eta^i = -\alpha^i\pi^*\omega$ . Then  $M$  admits an almost  $\mathcal{S}$ -structure.*

*Proof.* The proof is essentially the same as the proof given in [4] when  $N$  is Kähler, if we omit the proof of normality. Given a choice of compatible almost complex structure  $J$  and associated metric  $G$ , we can define an  $f$ -structure  $\varphi$  by  $\varphi X = \tilde{\pi}J\pi_*X$ , where  $\tilde{\pi}$  denotes the horizontal lift with respect to  $\boldsymbol{\eta}$ . If we let  $\xi_1, \dots, \xi_k$  denote vertical vectors such that  $\eta^i(\xi_j) = \delta_j^i$ , and define the metric  $g$  by

$$g(X, Y) = \pi^*G(X, Y) + \sum \eta^i(X)\eta^i(Y), \quad (4.3)$$

then it is straightforward to check that the data  $(\varphi, \xi_i, \eta^i, g)$  defines an almost  $\mathcal{S}$ -structure on  $M$ . (Note that  $\Phi_g = \pi^*\omega$ , so that  $d\eta^i = -\alpha^i\Phi_g$ .)  $\square$

*Remark 4.4.* We can also use the results of Tanno [7] to show that the vector fields  $\xi_1, \dots, \xi_k$  of an almost  $\mathcal{S}$ -structure are Killing. Let  $\tilde{\pi}$  denote the horizontal lift defined by  $\boldsymbol{\eta}$ . Then we can define a Riemannian metric  $G$  on  $N$  by  $G(X, Y) = g(\tilde{\pi}X, \tilde{\pi}Y)$  for any  $X, Y \in \Gamma(N, TN)$ , where  $g$  is the metric of the almost  $\mathcal{S}$ -structure on  $M$ . It follows that  $g = \pi^*G + \sum \eta^i \otimes \eta^i$ , whence  $g(\xi_i, \xi_i) = 1$  and  $\mathcal{L}(\xi_i)g = 0$  for  $i = 1, \dots, k$ . Moreover, the endomorphism field  $J \in \Gamma(N, \text{End}(TN))$  defined by  $JX = \pi_*\varphi\tilde{\pi}X$  is easily seen to be an almost complex structure on  $N$  that is compatible with  $G$ , and the symplectic form  $\Omega$  then satisfies  $\Omega(X, Y) = G(X, JY)$ .

*Remark 4.5.* If  $M$  is only an almost  $\mathcal{K}$ -manifold, it is not clear that we can expect any analogous result to hold, since the proof in [5] for a  $\mathcal{K}$ -manifold does not work without normality, and Tanno's theorem cannot be applied if  $\mathcal{L}(\xi_i)\eta^j \neq 0$  for all  $i, j$ , and this need not hold if  $d\eta^j$  is not a multiple of  $\Phi_g$ .

*Remark 4.6.* If  $M$  is noncompact, then as noted below the statement of Tanno's theorem, the period  $\lambda_i$  of one of the  $\xi_i$  could be infinite, in which case  $\xi_i$  generates an  $\mathbb{R}$ -action on  $M$  instead of an  $S^1$ -action.

## 5. Symplectization and Jacobi Structures

We conclude this paper with a discussion of the relationship between almost  $\mathcal{S}$ -structures and related geometries intended to reinforce the view that almost  $\mathcal{S}$ -structures deserve to be viewed as higher corank analogues of contact structures. (However, see also [16] for the notion of  $k$ -contact structures, which, from the point of view of Heisenberg calculus, are also deserving of the title of higher corank contact structure. From this perspective, almost  $\mathcal{S}$ -structures are perhaps more analogous to contact metric structures, or even strongly pseudoconvex CR structures, although they are not CR-integrable in general.)

Recall that a stable complex structure on a manifold  $M$  is a complex structure defined on the fibres of  $TM \oplus \mathbb{R}^k$  for some  $k$ . Given an  $f$ -pk-structure  $(\varphi, \xi_i, \eta^j)$  on  $M$ , we obtain a stable complex structure  $J \in \Gamma(M, \text{End}(TM \oplus \mathbb{R}^k))$  by setting  $JX = \varphi X$  for  $X \in \Gamma(M, E)$ , and defining  $J\xi_i = \tau_i$  and  $J\tau_i = -\xi_i$ , where  $\tau_1, \dots, \tau_k$  is a basis for  $\mathbb{R}^k$ . As explained in [17], a stable complex structure determines a  $\text{Spin}^c$ -structure on  $M$ .

Alternatively, (and with some abuse of notation), we can think of the above complex structure on each fibre  $T_x M \times \mathbb{R}^k$  as coming from an almost complex structure on  $M \times \mathbb{R}^k$  obtained from the  $f$ -structure  $\varphi$ . With this point of view, we note that it is possible to define a ‘‘symplectization’’ analogous to the symplectization of a cooriented contact manifold, provided that our  $f$ -pk-structure is an almost  $\mathcal{S}$ -structure, with at least one of the  $\alpha^i$  (such that  $d\eta^j = -\alpha^j \Phi_g$ ) nonzero. As above, we let  $TM = E \oplus T$  denote the splitting of the tangent bundle determined by the  $f$ -structure, and let  $E^0 \cong T^* = \text{span}\{\eta^i\} \cong M \times \mathbb{R}^k$  denote the annihilator of  $E$ . It is then possible to find an open connected symplectic submanifold  $E_+^0$  of  $T^*M$  whose tangent bundle is  $T_x M \times \mathbb{R}^k$ . For concreteness, let us use the identification  $E^0 \cong M \times \mathbb{R}^k$ , and with respect to coordinates  $(x, t_1, \dots, t_k)$ , let

$$\alpha = \sum_{i=1}^k t_i \eta^i, \quad (5.1)$$

and define  $\omega = -d\alpha$ . (We are abusing notation here slightly; technically we should write  $\pi^* \eta^i$  in place of  $\eta^i$ , where  $\pi : M \times \mathbb{R}^k \rightarrow M$  is the projection onto the first factor.) Using the fact that  $d\eta^i = -\alpha^i \Phi_g$  for each  $i$ , we have

$$\omega = \sum \eta^j \wedge dt_j + \left( \sum t_j \alpha^j \right) \Phi_g. \quad (5.2)$$

Define  $\tau \in C^\infty(E^0)$  to be the function given in coordinates by  $\tau = \sum \alpha^j t_j$ . Note that since  $\eta^i \wedge \eta^i = dt_i \wedge dt_i = 0$ , we have

$$\left( \sum_{i=1}^k \eta^i \wedge dt_i \right)^k = k! \eta^1 \wedge dt_1 \wedge \dots \wedge \eta^k \wedge dt_k. \quad (5.3)$$

We also note that  $\Phi_g^m = 0$  for  $m > n$ . Thus, using the binomial theorem, we find that the top-degree form  $\omega^{n+k}$  has only one nonzero term; namely,

$$\omega^{n+k} = \frac{(n+k)!}{n!} \eta^1 \wedge dt_1 \wedge \dots \wedge \eta^k \wedge dt_k \wedge (\tau \Phi_g)^n. \quad (5.4)$$

Thus,  $\omega^{n+k}$  is a volume form on the open subset  $E_+^0$  of  $E^0$  defined by  $\tau > 0$ , and hence  $\omega$  is a symplectic form on  $E_+^0$ .

Next, we will show that for certain choices of section  $\eta \in \Gamma(M, E^0)$  we obtain a Jacobi structure on  $M$  defined in a manner analogous to the Jacobi structure associated to a choice of contact form on a contact manifold. We recall that a Jacobi structure on  $M$  is given by a Lie bracket  $\{\cdot, \cdot\}$  on  $C^\infty(M)$  such that for any  $f, g \in C^\infty(M)$  the support of  $\{f, g\}$  is contained in the intersection of the supports of  $f$  and  $g$ . Jacobi structures were introduced independently by Kirillov [18] and Lichnerowicz [19]; a good introduction can be found in [20].

Again, we assume  $M$  is equipped with an almost  $\mathcal{S}$ -structure with the constants  $\alpha^j$  such that  $d\eta^j = -\alpha^j\Phi_g$  not all zero. Our first goal is to define a notion of a Hamiltonian vector field  $X_f$  associated to each function  $f \in C^\infty(M)$ . To begin with, let  $\xi = \sum b^j\xi_j$  be an arbitrary section of  $T = \ker\varphi$ , and let  $\eta = \sum c_j\eta^j$  be an arbitrary section of  $E^0 \cong T^*$ . We will narrow down the possibilities for  $\xi$  and  $\eta$  as we consider the properties we wish the vector fields  $X_f$  to satisfy. The idea is to generalize the approach used to define Hamiltonian vector fields on a contact manifold  $(M, \eta)$ . Recall that on manifold equipped with a contact form  $\eta$ , where we define  $\Phi = -d\eta$ , the Reeb vector field  $\xi$  is defined by  $\iota(\xi)\eta = 1$  and  $\iota(\xi)\Phi = 0$ . A contact Hamiltonian vector field  $X_f$  satisfies the equations  $\iota(X_f)\eta = f$  and  $\iota(X_f)\Phi = df - (\xi \cdot f)\eta$ . Lichnerowicz showed in [21] that these are the necessary and sufficient conditions for each  $X_f$  to be an infinitesimal symmetry of the contact structure: it follows that for each  $f \in C^\infty(M)$ ,  $\mathcal{L}(X_f)\eta = (\xi \cdot f)\eta$ .

We wish to impose similar conditions on  $\xi$ ,  $\eta$  and (the yet to be defined)  $X_f$  in the case of almost  $\mathcal{S}$ -manifolds. We already know that  $\iota(\xi)\Phi_g = 0$ , by Lemma 3.1, so we begin by adding the requirement that  $\eta(\xi) = \sum b^jc_j = 1$ . Next, we give our definition of a Hamiltonian vector field.

*Definition 5.1.* Let  $\eta$  and  $\xi$  be as above. For any  $f \in C^\infty(M)$ , we define the *Hamiltonian vector field* associated to  $f$  by the equations

$$\iota(X_f)\eta^j = \alpha^j f, \quad \text{for } j = 1, \dots, k, \quad (5.5)$$

$$\iota(X_f)\Phi_g = df - (\xi \cdot f)\eta. \quad (5.6)$$

*Remark 5.2.* Note that the above equations uniquely define  $X_f$ , by the nondegeneracy of the restriction of  $\Phi$  to  $E = \text{im}\varphi$ . The constants  $\alpha^j$  are the same ones such that  $d\eta^j = -\alpha^j\Phi_g$ . One can check that if we began with  $a^j$  in place of the  $\alpha^j$ , we would be forced to take  $a^j = \alpha^j$  for consistency reasons. (In particular, this will be necessary if the bracket we define below is to be a Lie bracket.) Moreover, this gives us the identity

$$\mathcal{L}(X_f)\eta^j = \alpha^j(\xi \cdot f)\eta \quad (5.7)$$

for each  $j = 1, \dots, k$ ; we would otherwise have an unwanted term of the form  $(a^j - \alpha^j)df$ . Note that on the right-hand side of the above equation we have  $\eta$  and not  $\eta^j$ ; this is unavoidable with our definition of  $X_f$ .

We can fix the coefficients of  $\xi$  by requiring that  $\xi$  be the Hamiltonian vector field associated to the constant function 1, as is standard for Jacobi structures (see [20]). It is easy to see that (5.5) then immediately forces us to take  $\xi = \sum \alpha^j\xi_j$ ; that is, the coefficients  $b^j$  are

equal the constants  $\alpha^j$ . Thus,  $\xi$  is essentially determined by the almost  $\mathcal{S}$ -structure, although  $\eta$  is constrained only by the condition  $\eta(\xi) = 1$ , so the Jacobi structure we define below cannot be considered entirely canonical (as one might expect). From the requirement that  $\eta(\xi) = 1$ , it follows that for each  $f \in C^\infty(M)$ , we have

$$\mathcal{L}(X_f)\eta = \sum c_j \mathcal{L}(X_f)\eta^j = \sum c_j \alpha^j (\xi \cdot f)\eta = (\xi \cdot f)\eta, \quad (5.8)$$

again in analogy with the contact case. Note that the normalization  $\eta(\xi) = 1$  also implies that  $d\eta = -\Phi_g$ . We are now ready to define our bracket on  $C^\infty(M)$ .

*Definition 5.3.* Let  $M$  be a manifold with almost  $\mathcal{S}$ -structure, with constants  $\alpha^j$  not all zero. Let  $\xi = \sum \alpha^j \xi_j$ , and let  $\eta$  be a section of  $E^0$  such that  $\eta(\xi) = 1$ . We then define a bracket on  $C^\infty(M)$  by

$$\{f, g\} = \iota([X_f, X_g])\eta. \quad (5.9)$$

The bracket is clearly antisymmetric, and one checks (using the identity  $\iota([X, Y]) = [\mathcal{L}(X), \iota(Y)]$ ) that

$$\{f, g\} = X_f \cdot g - X_g \cdot f + \Phi_g(X_f, X_g) = X_f \cdot g - (\xi \cdot f)g. \quad (5.10)$$

Note that since the definition of the Hamiltonian vector fields depended on the choice of  $\eta$ , the bracket depends on  $\eta$ , even though  $\eta$  no longer appears explicitly in either of the above expressions for the bracket. From the latter equality we see that the support of  $\{f, g\}$  is contained in the support of  $g$ , and by antisymmetry it must be contained in the support of  $f$  as well. Thus, the bracket given by (5.9) is a Jacobi bracket provided we can verify the Jacobi identity. Since the Jacobi identity is valid for the Lie bracket on vector fields, it suffices to prove the following proposition.

**Proposition 5.4.** *Let  $\{f, g\}$  be the bracket on  $C^\infty(M)$  given by (5.9). Then the vector field  $X_{\{f, g\}}$  corresponding to the function  $\{f, g\}$  is given by  $X_{\{f, g\}} = [X_f, X_g]$ .*

**Lemma 5.5.** *For each  $i = 1, \dots, k$ , we have  $[\xi_i, X_f] = X_{\xi_i \cdot f}$ .*

*Proof.* From Propositions 3.3 and 3.4, we know that  $[\xi_i, \xi_j] = 0$  and  $\mathcal{L}(\xi_i)\eta^j = 0$  for any  $i, j \in \{1, \dots, k\}$ ; from the latter, it follows easily that  $\mathcal{L}(\xi_i)\Phi_g = 0$  as well. The result then follows from the uniqueness of Hamiltonian vector fields, since

$$\begin{aligned} \iota([\xi_i, X_f])\eta^j &= [\mathcal{L}(\xi_i), \iota(X_f)]\eta^j = \alpha^j \xi_i \cdot f, \\ \iota([\xi_i, X_f])\Phi_g &= \mathcal{L}(\xi_i)(df - (\xi \cdot f)\eta) = d(\xi_i \cdot f) - (\xi \cdot (\xi_i \cdot f))\eta. \end{aligned} \quad (5.11)$$

□

**Lemma 5.6.** *For each  $i = 1, \dots, k$ , we have  $\xi_i \cdot \{f, g\} = \{\xi_i \cdot f, g\} + \{f, \xi_i \cdot g\}$ .*

*Proof.* We have, using Lemma 5.5 and the fact that  $[\xi_i, \xi] = 0$  in the second line,

$$\begin{aligned}\xi_i \cdot \{f, g\} &= \xi_i \cdot (X_f \cdot g) - \xi_i \cdot ((\xi \cdot f)g) \\ &= X_f \cdot (\xi_i \cdot g) - (\xi \cdot f)(\xi_i \cdot g) + X_{\xi_i \cdot f} \cdot g - \xi(\xi_i \cdot f)g \\ &= \{f, \xi_i \cdot g\} + \{\xi_i \cdot f, g\}.\end{aligned}\tag{5.12}$$

□

*Proof of Proposition 5.4.* We need to show that  $\iota([X_f, X_g])\eta^j = \alpha^j \{f, g\}$  for each  $j = 1, \dots, k$ , and that  $\iota([X_f, X_g])\Phi = d\{f, g\} - (\xi \cdot \{f, g\})\eta$ . First, since  $\iota(X_g)\eta = \sum c_j \alpha^j g = g$ , we have

$$\iota([X_f, X_g])\eta^j = \mathcal{L}(X_f)\eta^j(X_g) - \iota(X_g)\mathcal{L}(X_f)\eta^j = \alpha^j X_f \cdot g - \iota(X_g)(\alpha^j \xi \cdot f)\eta = \alpha^j \{f, g\}.\tag{5.13}$$

From Lemma 5.6, we have  $\xi \cdot \{f, g\} = \{f, \xi \cdot g\} - \{g, \xi \cdot f\} = X_f \cdot (\xi \cdot g) - X_g \cdot (\xi \cdot f)$ , and, thus,

$$\begin{aligned}\iota([X_f, X_g])\Phi_g &= \mathcal{L}(X_f)(dg - (\xi \cdot g)\eta) - \iota(X_g)(-d(\xi \cdot f) \wedge \eta + (\xi \cdot f)\Phi_g) \\ &= d(X_f \cdot g) - X_f \cdot (\xi \cdot g) - (\xi \cdot g)(\xi \cdot f)\eta + X_g \cdot (\xi \cdot f)\eta \\ &\quad - gd(\xi \cdot f) - (\xi \cdot f)(dg - (\xi \cdot g)\eta) \\ &= d(X_f \cdot g - (\xi \cdot f)g) - (X_f \cdot (\xi \cdot g) - X_g \cdot (\xi \cdot f))\eta \\ &= d\{f, g\} - \xi \cdot \{f, g\}\eta.\end{aligned}\tag{5.14}$$

□

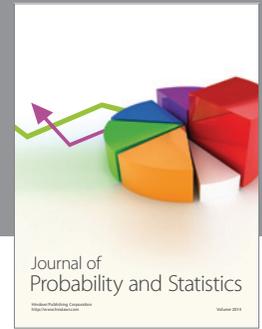
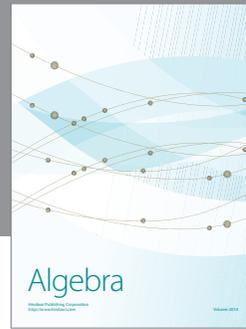
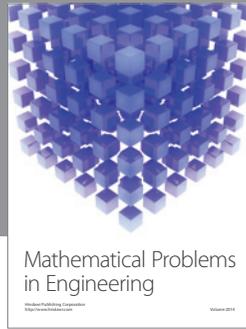
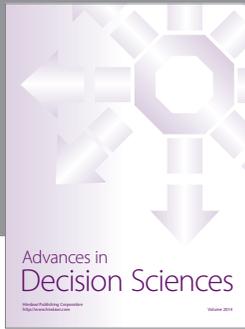
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